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INJECTIVE PARTIAL TRANSFORMATIONS WITH INFINITE DEFECTS

Boorapa Singha, Jintana Sanwong, and Robert Patrick Sullivan

To K. P. Shum on his 70th birthday, a respected mentor for mathematics in Asia

Abstract. In 2003, Marques-Smith and Sullivan described the join Ω of the 'natural order' $\leq$ and the 'containment order' $\subseteq$ on $P(X)$, the semigroup under composition of all partial transformations of a set X. And, in 2004, Pinto and Sullivan described all automorphisms of $PS(q)$, the partial Baer-Levi semigroup consisting of all injective $\alpha \in P(X)$ such that $|X \setminus X\alpha| = q$, where $\aleph_0 \leq q \leq |X|$. In this paper, we describe the group of automorphisms of $R(q)$, the largest regular subsemigroup of $PS(q)$. In 2010, we studied some properties of $\leq$ and $\subseteq$ on $PS(q)$. Here, we characterize the meet and join under those orders for elements of $R(q)$ and $PS(q)$. In addition, since $\leq$ does not equal Ω on $I(X)$, the symmetric inverse semigroup on X, we formulate an algebraic version of Ω on arbitrary inverse semigroups and discuss some of its properties in an algebraic setting.

1. Introduction

Suppose X is a non-empty set, and let $P(X)$ denote the semigroup (under composition) of all *partial* transformations of X (that is, all mappings $\alpha : A \to B$, where $A, B \subseteq X$). For any $\alpha \in P(X)$, we let $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$ denote the *domain* of α and *range* of α, respectively. We also write

$$g(\alpha) = |X \setminus \operatorname{dom}\alpha|, \quad d(\alpha) = |X \setminus \operatorname{ran}\alpha|,$$

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and refer to these cardinals as the *gap* and the *defect* of α, respectively. And, as usual, $I(X)$ denotes the *symmetric inverse semigroup* on X (see [2, vol 1, p. 29]): that is, the set of all injective mappings in $P(X)$. If $|X| = p \geq q \geq \aleph_0$, we write

$$PS(q) = \{\alpha \in I(X) : d(\alpha) = q\}$$

and call this the *partial Baer-Levi semigroup* on X (as first defined in [12, p. 82]). When necessary, we will use the notation $PS(X, p, q)$ to highlight the set X and its cardinal p.

In [9, Theorem 2], the authors proved that $\mathrm{Aut}PS(q)$, the group of all automorphisms of $PS(q)$, is isomorphic to $G(X)$, the symmetric group on X. They also showed that, if X and Y are sets such that $|X| = p \geq q \geq \aleph_0$ and $|Y| = r \geq s \geq \aleph_0$, then $PS(X, p, q)$ is isomorphic to $PS(Y, r, s)$ if and only if $p = r$ and $q = s$ (see [9, Theorem 3]). In addition, as shown in [9, Corollary 1], $PS(q)$ contains an inverse semigroup

$$R(q) = \{\alpha \in PS(q) : g(\alpha) = q\}$$

which consists of all the regular elements of $PS(q)$. By following the ideas in [9, Section 2], we show in Section 3 that these results about automorphisms and isomorphisms also hold for $R(q)$.

In [8] Mitsch defined a partial order on an arbitrary semigroup S by

$$a \leq b \quad \text{if and only if} \quad a = xb = by \text{ and } a = ay \text{ for some } x, y \in S^1,$$

and now this is called the *natural order* on S. Later in [5] the authors studied various properties of this order on the semigroup $T(X)$ consisting of all *total* transformations of X (that is, all $\alpha \in P(X)$ for which $\operatorname{dom}\alpha = X$). Then in [7] Marques-Smith and Sullivan extended some of the previous work to the ordered semigroups $(P(X), \leq)$ and $(P(X), \subseteq)$, where $\subseteq$ denotes the *containment order* on $P(X)$: that is, the partial order defined by

$$\alpha \subseteq \beta \quad \text{if and only if} \quad \operatorname{dom}\alpha \subseteq \operatorname{dom}\beta \;\text{ and }\; x\alpha = x\beta \text{ for all } x \in \operatorname{dom}\alpha.$$

They also defined partial orders Ω' and Ω on $P(X)$ as follows.

$$(\alpha, \beta) \in \Omega' \text{ if and only if } X\alpha \subseteq X\beta,\; \operatorname{dom}\alpha \subseteq \operatorname{dom}\beta \text{ and } \alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) \subseteq \alpha\alpha^{-1},$$

$$(\alpha, \beta) \in \Omega \text{ if and only if } (\alpha, \beta) \in \Omega' \text{ and } \beta\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) \subseteq \alpha\alpha^{-1}.$$

And, in [7, Theorem 7], they proved that Ω equals the join of $\leq$ and $\subseteq$ in the poset of all partial orders on $P(X)$.

In [11] the authors observed that $\leq \,=\, \subseteq$ and $\Omega = \Omega'$ on $I(X)$, but $\leq$ does not equal Ω on $I(X)$. In Section 6, we define a new partial order on any inverse semigroup, show that it equals Ω on $I(X)$ and discuss some of its algebraic properties. On the other hand, it was shown in [11] that, when restricted to $PS(q)$, $\leq$ is properly contained in $\subseteq$, and $\subseteq$ is properly contained in Ω. In Sections 4 and 5, we characterize the meet and join for elements of $R(q)$ and

$PS(q)$ under $\leq$ and $\subseteq$. We leave the more complicated problem about meets and joins in these semigroups under Ω to a subsequent paper.

2. Preliminary notation and results

In this paper, $Y = A \dot{\cup} B$ means Y is a *disjoint* union of A and B. As usual, $\emptyset$ denotes the empty (one-to-one) mapping which acts as a zero for $P(X)$. For each non-empty $A \subseteq X$, we write id_A for the identity transformation on A: these mappings constitute all the idempotents in $I(X)$ and belong to $PS(q)$ precisely when $|X \setminus A| = q$.

It is well-known that, for each non-zero $\alpha \in I(X)$, $\alpha\alpha^{-1} = \mathrm{id}_{\mathrm{dom}\,\alpha}$ and $\alpha^{-1}\alpha = \mathrm{id}_{\mathrm{ran}\,\alpha}$. Consequently, this is also true for $PS(q)$ and we use this fact without further mention.

We modify the convention introduced in [2, vol 2, p. 241]: namely, if $\alpha \in I(X)$ is non-zero, then we write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{x_i\}$ denotes $\{x_i : i \in I\}$, and that $\mathrm{ran}\,\alpha = \{x_i\}$, $x_i\alpha^{-1} = \{a_i\}$ and $\mathrm{dom}\,\alpha = \{a_i : i \in I\}$. For simplicity, if $A \subseteq X$, we sometimes write $A\alpha$ in place of $(A \cap \mathrm{dom}\,\alpha)\alpha$. In addition, we let x_a denote the mapping with domain $\{x\}$ and range $\{a\}$.

In [1], the authors showed that, if $|X| = p \geq q \geq \aleph_0$, then

$$A(X) = \{\alpha \in I(X) : g(\alpha) = d(\alpha)\}$$

is a *factorisable* inverse semigroup (that is, $A(X) = GE$, where G is the group of units and E is the set of idempotents in $A(X)$). And, in [10, Theorem 3], it was shown that any factorisable inverse semigroup S can be embedded in $A(S)$.

Although $R(q)$ is an inverse subsemigroup of $A(X)$, we assert that it is never factorisable. To see this, suppose there exists $\varepsilon \in R(q)$ such that $\alpha\varepsilon = \varepsilon\alpha = \alpha$ for all $\alpha \in R(q)$, and write $X = B \dot{\cup} C \dot{\cup} \{x\}$ where $|B| = p$ and $|C| = q$. Then, from $\mathrm{id}_{B\cup\{x\}} \in R(q)$ and $\mathrm{id}_{B\cup\{x\}} \circ \varepsilon = \mathrm{id}_{B\cup\{x\}}$, we deduce that $x \in \mathrm{ran}\,\varepsilon$ for all $x \in X$. Since ε is idempotent, it follows that $\varepsilon = \mathrm{id}_X$ which does not belong to $R(q)$. That is, $R(q)$ does not contain an identity and so, by [1, Lemma 2.1], $R(q)$ is not factorisable.

In [4] Howie used $R(q)$, for $q < p$, to construct a class of bisimple congruence-free inverse semigroups, something that "seems rarely to be easy" ([4, p. 337]). On the other hand, in [13, Corollary 4], Sullivan proved that $\alpha \in I(X)$ is a product of nilpotents in $I(X)$ if and only if $d(\alpha) = g(\alpha) = p$. As in the proof of [9, Theorem 1], it is easy to see that $R(q)$ contains a zero precisely when $q = p$ and, in this case, the zero is $\emptyset$. Hence, if $q < p$, then no element of $R(q)$ is a product of nilpotents in $R(q)$ (since any nilpotent in $R(q)$ is also nilpotent in $I(X)$). However, $R(p)$ equals the semigroup generated by all of the nilpotents in $I(X)$. Also, as in [11] Remark (with a small correction), if $p = q$, then $PS(p)$ is the union of $R(p)$ and the set of elements in $PS(p)$ which are maximal under $\leq$, and the latter set forms a semigroup.

