Bull. Korean Math. Soc. **49** (2012), No. 1, pp. 205–211
http://dx.doi.org/10.4134/BKMS.2012.49.1.205

ON GRADED KRULL OVERRINGS OF A GRADED NOETHERIAN DOMAIN

Eun Kyung Lee and Mi Hee Park

Abstract. Let R be a graded Noetherian domain and A a graded Krull overring of R. We show that if $\text{h-dim}\, R \leq 2$, then A is a graded Noetherian domain with $\text{h-dim}\, A \leq 2$. This is a generalization of the well-known theorem that a Krull overring of a Noetherian domain with dimension ≤ 2 is also a Noetherian domain with dimension ≤ 2.

Let $R = \oplus_{\gamma \in \Gamma} R_\gamma$ be a commutative ring with identity graded by an arbitrary torsionless grading monoid Γ. That is, Γ is a commutative cancellative additive monoid with torsion-free quotient group $G = \langle \Gamma \rangle$. A torsionless grading monoid can be given a total order compatible with the monoid operation [2, Corollary 3.4]. This is used in many referenced materials.

This paper continues the study of graded Noetherian rings begun in [7, 8]. In particular, we will show that, if R is a graded Noetherian domain with $\text{h-dim}\, R \leq 2$, then each graded Krull overring of R is graded Noetherian with h-dimension ≤ 2. This is a graded version of the following Heinzer Theorem [3, Theorem 9]: If D is a Noetherian domain with $\dim D \leq 2$, then each Krull overring of D is again Noetherian with dimension ≤ 2.

The graded ring R is called a *graded Noetherian ring* if R satisfies the ascending chain condition on homogeneous ideals, or equivalently, if each