3. Automorphisms and isomorphisms

In [12, Theorem 3], Sullivan showed that $\mathrm{Aut}PS(q)$ and $G(X)$ are isomorphic when $p = q$. Later, in [9, Theorem 2], Pinto and Sullivan showed that this is also true when $p > q$. Here, we first consider the problem of describing all automorphisms of $R(q)$.

As in [12], a subsemigroup S of $P(X)$ is $G(X)$-*normal* if $\beta\alpha\beta^{-1} \in S$ for all $\alpha \in S$ and all $\beta \in G(X)$. It is easy to see that $PS(q)$ is $G(X)$-normal, and consequently the same is true for $R(q)$ (since $R(q)$ is the set of all regular elements of $PS(q)$).

When $p = q$, we know $R(q)$ *covers* X: that is, for each $x \in X$, there is a constant idempotent (namely $\mathrm{id}_{\{x\}}$) in $R(q)$ with range $\{x\}$. So, in this case, [12, Theorem 1] implies that φ is *inner* for all $\varphi \in \mathrm{Aut}R(q)$: that is, there exists $\beta \in G(X)$ such that $\alpha\varphi = \beta\alpha\beta^{-1}$ for all $\alpha \in R(q)$. Also, by [12, Theorem 2], $\mathrm{Aut}R(q)$ is isomorphic to $G(X)$.

We now consider the same problem when $p > q$. In fact, in [6, Theorem 3.18], Levi proved that, if S is a constant-free $G(X)$-normal subsemigroup of $P(X)$ which contains a non-total transformation, then every automorphism of S is inner. So, every automorphism of $R(q)$ is inner when $p > q$. By using arguments similar to those in [9, Section 2], we obtain the following results.

Lemma 1. *For each $\varphi \in \mathrm{Aut}R(q)$, there exists a unique $\gamma \in G(X)$ such that $\alpha\varphi = \gamma^{-1}\alpha\gamma$ for all $\alpha \in R(q)$ and, in this event, we write $\gamma = \gamma_\varphi$.*

Proof. Let $\varphi \in \mathrm{Aut}R(q)$. Then φ is inner, so there exists $\gamma \in G(X)$ such that $\alpha\varphi = \gamma^{-1}\alpha\gamma$ for all $\alpha \in R(q)$. Suppose there exists $\mu \in G(X)$ such that $\gamma^{-1}\alpha\gamma = \alpha\varphi = \mu^{-1}\alpha\mu$ for all $\alpha \in R(q)$. Let $x \in X$ and write $X = A \,\dot\cup\, B \,\dot\cup\, \{x\}$ where $|A| = p$ and $|B| = q$. If $\alpha = \mathrm{id}_A$ and $\beta = \mathrm{id}_{A \,\dot\cup\, \{x\}}$, then $\alpha, \beta \in R(q)$. This implies that

$$A\gamma = X\gamma^{-1}\alpha\gamma = X\mu^{-1}\alpha\mu = A\mu$$

and

$$(A \,\dot\cup\, \{x\})\gamma = X\gamma^{-1}\beta\gamma = X\mu^{-1}\beta\mu = (A \,\dot\cup\, \{x\})\mu.$$

Since γ and μ are injective, we have

$$A\gamma \,\dot\cup\, \{x\gamma\} = A\mu \,\dot\cup\, \{x\mu\},$$

where $A\gamma = A\mu$. Thus $x\gamma = x\mu$ for all $x \in X$, that is, $\gamma = \mu$. □

The proof of the next result is identical to that for [9, Theorem 2] (after replacing $PS(q)$ by $R(q)$), so we omit the details.

Theorem 1. *If $p > q$, then* $\mathrm{Aut}R(q) \to G(X)$, $\varphi \to \gamma_\varphi$, *is an isomorphism.*

Since $R(X, p, q)$ played an important role in both [4] and [9], it is natural to ask whether any of the semigroups $R(X, p, q)$ are isomorphic for different cardinals p and q (here and below, we write $R(q)$ as $R(X, p, q)$ to highlight the set X and its cardinal p). To answer this question, we first need a result for $R(q)$ which corresponds to [9, Lemma 1] for $PS(q)$. Since the proof is almost verbatim, we omit the details.

Lemma 2. *If $\alpha, \beta \in R(q)$, then the following are equivalent.*
(a) $\mathrm{ran}\,\alpha \subseteq \mathrm{ran}\,\beta$,
(b) *for each $\gamma \in R(q)$, $\beta\gamma = \beta$ implies $\alpha\gamma = \alpha$.*

Corollary 1. *Suppose $|X| = p \geq q \geq \aleph_0$ and $|Y| = r \geq s \geq \aleph_0$. If $\varphi : R(X, p, q) \to R(Y, r, s)$ is an isomorphism, then, for each $\alpha, \beta \in R(X, p, q)$,* $\mathrm{ran}\,\alpha \subseteq \mathrm{ran}\,\beta$ *if and only if* $\mathrm{ran}(\alpha\varphi) \subseteq \mathrm{ran}(\beta\varphi)$.

Proof. Suppose $\alpha, \beta \in R(X, p, q)$. Then, since φ is an isomorphism, Lemma 2 provides the following equivalences:

$$\begin{aligned} \mathrm{ran}\,\alpha \subseteq \mathrm{ran}\,\beta &\iff \text{for each } \gamma \in R(X,p,q),\ \beta\gamma = \beta \ \text{ implies } \ \alpha\gamma = \alpha \\ &\iff \text{for each } \gamma \in R(X,p,q),\ \beta\varphi.\gamma\varphi = \beta\varphi \ \text{ implies } \ \alpha\varphi.\gamma\varphi = \alpha\varphi \\ &\iff \text{for each } \gamma' \in R(Y,r,s),\ \beta\varphi.\gamma' = \beta\varphi \ \text{ implies } \ \alpha\varphi.\gamma' = \alpha\varphi \\ &\iff \mathrm{ran}(\alpha\varphi) \subseteq \mathrm{ran}(\beta\varphi). \end{aligned}$$ □

Theorem 2. *The semigroups $R(X, p, q)$ and $R(Y, r, s)$ are isomorphic if and only if $p = r$ and $q = s$. Moreover, for each isomorphism φ, there is a bijection $\gamma : X \to Y$ such that $\alpha\varphi = \gamma^{-1}\alpha\gamma$ for each $\alpha \in R(X, p, q)$.*

Proof. Clearly, if the cardinals are equal as stated, then any bijection from X onto Y will induce an isomorphism between the semigroups. So, we assume there is an isomorphism $\varphi : R(X, p, q) \to R(Y, r, s)$ and write

$$U = \{\mathrm{ran}\,\alpha : \alpha \in R(X,p,q)\}, \quad V = \{\mathrm{ran}\,\beta : \beta \in R(Y,r,s)\}.$$

Let $\Gamma : U \to V$ be defined by $(\mathrm{ran}\,\alpha)\Gamma = \mathrm{ran}(\alpha\varphi)$. Then, by Corollary 1, Γ is an order-monomorphism: that is, Γ is injective and $A \subseteq B$ if and only if $A\Gamma \subseteq B\Gamma$ for all $A, B \in U$. Next, if $C = \mathrm{ran}\,\beta$ for some $\beta \in R(Y, r, s)$, then $\beta = \alpha\varphi$ for some $\alpha \in R(X, p, q)$ (since φ is onto). Thus $(\mathrm{ran}\,\alpha)\Gamma = \mathrm{ran}(\alpha\varphi) = \mathrm{ran}\,\beta = C$, so Γ is onto. In fact, if

$$\mathcal{B}(X,q) = \{A \subseteq X : |X \setminus A| = q\}, \quad \mathcal{B}(Y,s) = \{B \subseteq Y : |Y \setminus B| = s\},$$

then $U = \mathcal{B}(X, q)$ and $V = \mathcal{B}(Y, s)$, since $\mathrm{id}_A \in R(X, p, q)$ and $\mathrm{id}_B \in R(Y, r, s)$ for all $A \in \mathcal{B}(X, q)$ and $B \in \mathcal{B}(Y, s)$. That is, Γ is an order-isomorphism from $\mathcal{B}(X, q)$ onto $\mathcal{B}(Y, s)$. Thus by [9, Lemma 2], there exists a bijection $\gamma : X \to Y$ such that $A\Gamma = A\gamma$ for all $A \in \mathcal{B}(X, q)$, so $p = r$. By using the same argument as in the proof of [9, Theorem 3], we have $\alpha\varphi = \gamma^{-1}\alpha\gamma$ for all $\alpha \in R(X, p, q)$. Finally, since $\alpha\varphi \in R(Y, r, s)$, we have $s = |Y \setminus Y(\alpha\varphi)| = |Y \setminus Y\gamma^{-1}\alpha\gamma| = |X\gamma \setminus X\alpha\gamma| = |(X \setminus X\alpha)\gamma| = q$. □

Although we have used some ideas from [9, Section 2], a careful reading of the above discussion shows that we have not used [9, Theorem 3]: namely, the characterization of when $PS(X, p, q)$ is isomorphic to $PS(Y, r, s)$. Therefore, since $R(X, p, q)$ is the largest regular subsemigroup of $PS(X, p, q)$, we can deduce the following result. However, an explicit description of all isomorphisms between $PS(X, p, q)$ and $PS(Y, r, s)$ in terms of associated bijections between X and Y seems to require an argument like that in the proof of [9, Theorem 3].

Corollary 2. *The semigroups $PS(X, p, q)$ and $PS(Y, r, s)$ are isomorphic if and only if $p = r$ and $q = s$.*

4. Meets

In this section, we study the existence of a *meet* $\alpha \wedge \beta$ for α, β in the semigroups $I(X)$, $PS(q)$ and $R(q)$ for each of the orders $\leq$ and $\subseteq$. To do this, we first define the *equaliser* of $\alpha, \beta \in I(X)$ (compare [14, p. 416] for linear transformations) as follows.

$$E(\alpha, \beta) = \{x \in \operatorname{dom}\alpha \cap \operatorname{dom}\beta : x\alpha = x\beta\}.$$

The next result may be well-known, but we do not know a reference in the literature (recall that $\subseteq$ equals $\leq$ on $I(X)$).

Theorem 3. *Let $\alpha, \beta \in I(X)$ and $E = E(\alpha, \beta)$. Then, under $\subseteq$, $\alpha \wedge \beta = \alpha|E = \beta|E$.*

Proof. As discussed in [7], each $\alpha \in P(X)$ can be regarded as a special subset of $X \times X$. With this in mind, if $\alpha, \beta \in I(X)$, then $\alpha \cap \beta \in I(X)$ and clearly $\alpha \cap \beta = \alpha \wedge \beta$ (as sets). Also, $E = \emptyset$ if and only if $\alpha \cap \beta = \emptyset$; and, if $E \neq \emptyset$, then $\alpha \cap \beta = \alpha|E = \beta|E$. □

Recall that $\leq$ is properly contained in $\subseteq$ on $PS(q)$. Thus, unlike for Theorem 3, we expect a characterization of meets in $(PS(q), \subseteq)$ to involve an additional condition. As stated in Section 2, if $A \subseteq X$ and $\alpha \in I(X)$, then $A\alpha$ denotes $(A \cap \operatorname{dom}\alpha)\alpha$.

Theorem 4. *Let $\alpha, \beta \in PS(q)$ and $E = E(\alpha, \beta)$. Then $\gamma \subseteq \alpha, \beta$ for some non-empty $\gamma \in PS(q)$ if and only if*
(a) *$E \neq \emptyset$, and*
(b) *$\max(|X\alpha \setminus E\alpha|, |X\beta \setminus E\beta|) \leq q$.*

Moreover, when this occurs, $\alpha \cap \beta$ is the non-empty meet of α, β under $\subseteq$.

Proof. Suppose $\emptyset \neq \gamma \subseteq \alpha, \beta$ in $PS(q)$. Then $\emptyset \neq \operatorname{dom}\gamma \subseteq \operatorname{dom}\alpha \cap \operatorname{dom}\beta$ and $x\alpha = x\gamma = x\beta$ for all $x \in \operatorname{dom}\gamma$. That is, $\emptyset \neq \operatorname{dom}\gamma \subseteq E$ and this implies $X\gamma = E\gamma$. Now $E\gamma = (E \cap \operatorname{dom}\gamma)\gamma \subseteq E\alpha \subseteq X\alpha$ and so

$$|X\alpha \setminus E\alpha| \leq |X\alpha \setminus E\gamma| \leq |X \setminus X\gamma| = q.$$

Similarly, $|X\beta \setminus E\beta| \leq q$ and hence the conditions hold. Conversely, if the conditions hold, then $\gamma = \alpha \cap \beta$ is a non-empty element of $I(X)$ with domain $E = E(\alpha, \beta)$ and $\gamma \subseteq \alpha, \beta$. Moreover, since $X\gamma = E\gamma = E\alpha \subseteq X\alpha$, we have $X \setminus X\gamma = (X \setminus X\alpha) \,\dot\cup\, (X\alpha \setminus E\alpha)$ and it follows that $d(\gamma) = q$. That is, $\gamma \in PS(q)$. $\square$

Of course, when we turn to $R(q)$, we expect a further condition to be needed in order to characterize meets in $R(q)$ under $\subseteq$.

Theorem 5. *Let $\alpha, \beta \in R(q)$ and $E = E(\alpha, \beta)$. Then $\gamma \subseteq \alpha, \beta$ for some non-empty $\gamma \in R(q)$ if and only if*
(a) $E \neq \emptyset$,
(b) $\max(|X\alpha \setminus E\alpha|, |X\beta \setminus E\beta|) \leq q$, *and*
(c) $\max(|\operatorname{dom}\alpha \setminus E|, |\operatorname{dom}\beta \setminus E|) \leq q$.

Moreover, when this occurs, $\alpha \cap \beta$ is the non-empty meet of α, β under $\subseteq$.

Proof. Suppose $\emptyset \neq \gamma \subseteq \alpha, \beta$. Since $R(q) \subseteq PS(q)$, Theorem 4 implies that (a) and (b) hold. Since $\operatorname{dom}\gamma \subseteq E \subseteq \operatorname{dom}\alpha$, we have

$$|\operatorname{dom}\alpha \setminus E| \leq |\operatorname{dom}\alpha \setminus \operatorname{dom}\gamma| \leq |X \setminus \operatorname{dom}\gamma| = q.$$

Similarly, $|\operatorname{dom}\beta \setminus E| \leq q$ and hence (c) holds. Conversely, suppose the conditions hold. By Theorem 4 again, (a) and (b) imply that $\gamma = \alpha \cap \beta$ is a non-empty element of $PS(q)$ and it is also the meet of α, β in $PS(q)$ under $\subseteq$. Also, since $\operatorname{dom}\gamma = E \subseteq \operatorname{dom}\alpha$, we have

$$X \setminus \operatorname{dom}\gamma = (X \setminus \operatorname{dom}\alpha) \ \dot\cup \ (\operatorname{dom}\alpha \setminus E).$$

Then (c) implies that $g(\gamma) = q$, hence $\gamma \in R(q)$. $\square$

In [11, Theorem 2.4], the authors proved that $\leq$ equals $\subseteq \cap\, \mathbb{L}$ on $PS(q)$, where $\mathbb{L}$ is the relation defined on $PS(q)$ by

$$(\alpha, \beta) \in \mathbb{L} \Longleftrightarrow \alpha = \beta \text{ or } X\alpha \subseteq X\beta \text{ and } q \leq \max(g(\beta), |X\beta \setminus X\alpha|) \leq \max(g(\alpha), q).$$

Note that if $\alpha \wedge \beta = \emptyset$ in $PS(q)$ under $\leq$, then $p = q$. In this case, if $x \in E = E(\alpha, \beta)$ and $x\alpha = x\beta = y$, then $x_y \in PS(q)$ and $x_y \subseteq \alpha, \beta$. Also, since $|X\alpha \setminus \{y\}| = |\operatorname{dom}\alpha \setminus \{x\}|$ and $g(\alpha) = |X \setminus \operatorname{dom}\alpha|$, we have

$$q = p = \max(g(\alpha), |X\alpha \setminus \{y\}|) \leq \max(g(x_y), q) = p = q.$$

That is, $x_y \leq \alpha, \beta$, so $x_y \leq \alpha \wedge \beta = \emptyset$, a contradiction. In other words, if $\alpha \wedge \beta = \emptyset$, then $E = \emptyset$ and so $\alpha \cap \beta = \emptyset$. Consequently, $\alpha \wedge \beta = \alpha \cap \beta$ when one of these equals $\emptyset$.

In essence, condition (b) in the next result ensures that, when $\alpha \cap \beta$ equals $\alpha \wedge \beta$ under $\subseteq$ on $PS(q)$, then it also equals $\alpha \wedge \beta$ under $\leq$ on $PS(q)$. As usual, if $\preceq$ is a partial order on a set S, we say $a, b \in S$ are *non-comparable* if $a \not\preceq b$ and $b \not\preceq a$.

Theorem 6. *Suppose $\alpha, \beta \in PS(q)$ are non-comparable under $\leq$ and let $E = E(\alpha, \beta)$. Then $\gamma \leq \alpha, \beta$ for some non-empty $\gamma \in PS(q)$ if and only if there exists a non-empty $Y \subseteq E$ such that*
(a) $\max(|X\alpha \setminus Y\alpha|, |X\beta \setminus Y\beta|) \leq q$ *and*
(b) $q \leq \max(g(\alpha), |X\alpha \setminus Y\alpha|)$ *and* $q \leq \max(g(\beta), |X\beta \setminus Y\beta|)$.

In this event, $\gamma = \alpha|Y = \beta|Y$. Hence, $\alpha \wedge \beta$ exists in $PS(q)$ under $\leq$ and it is non-empty precisely when α and β satisfy conditions (a) *and* (b) *and $Y = E$, in which case $\alpha \wedge \beta = \alpha|E = \beta|E$.*

Proof. Suppose $\emptyset \neq \gamma \leq \alpha, \beta$ and let $Y = \operatorname{dom}\gamma$. Then $\gamma \subseteq \alpha, \beta$ and so $x\alpha = x\gamma = x\beta$ for all $x \in Y$. That is, $Y \subseteq E$ and $X\gamma = Y\gamma = Y\alpha = Y\beta$. Since $d(\gamma) = q$, we see that $|X\alpha \setminus Y\alpha| \leq |X \setminus X\gamma| = q$ and likewise $|X\beta \setminus Y\beta| \leq q$, so (a) holds. Also $(\gamma, \alpha) \in \mathbb{L}$ and $(\gamma, \beta) \in \mathbb{L}$ imply

$$q \leq \max(g(\alpha), |X\alpha \setminus Y\alpha|) \quad \text{and} \quad q \leq \max(g(\beta), |X\beta \setminus Y\beta|).$$

Conversely, suppose the conditions hold and write

$$(1) \qquad \alpha = \begin{pmatrix} y_i & e_j & u_m \\ a_i & a_j & a_m \end{pmatrix}, \quad \beta = \begin{pmatrix} y_i & e_j & v_n \\ a_i & a_j & b_n \end{pmatrix}, \quad \gamma = \begin{pmatrix} y_i \\ a_i \end{pmatrix},$$

where $Y = \{y_i\}$ and $E = Y \dot{\cup} \{e_j\}$ (possibly $J = \emptyset$). Then $d(\gamma) = |J| + |M| + d(\alpha) = q$ (by supposition since $|J| + |M| = |X\alpha \setminus Y\alpha|$), so $\gamma \in PS(q)$. Clearly, $\gamma \subseteq \alpha, \beta$. Also, $g(\gamma) = |J| + |M| + g(\alpha) \geq g(\alpha)$. Now, if $g(\gamma) \leq q$, then condition (a) implies that

$$\max(g(\alpha), |X\alpha \setminus X\gamma|) \leq q = \max(g(\gamma), q);$$

and if $q < g(\gamma)$, then, since $|X\alpha \setminus X\gamma| = |X\alpha \setminus Y\alpha| \leq q$, we have:

$$\max(g(\alpha), |X\alpha \setminus X\gamma|) \leq g(\gamma) = \max(g(\gamma), q).$$

Hence, the above and condition (b) imply that $(\gamma, \alpha) \in \mathbb{L}$ and similarly $(\gamma, \beta) \in \mathbb{L}$. Thus, we have shown that $\gamma \leq \alpha, \beta$.

Finally, suppose $\gamma = \alpha \wedge \beta$ exists and is non-empty, and write α, β as in (1). If $g(\gamma) < q$, then [11, Theorem 4.3] implies that γ is maximal under $\leq$ and so $\gamma = \alpha = \beta$, contradicting the supposition. Hence $g(\gamma) \geq q$. Now $\gamma \leq \alpha, \beta$, so $Y = \operatorname{dom}\gamma \subseteq E$ and hence α and β satisfy (a) and (b). If there exists $e_0 \in E \setminus Y$ for some $0 \in J$, we can define $\gamma' \in PS(q)$ by

$$\gamma' = \begin{pmatrix} y_i & e_0 \\ a_i & a_0 \end{pmatrix}.$$

Then $\gamma \subseteq \gamma' \subseteq \alpha$ and $|X\gamma' \setminus X\gamma| = 1$, and we see that

$$\begin{aligned} g(\gamma) &= |J| + |M| + g(\alpha), \\ g(\gamma') &= |J \setminus \{0\}| + |M| + g(\alpha). \end{aligned}$$

Thus, if $|J| + |M| \geq \aleph_0$, then $g(\gamma) = g(\gamma') \geq g(\alpha)$; and if $|J| + |M| < \aleph_0$, then $\gamma \leq \alpha$ implies $q \leq \max(g(\alpha), |J| + |M|)$, so $g(\alpha) \geq q$ and hence $g(\gamma) = g(\gamma') \geq g(\alpha)$. Similarly, in both cases, $g(\gamma') \geq |J| + |M|$. Therefore,

$$q \leq g(\gamma) = \max(g(\alpha), |J| + |M|) \leq \max(g(\gamma'), 1) = g(\gamma) \leq \max(g(\gamma), q).$$

Thus $(\gamma, \gamma') \in \mathbb{L}$ and likewise $(\gamma', \alpha) \in \mathbb{L}$. In other words, we can show that $\gamma < \gamma' \leq \alpha, \beta$, a contradiction. Hence, it follows that $Y = E$. Conversely, suppose $Y = E$ and α and β satisfy (a) and (b). Then, by the first part of this proof, $\gamma \leq \alpha, \beta$ where $\gamma = \alpha|E = \beta|E \in PS(q)$. Moreover, if $\gamma \leq \gamma' \leq \alpha, \beta$ for some $\gamma' \in PS(q)$, then $x\gamma' = x\alpha = x\beta$ for all $x \in \operatorname{dom}\gamma'$, so $E = \operatorname{dom}\gamma \subseteq \operatorname{dom}\gamma' \subseteq E$, and it follows that $\gamma = \gamma'$. That is, $\gamma = \alpha \wedge \beta$. □

In effect, by [11, Theorem 4.3], the next result determines when two elements of $PS(q)$, which are maximal under $\leq$, possess a meet under $\leq$.

Corollary 3. *Suppose $\alpha, \beta \in PS(q)$ are non-comparable under $\leq$ and let $E = E(\alpha, \beta)$. If $g(\alpha) < q$ and $g(\beta) < q$, then $\alpha \wedge \beta$ exists in $PS(q)$ under $\leq$ if and only if $|X\alpha \setminus E\alpha| = q = |X\beta \setminus E\beta|$.*

Proof. Suppose $g(\alpha) < q$. If $\alpha \wedge \beta$ exists under $\leq$, then Theorem 6(b) implies that $q \leq |X\alpha \setminus E\alpha|$ which is at most q by Theorem 6(a). Thus $|X\alpha \setminus E\alpha| = q$ and likewise $g(\beta) < q$ implies $|X\beta \setminus E\beta| = q$. Conversely, if $|X\alpha \setminus E\alpha| = q = |X\beta \setminus E\beta|$, then both (a) and (b) hold for $E = E(\alpha, \beta)$ in Theorem 6, so $\alpha \wedge \beta$ exists. □

Example 1. Suppose $X = M \,\dot\cup\, N \,\dot\cup\, \{b, c\}$, where $|M| = p, |N| = q$ and

$$\alpha = \begin{pmatrix} M \cup N & b \\ M & b \end{pmatrix}, \quad \beta = \begin{pmatrix} M \cup N & c \\ M & c \end{pmatrix},$$

where $E = E(\alpha, \beta) = M \cup N$. Then $d(\alpha) = q = d(\beta)$, so $\alpha, \beta \in PS(q)$ and $\alpha \cap \beta = \alpha|E \in PS(q)$. But, $|X\alpha \setminus E\alpha| = 1 = |X\beta \setminus E\beta|$ and $g(\alpha) = 1 = g(\beta)$, so E satisfies condition (a) in Theorem 6 but not condition (b), and hence $\alpha \wedge \beta$ does not exist in $(PS(q), \leq)$. That is, although $\alpha \cap \beta$ may be the greatest lower bound under $\subseteq$, that may not be true for $\leq$ since $\leq \neq \subseteq$ on $PS(q)$.

Remark 1. Suppose S is any inverse subsemigroup of $I(X)$. If $\alpha \leq \beta$ in S, then $\alpha = \mathrm{id}_A \circ \beta$ for some $A \subseteq X$ and we deduce that $\alpha \subseteq \beta$. On the other hand, if $\alpha \subseteq \beta$ in the inverse semigroup $R(q)$, then $\alpha = \mathrm{id}_{\operatorname{dom}\alpha} \circ \beta$, where $\mathrm{id}_{\operatorname{dom}\alpha} \in R(q)$, and so $\alpha \leq \beta$ in $R(q)$. That is, $\leq\, =\, \subseteq$ on $R(q)$.

5. Joins

In this section, we study the existence of a *join* $\alpha \vee \beta$ for α, β in the semigroups $I(X)$, $PS(q)$ and $R(q)$ for each of the orders $\leq$ and $\subseteq$.

Theorem 7. *Let $\alpha, \beta \in I(X)$ under $\subseteq$. Then $\alpha, \beta \subseteq \gamma$ for some $\gamma \in I(X)$ if and only if*

(a) $\operatorname{dom}\alpha \cap \operatorname{dom}\beta \subseteq E(\alpha, \beta)$ *and*

(b) $(\operatorname{dom}\alpha \setminus \operatorname{dom}\beta)\alpha \cap (\operatorname{dom}\beta \setminus \operatorname{dom}\alpha)\beta = \emptyset$.

Moreover, in this case, $\alpha \vee \beta$ exists and equals $\alpha \cup \beta$.

Proof. Suppose $\alpha, \beta \subseteq \gamma \in I(X)$. If $x \in \operatorname{dom}\alpha \cap \operatorname{dom}\beta$, then $x\alpha = x\gamma = x\beta$, and so $x \in E(\alpha, \beta)$. On the other hand, if there exist $y \in \operatorname{dom}\alpha \setminus \operatorname{dom}\beta$ and $z \in \operatorname{dom}\beta \setminus \operatorname{dom}\alpha$ such that $y\alpha = z\beta$, then $y\gamma = z\gamma$. Since γ is injective, this implies that $y = z$, a contradiction.

Conversely, suppose the conditions hold and let $\gamma = \alpha \cup \beta$ (as sets). Then (a) says that γ is a mapping and (b) says it is injective, so $\gamma \in I(X)$ and clearly it is an upper bound of $\{\alpha, \beta\}$. Moreover, if (a) and (b) hold, then $\gamma = \alpha \vee \beta$, since $\alpha, \beta \subseteq \lambda \in I(X)$ implies $\alpha, \beta \subseteq \alpha \cup \beta \subseteq \lambda$ (as sets) where $\alpha \cup \beta \in I(X)$. □

Like before, the result for joins in $PS(q)$ under $\subseteq$ involves an extra condition.

Theorem 8. *Let $\alpha, \beta \in PS(q)$ under $\subseteq$. Then $\alpha, \beta \subseteq \gamma$ for some $\gamma \in PS(q)$ if and only if the following conditions hold.*
(a) $\operatorname{dom}\alpha \cap \operatorname{dom}\beta \subseteq E(\alpha, \beta)$,
(b) $(\operatorname{dom}\alpha \setminus \operatorname{dom}\beta)\alpha \cap (\operatorname{dom}\beta \setminus \operatorname{dom}\alpha)\beta = \emptyset$, *and*
(c) $|X \setminus (X\alpha \cup X\beta)| = q$.

Moreover, in this case, $\alpha \vee \beta$ exists and equals $\alpha \cup \beta$.

Proof. Suppose $\alpha, \beta \subseteq \gamma$ in $PS(q)$. Then, conditions (a) and (b) hold since $PS(q) \subseteq I(X)$. Since $X\alpha \cup X\beta \subseteq X\gamma$, we also have

$$q = |X \setminus X\gamma| \leq |X \setminus (X\alpha \cup X\beta)| \leq |X \setminus X\alpha| = q.$$

Hence (c) holds. Conversely, suppose (a), (b) and (c) hold and let $\gamma = \alpha \cup \beta$. Then (a) and (b) imply that $\gamma \in I(X)$, and (c) implies that $d(\gamma) = q$, that is, $\gamma \in PS(q)$. Finally, as in Theorem 7, we can show that $\alpha \vee \beta = \gamma$. □

Theorem 9. *Let $\alpha, \beta \in R(q)$. Then $\alpha, \beta \subseteq \gamma$ for some $\gamma \in R(q)$ if and only if the following conditions hold.*
(a) $\operatorname{dom}\alpha \cap \operatorname{dom}\beta \subseteq E(\alpha, \beta)$,
(b) $(\operatorname{dom}\alpha \setminus \operatorname{dom}\beta)\alpha \cap (\operatorname{dom}\beta \setminus \operatorname{dom}\alpha)\beta = \emptyset$,
(c) $|X \setminus (X\alpha \cup X\beta)| = q$, *and*
(d) $|X \setminus (\operatorname{dom}\alpha \cup \operatorname{dom}\beta)| = q$.

Moreover, when this occurs, $\alpha \cup \beta$ is the join of α, β under $\subseteq$.

Proof. Suppose $\alpha, \beta \subseteq \gamma$ in $R(q)$. Since $R(q) \subseteq PS(q)$, Theorem 8 implies that (a), (b) and (c) hold. Since $\operatorname{dom}\alpha \cup \operatorname{dom}\beta \subseteq \operatorname{dom}\gamma$, we have

$$q = |X \setminus \operatorname{dom}\gamma| \leq |X \setminus (\operatorname{dom}\alpha \cup \operatorname{dom}\beta)| \leq |X \setminus \operatorname{dom}\alpha| = q.$$

Hence (d) holds. Conversely, suppose the conditions hold. By Theorem 8 again, (a), (b) and (c) imply that $\gamma = \alpha \cup \beta$ is an element of $PS(q)$ and it is also a join of α, β under $\subseteq$. Also, (d) implies that $g(\gamma) = q$, so $\gamma \in R(q)$. □

To characterize joins in $PS(q)$ under $\leq$, we need two lemmas. In effect, the first provides a description of $\leq$ in terms of $\subseteq$ which differs from that in [11, Theorem 2.4].

Lemma 3. *Suppose $\alpha, \beta \in PS(q)$ and $\alpha \neq \beta$. Then $\alpha < \beta$ if and only if $\alpha \subset \beta$ and $g(\alpha) \geq q$.*

Proof. If $\alpha < \beta$, then $\alpha \subset \beta$ and $(\alpha, \beta) \in \mathbb{L}$. Therefore, $\operatorname{dom}\alpha \subset \operatorname{dom}\beta$ and $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$, and hence

$$X \setminus \operatorname{dom}\alpha = (X \setminus \operatorname{dom}\beta) \,\dot{\cup}\, (\operatorname{dom}\beta \setminus \operatorname{dom}\alpha), \text{ and} \tag{2}$$

$$X\beta = [(\operatorname{dom}\beta \setminus \operatorname{dom}\alpha)\beta] \,\dot{\cup}\, [(\operatorname{dom}\alpha)\beta].$$

Now, $(\operatorname{dom}\alpha)\beta = (\operatorname{dom}\alpha)\alpha = X\alpha$ (since $\alpha \subset \beta$) and so

$$|X\beta \setminus X\alpha| = |(\operatorname{dom}\beta \setminus \operatorname{dom}\alpha)\beta| = |\operatorname{dom}\beta \setminus \operatorname{dom}\alpha|. \tag{3}$$

By [11, Theorem 2.3], we also know that

$$q \leq \max(g(\beta), |X\beta \setminus X\alpha|) \leq \max(g(\alpha), q).$$

Hence, if $\max(g(\beta), |X\beta \setminus X\alpha|) = g(\beta)$, then $q \leq g(\beta) \leq g(\alpha)$ by (2); and if $\max(g(\beta), |X\beta \setminus X\alpha|) = |X\beta \setminus X\alpha|$, then $q \leq |\operatorname{dom}\beta \setminus \operatorname{dom}\alpha| \leq g(\alpha)$ by (3). That is, the conditions hold.

Conversely, suppose the conditions hold. Then $\max(g(\alpha), q) = g(\alpha) \geq g(\beta)$ by (2) and $X\alpha \subseteq X\beta$. Also, $|X\beta \setminus X\alpha| \leq g(\alpha)$ by (3). Moreover, if $|X\beta \setminus X\alpha| < q$, then (2) and (3) imply that $g(\beta) \geq q$. Consequently,

$$q \leq \max(g(\beta), |X\beta \setminus X\alpha|) \leq \max(g(\alpha), q),$$

and so $(\alpha, \beta) \in \mathbb{L}$. By [11, Theorem 2.4], it follows that $\alpha < \beta$. □

Lemma 4. *Suppose $\alpha, \beta \in PS(q)$ are non-comparable under $\leq$. Then $\alpha, \beta \leq \gamma$ for some $\gamma \in PS(q)$ if and only if*
(a) *$\alpha, \beta \subseteq \theta$ for some $\theta \in PS(q)$, and*
(b) *$g(\alpha) \geq q$ and $g(\beta) \geq q$.*

Proof. If $\alpha, \beta \leq \gamma$, then $\alpha, \beta \subseteq \gamma$, so (a) holds. In addition, if $g(\alpha) < q$, then α is maximal under $\leq$ (by [11, Theorem 4.3]). Hence $\alpha \leq \gamma$ implies $\alpha = \gamma$ and so $\beta \leq \alpha$, contradicting the supposition. Therefore, $g(\alpha) \geq q$ and $g(\beta) \geq q$. That is, (b) holds.

Conversely, suppose (a) and (b) hold. Then $\alpha, \beta \subseteq \alpha \cup \beta = \pi$ (say, as relations) and, from (a) and Theorem 8, we deduce that $\pi \in PS(q)$. If $\alpha = \pi$, then $\operatorname{dom}\beta \subseteq \operatorname{dom}\pi = \operatorname{dom}\alpha$ and $x\beta = x\pi = x\alpha$ for each $x \in \operatorname{dom}\beta$. Thus, $\beta \subsetneq \alpha$ and $g(\beta) \geq q$, so $\beta < \alpha$ by Lemma 3, which contradicts the supposition. Therefore, $\alpha \subsetneq \pi$ and $g(\alpha) \geq q$, so $\alpha < \pi$ by Lemma 3 again. Similarly, $\beta < \pi$ and so α, β have an upper bound in $PS(q)$ under $\leq$. □

Example 2. Surprisingly, (a) and (b) in Lemma 4 do not ensure that $\alpha \cup \beta$ equals $\alpha \vee \beta$ in $PS(q)$ under $\leq$. For example, write $X = A \dot{\cup} B \dot{\cup} C \dot{\cup} D \dot{\cup} \{a\}$ where $|A| = p = |X|$ and $|B| = |C| = |D| = q$. Let

$$\alpha = \begin{pmatrix} A \cup B \\ A \end{pmatrix} \cup \mathrm{id}_C, \quad \beta = \begin{pmatrix} A \cup B \\ A \end{pmatrix} \cup \mathrm{id}_D,$$

where $x\alpha = x\beta$ for all $x \in A \cup B$. Then $\alpha, \beta \in PS(q)$ and they are non-comparable under $\leq$ (since $\alpha \not\subseteq \beta$ and $\beta \not\subseteq \alpha$). If $\theta = \alpha \cup \beta$, then $\alpha, \beta \subseteq \theta \in PS(q)$ (since $d(\theta) = |B| = q$), hence α and β satisfy (a). Also, $g(\alpha) = |D| = q = |C| = g(\beta)$, and hence α and β satisfy (b). By Lemma 3, $\alpha, \beta < \theta' = \theta \cup \mathrm{id}_{\{a\}} \in PS(q)$, but $\theta \not\leq \theta'$ since $g(\theta) = 1 \not\geq q$, and thus $\alpha \cup \beta$ does not equal $\alpha \vee \beta$.

Theorem 10. *Suppose $\alpha, \beta \in PS(q)$ are non-comparable under $\leq$. Then $\alpha \vee \beta$ exists if and only if*

(a) *$\alpha, \beta < \theta$ for some $\theta \in PS(q)$, and*

(b) *either $X = \mathrm{dom}\,\alpha \cup \mathrm{dom}\,\beta$ or $|X \setminus (\mathrm{dom}\,\alpha \cup \mathrm{dom}\,\beta)| \geq q$.*

Moreover, when this occurs, $\alpha \vee \beta$ equals $\alpha \cup \beta$.

Proof. Suppose $\alpha \vee \beta$ exists under $\leq$ and write $\gamma = \alpha \vee \beta$. Then $\alpha, \beta < \gamma$, so (a) holds. Consequently, $\alpha, \beta \subset \gamma$ and so Theorem 8 implies that $\pi = \alpha \cup \beta \in PS(q)$ and clearly $\pi \subseteq \gamma$. Now, to prove (b), suppose $\mathrm{dom}\,\alpha \cup \mathrm{dom}\,\beta \subsetneq X$. Choose $a \in X \setminus (\mathrm{dom}\,\alpha \cup \mathrm{dom}\,\beta) = X \setminus \mathrm{dom}\,\pi$ and, for any $x \in X \setminus X\pi$ (non-empty since $d(\pi) = q$), we let

$$\mu_x = \begin{pmatrix} \mathrm{dom}\,\pi & a \\ X\pi & x \end{pmatrix},$$

where $\mu_x | \mathrm{dom}\,\pi = \pi$. Then $\mu_x \in PS(q)$ since $d(\mu_x) = |X \setminus X\pi| = d(\pi) = q$. Clearly, $\alpha \subseteq \mu_x$ and $\alpha \neq \mu_x$ (since $a \in \mathrm{dom}\,\mu_x \setminus \mathrm{dom}\,\alpha$). Therefore, since $g(\alpha) \geq q$ by Lemma 3 (using the fact that $\alpha < \gamma$), we deduce that $\alpha < \mu_x$ by Lemma 3 again. Similarly, $\beta < \mu_x$ and thus $\gamma \leq \mu_x$ for all $x \in X \setminus X\pi$. If $\gamma = \mu_x$ for all $x \in X \setminus X\pi$, then $\mu_x = \mu_y$ for all $x \neq y$ in $X \setminus X\pi$, a contradiction. Hence, $\gamma < \mu_x$ for some $x \in X \setminus X\pi$, and so γ is not maximal. Therefore, by [11, Theorem 4.3], $q \leq g(\gamma) \leq g(\pi) = |X \setminus (\mathrm{dom}\,\alpha \cup \mathrm{dom}\,\beta)|$, and so we have proved (b).

Conversely, suppose the conditions hold. Then Lemma 4(a) and Theorem 8 imply that (say) $\pi = \alpha \cup \beta \in PS(q)$ and we claim that $\pi = \alpha \vee \beta$ under $\leq$. If $\pi = \alpha$, then $\beta \subseteq \alpha$ and, as in the proof of Lemma 4, a contradiction follows. Hence, $\alpha \subset \pi$. In addition, since $\alpha, \beta < \theta$ for some $\theta \in PS(q)$ by (a), Lemma 3 implies that $g(\alpha) \geq q$. By Lemma 3, we deduce that $\alpha < \pi$ and similarly $\beta < \pi$. Finally, if $\alpha, \beta \leq \mu$ for some $\mu \in PS(q)$, then $\alpha, \beta \subseteq \mu$ and so $\pi \subseteq \mu$. Since (b) holds, if $X = \mathrm{dom}\,\alpha \cup \mathrm{dom}\,\beta$, then $X = \mathrm{dom}\,\pi$ and so $\pi = \mu$. Consequently, if $\pi \neq \mu$, then $|X \setminus (\mathrm{dom}\,\alpha \cup \mathrm{dom}\,\beta)| \geq q$, so $\pi < \mu$ by Lemma 3. In other words, π is the join of α and β in $PS(q)$ under $\leq$. ☐

6. A partial order on an inverse semigroup

The Vagner-Preston Theorem states that any inverse semigroup S can be embedded in $I(S)$ via the mapping given by

$$\rho : S \to I(S),\ a \to \rho_a,$$

where, for each $a \in S$, $\rho_a : Sa^{-1} \to Sa$, $x \to xa$ (see [2, vol 1, Theorem 1.20]). In fact, the embedding is $\leq$*–preserving* in the sense that $a \leq b$ in S if and only if $\rho_a \leq \rho_b$ in $I(S)$. Probably the next result is well-known but we cannot find a reference for it.

Theorem 11. *Let S be an inverse semigroup and $a, b \in S$. Then $a \leq b$ in S if and only if $\rho_a \leq \rho_b$ in $I(S)$.*

Proof. If $a \leq b$, then $a = eb$ for some idempotent $e \in S$. Hence, $\rho_a = \rho_e \rho_b$, where ρ_e is an idempotent in $I(S)$, so $\rho_a \leq \rho_b$ in $I(S)$. Conversely, if $\rho_a \leq \rho_b$, then $\rho_a \subseteq \rho_b$ by [3, Proposition V.2.3], so $(aa^{-1})a = (aa^{-1})b$ and hence $a \leq b$ by [3, Proposition V.2.2]. □

On any inverse semigroup S, the natural partial order can be defined by

$$a \leq b \quad \text{if and only if} \quad ab^{-1} = aa^{-1}. \tag{4}$$

In addition, for any set X we have $\leq\, = \,\subseteq$ on $I(X)$ (see [3, Proposition V.2.3]), but $\leq$ is properly contained in Ω on $I(X)$ for $|X| > 1$ (see [11, p. 198]), where Ω can be defined on $I(X)$ as follows (recall our comments at the end of Section 1).

$$(\alpha, \beta) \in \Omega \quad \text{if and only if} \quad X\alpha \subseteq X\beta, \qquad \operatorname{dom}\alpha \subseteq \operatorname{dom}\beta \text{ and}$$
$$\alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) \subseteq \alpha\alpha^{-1}.$$

Consequently, there are two obvious questions: is there an algebraic formulation of Ω on any inverse semigroup? And, does the Vagner-Preston embedding preserve that formulation of Ω? To answer these questions, we define a relation $\ll$ on any inverse semigroup S by

$$a \ll b \quad \text{if and only if} \quad aa^{-1} \leq bb^{-1},\ a^{-1}a \leq b^{-1}b \text{ and } ab^{-1}.aa^{-1} \leq aa^{-1}.$$

For the proof of the next three results, recall that $\leq$ is both left and right compatible on S and that $x \leq y$ in S implies $x^{-1} \leq y^{-1}$ (see [3, Proposition V.2.4]).

From now on, the semigroup S and the set X we consider can be finite or infinite.

Theorem 12. *Let S be an inverse semigroup. Then $\ll$ is a partial order on S which contains $\leq$. Also, $a \ll b$ implies $a^{-1} \ll b^{-1}$.*

Proof. Clearly $\ll$ is reflexive, and it contains $\leq$ since $a \leq b$ implies $aa^{-1} \leq bb^{-1}$, $a^{-1}a \leq b^{-1}b$ and $ab^{-1} \leq bb^{-1}$, hence $ab^{-1}aa^{-1} \leq bb^{-1}aa^{-1} = aa^{-1}$. Suppose $a \ll b$ and $b \ll a$. That is, $aa^{-1} = bb^{-1}, a^{-1}a = b^{-1}b$ and

$$ab^{-1}.aa^{-1} \leq aa^{-1}, \quad ba^{-1}.bb^{-1} \leq bb^{-1}.$$

Then $ab^{-1} = ab^{-1}.bb^{-1} = ab^{-1}.aa^{-1} \leq aa^{-1}$ and similarly $ba^{-1} \leq bb^{-1}$. Thus, taking inverses, we also have $ba^{-1} \leq aa^{-1}$ and $ab^{-1} \leq bb^{-1}$. Hence

$$b = bb^{-1}.b \geq ab^{-1}.b = a.a^{-1}a = a$$

and similarly $b \leq a$, so $\ll$ is antisymmetric. To show $\ll$ is transitive, suppose $a, b, c \in S$ and

$$\begin{aligned} aa^{-1} \leq bb^{-1} \leq cc^{-1}, \qquad & a^{-1}a \leq b^{-1}b \leq c^{-1}c, \\ ab^{-1}.aa^{-1} \leq aa^{-1}, \qquad & bc^{-1}.bb^{-1} \leq bb^{-1}. \end{aligned}$$

Then $a = a.a^{-1}a \leq ab^{-1}b$, so $ac^{-1} \leq ab^{-1}bc^{-1}$ and hence

$$ac^{-1}.aa^{-1} \leq ab^{-1}bc^{-1}.aa^{-1} \leq ab^{-1}.bc^{-1}.bb^{-1} \leq ab^{-1}.bb^{-1} = ab^{-1}.$$

Therefore, multiplying on the right, we get $ac^{-1}.aa^{-1} \leq ab^{-1}.aa^{-1} \leq aa^{-1}$.

Finally, $ab^{-1}.aa^{-1} \leq aa^{-1}$ is equivalent to $ab^{-1}a \leq a$ and to $a^{-1}ba^{-1} \leq a^{-1}$, and hence to $a^{-1}(b^{-1})^{-1}.a^{-1}a \leq a^{-1}a$. Thus, we easily see that, if $a \ll b$, then $a^{-1} \ll b^{-1}$. □

Theorem 13. *$\ll$ equals Ω on $I(X)$.*

Proof. Let $\alpha, \beta \in I(X)$ and recall that $\leq$ equals $\subseteq$ on $I(X)$. It is easy to see that $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$ if and only if $\alpha\alpha^{-1} = \mathrm{id}_{\operatorname{dom}\alpha} \subseteq \mathrm{id}_{\operatorname{dom}\beta} = \beta\beta^{-1}$, and $X\alpha \subseteq X\beta$ if and only if $\alpha^{-1}\alpha = \mathrm{id}_{X\alpha} \subseteq \mathrm{id}_{X\beta} = \beta^{-1}\beta$. Thus, it remains to show that

$$(5) \quad \alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) \subseteq \alpha\alpha^{-1} \quad \text{if and only if} \quad \alpha\beta^{-1} \circ \alpha\alpha^{-1} \subseteq \alpha\alpha^{-1}.$$

In fact, if $\alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) \subseteq \alpha\alpha^{-1}$ and $(x, y) \in \alpha\beta^{-1} \circ \alpha\alpha^{-1}$, then $(x, y) \in \alpha\beta^{-1}$ and $x, y \in \operatorname{dom}\alpha$, so $(x, y) \in \alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha)$ and hence $(x, y) \in \alpha\alpha^{-1}$: that is, the containment on the left of (5) implies the one on the right. Conversely, if $\alpha\beta^{-1} \circ \alpha\alpha^{-1} \subseteq \alpha\alpha^{-1}$ and $(x, y) \in \alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha)$, then $(x, y) \in \alpha\beta^{-1}$ and $(y, y) \in \alpha\alpha^{-1}$, so $(x, y) \in \alpha\beta^{-1} \circ \alpha\alpha^{-1}$ and hence $x = y$: that is, the reverse implication in (5) also holds. □

Theorem 14. *Let S be an inverse semigroup and $a, b \in S$. Then $a \ll b$ in S if and only if $\rho_a \ll \rho_b$ in $I(S)$.*

Proof. First recall that, for each $a \in S$, $\rho_{a^{-1}} = \rho_a^{-1}$ (see [3, Theorem V.1.10]). By Theorem 11, we have $aa^{-1} \leq bb^{-1}$ if and only if $\rho_a\rho_a^{-1} = \rho_{aa^{-1}} \leq \rho_{bb^{-1}} = \rho_b\rho_b^{-1}$. Similarly, we deduce that $a^{-1}a \leq b^{-1}b$ if and only if $\rho_a^{-1}\rho_a = \rho_{a^{-1}a} \leq \rho_{b^{-1}b} = \rho_b^{-1}\rho_b$, and $ab^{-1} \cdot aa^{-1} \leq aa^{-1}$ if and only if $\rho_a\rho_b^{-1} \circ \rho_a\rho_a^{-1} = \rho_{ab^{-1}\cdot aa^{-1}} \leq \rho_{aa^{-1}} = \rho_a\rho_a^{-1}$. Hence $a \ll b$ if and only if $\rho_a \ll \rho_b$. □

As already noted, $\leq$ is left and right compatible on any inverse semigroup, but this is not true for Ω on $I(X)$. For convenience, we quote [11, Theorem 3.6] and, for comparison with what follows, we provide a slightly different proof of that result.

Theorem 15. *If $\gamma \in I(X)$ is non-zero, then*
(a) *γ is left compatible with Ω on $I(X)$ if and only if $\gamma^{-1}\gamma = \mathrm{id}_X$,*
(b) *γ is right compatible with Ω on $I(X)$ if and only if $\gamma\gamma^{-1} = \mathrm{id}_X$.*

Proof. For (a), suppose γ is left compatible with Ω and $a\gamma = x$. There is nothing to prove if $|X| = 1$, so we assume $|X| \geq 2$ and choose $y \neq x$ in X. Now define

$$\alpha = \begin{pmatrix} x \\ x \end{pmatrix}, \quad \beta = \begin{pmatrix} y & x \\ x & y \end{pmatrix}.$$

Clearly, $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$ and $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$. Also, $\alpha\beta^{-1} = x_y$ and $\alpha\alpha^{-1} = x_x$, so $\alpha\beta^{-1} \circ \alpha\alpha^{-1} = \emptyset \subseteq \alpha\alpha^{-1}$, and thus $(\alpha, \beta) \in \Omega$. Therefore, $(\gamma\alpha, \gamma\beta) \in \Omega$, where $\gamma\alpha = a_x$, so $x \in \operatorname{ran}\gamma\beta$ and hence $y \in \operatorname{ran}\gamma$. Since y is arbitrary, we conclude that $\operatorname{ran}\gamma = X$. The converse is the same as in the proof of [11, Theorem 3.6(a)], and the proof of (b) is similar. □

Suppose S is an inverse semigroup with zero 0 and identity 1, and let $M(S)$ denote the set of non-zero idempotents in S which are minimal under $\leq$. We say S is *right-pointed* if $M(S) \neq \emptyset$ and S has the following properties.

(R1) for each $g \in S$, if $g^{-1}gx = 0$ for all $x \in M(S)$, then $g^{-1}g = 0$,

(R2) for each $g \in S$, if $g^{-1}gx \neq 0$ for all $x \in M(S)$, then $g^{-1}g = 1$, and

(R3) for each $x, y \in M(S)$, there exists $b \in S$ such that $x \ll b$ and $bxby = y$.

Lemma 5. *Let S be a right-pointed inverse semigroup and suppose $g \in S$ is non-zero. Then g is left compatible with $\ll$ if and only if $g^{-1}g = 1$.*

Proof. We first note that in any inverse semigroup S, if $aa^{-1} \leq bb^{-1}$, then $g.aa^{-1}.g^{-1} \leq g.bb^{-1}.g^{-1}$ (by left and right compatibility of $\leq$), and so $ga(ga)^{-1} \leq gb(gb)^{-1}$. Also, $ab^{-1}.aa^{-1} \leq aa^{-1}$ implies

$$a(gb)^{-1}ga(ga)^{-1} = ab^{-1}.g^{-1}g.aa^{-1}.g^{-1} = ab^{-1}.aa^{-1}.g^{-1} \leq aa^{-1}.g^{-1} = a(ga)^{-1}.$$

Hence, by premultiplying this inequality by g, we obtain $ga(gb)^{-1}ga(ga)^{-1} \leq ga(ga)^{-1}$.

Now suppose g is left compatible with $\ll$. Since $g \neq 0$, (R1) implies that $g^{-1}gx \neq 0$ for some $x \in M(S)$, and so $gx \neq 0$. Also, by (R3), for each $y \in M(S)$, there exists $b \in S$ such that $x \ll b$ and $bxby = y$. Then $gx \ll gb$, so $(gx)^{-1}gx \leq (gb)^{-1}gb$. If $(gx)^{-1}gx = 0$, then $gx = 0$, a contradiction. So, $0 \neq (gx)^{-1}gx \leq x$ and, by the minimality of x under $\leq$, we deduce that $(gx)^{-1}gx = x$. Now, $xby \neq 0$ and, since $\leq$ is right compatible,

$$xby = x.xby \leq b^{-1}g^{-1}gb.xby = b^{-1}.g^{-1}gy.$$

Hence, $g^{-1}gy \neq 0$ for each $y \in M(S)$, and so $g^{-1}g = 1$ by (R2). Conversely, if $g^{-1}g = 1$ and $a^{-1}a \leq b^{-1}b$, then $(ga)^{-1}ga = a^{-1}a \leq b^{-1}b = (gb)^{-1}gb$, and this completes the proof. □

Example 3. It is easy to see that the non-zero minimal idempotents of $(I(X), \leq)$ are precisely the constant idempotents in $I(X)$ (compare [7, Theorem 13] for $P(X)$ under $\leq$). Thus, $I(X)$ clearly satisfies (R1) and (R2). Also, the proof of Theorem 15(a) shows that $I(X)$ satisfies (R3). However, although the set E of idempotents in $I(X)$ is an inverse semigroup which satisfies (R1) and (R2), it does not satisfy (R3) if $|X| \geq 2$. This is because the non-zero minimal elements of $(E, \leq)$ are the constants in E. Hence, if $x_x, y_y \in E$ are distinct, then $\beta x_x \beta y_y = \beta x_x y_y = \emptyset$ for each $\beta \in E$, so $\beta x_x \beta y_y \neq y_y$.

To determine conditions under which $\ll$ is right compatible, we say S is *left-pointed* if $M(S) \neq \emptyset$ and S has the following properties.

(L1) for each $g \in S$, if $xgg^{-1} = 0$ for all $x \in M(S)$, then $gg^{-1} = 0$,

(L2) for each $g \in S$, if $xgg^{-1} \neq 0$ for all $x \in M(S)$, then $gg^{-1} = 1$, and

(L3) for each $x, y \in M(S)$, there exists $b \in S$ such that $x \ll b$ and $ybxb = y$.

Lemma 6. *Let S be a left-pointed inverse semigroup and suppose $g \in S$ is non-zero. Then g is right compatible with $\ll$ if and only if $gg^{-1} = 1$.*

Proof. Clearly, $a^{-1}a \leq b^{-1}b$ always implies $(ag)^{-1}ag \leq (bg)^{-1}bg$. Also, if $ab^{-1}.aa^{-1} \leq aa^{-1}$, then $ab^{-1}a \leq a$ and so

$$\begin{aligned} ag(bg)^{-1}.ag(ag)^{-1} &= a.a^{-1}a.gg^{-1}.b^{-1}a.g(ag)^{-1} \\ &= agg^{-1}.a^{-1}.ab^{-1}a.g(ag)^{-1} \\ &\leq agg^{-1}.a^{-1}a.g(ag)^{-1} = ag(ag)^{-1}. \end{aligned}$$

The rest of the proof is the dual of that for Lemma 5. That is, we start with $xg \neq 0$ for some $x \in M(S)$ and observe that $0 \neq xg(xg)^{-1} \leq x$. Then, for each $y \in M(S)$ and some $b \in S$, we have

$$ybx = ybx.x \leq ybx.bgg^{-1}b^{-1} = ygg^{-1}.b^{-1}$$

and the remaining details follow like before. □

Remark 2. In fact, S is right-pointed if and only if it is left-pointed, and thus Lemmas 6 and properties (L1)−(L3) provide an algebraic formulation of Lemma 5 for what we may call *pointed* inverse semigroups. To see this, first note that (R1) is equivalent to the statement:

(R1.1) for each $g \in S$, if $gx = 0$ for all $x \in M(S)$, then $g = 0$.

Now, if (R1) holds and $xgg^{-1} = 0$ for all $x \in M(S)$, then $xg = 0$, so $g^{-1}x = 0$ for all $x \in M(S)$, and hence (R1.1) implies $g^{-1} = 0$: that is, (R1) implies (L1). Likewise, (R2) is equivalent to

(R2.1) for each $g \in S$, if $gx \neq 0$ for all $x \in M(S)$, then $g^{-1}g = 1$.

Now, if (R2) holds and $xgg^{-1} \neq 0$ for all $x \in M(S)$, then $xg \neq 0$, so $g^{-1}x \neq 0$ for all $x \in M(S)$, and hence (R2.1) implies $(g^{-1})^{-1}g^{-1} = 1$: that is, (R2) implies (L2). Finally, if (R3) holds and $x, y \in M(S)$, then there exists

$b \in S$ such that $x \ll b$ and $bxby = y$. Hence, $x \ll b^{-1}$ by Theorem 12, where $b^{-1} \in S$ and $yb^{-1}xb^{-1} = y$: that is, (R3) implies (L3). Clearly, the converse of each of these implications is also true.

Theorem 16. *If $\ll$ is right (or left) compatible on an inverse semigroup S, then $\ll$ equals $\leq$.*

Proof. We know $\leq$ is contained in $\ll$. Thus, we need only show that, if $\ll$ is right compatible and $a \ll b$, then $a \leq b$. If $a \ll b$, then

$$aa^{-1} \leq bb^{-1},\ a^{-1}a \leq b^{-1}b \text{ and } ab^{-1}.aa^{-1} \leq aa^{-1}. \tag{6}$$

From the proof of Lemma 5, we know the first inequality above is always left compatible. In addition, the third inequality is equivalent to $ab^{-1}a \leq a$ and to $a^{-1}ba^{-1} \leq a^{-1}$. Therefore, by supposition, $ag(ag)^{-1} \leq bg(bg)^{-1}$ for each $g \in S$, and hence (4) implies

$$agg^{-1}a^{-1}.bgg^{-1}b^{-1} = agg^{-1}a^{-1}. \tag{7}$$

In particular, if $g = a^{-1}$ in (7), then, from (6), we obtain

$$aa^{-1} = aa^{-1}a.a^{-1}ba^{-1}.ab^{-1} \leq a.a^{-1}.ab^{-1} = ab^{-1}. \tag{8}$$

Hence, post-multiplying this inequality by a, we obtain $a \leq ab^{-1}a$ and it follows that $ab^{-1}a = a$. Hence, $ab^{-1}.ab^{-1} = ab^{-1}$. Now, $ab^{-1} = aa^{-1}.ab^{-1}$ where ab^{-1} is an idempotent, so $ab^{-1} \leq aa^{-1}$ and we conclude from (8) that $ab^{-1} = aa^{-1}$. That is, $a \leq b$ and we have shown that, if $\ll$ is right compatible, then $\ll$ equals $\leq$. Likewise, if $\ll$ is left compatible, then the second inequality in (6) is left compatible with all $g \in S$, and an argument similar to the one above shows that $a^{-1}a \leq a^{-1}b$. Then, using the fact that $a^{-1}b(a^{-1}b)^{-1} \leq a^{-1}a$, we deduce that $a^{-1}b = a^{-1}a$ and thus $a \leq b$ by [3, Proposition V.2.2]. □

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Boorapa Singha
Department of Mathematics
Chiang Mai University
Chiangmai 50200, Thailand
E-mail address: boorapas@yahoo.com

Jintana Sanwong
Department of Mathematics
Chiang Mai University
Chiangmai 50200, Thailand
and
Material Science Research Center
Faculty of Science
Chiang Mai University
Thailand
E-mail address: scmti004@chiangmai.ac.th

Robert Patrick Sullivan
School of Mathematics & Statistics
University of Western Australia
Nedlands, 6009, Australia
E-mail address: bob@maths.uwa.edu.au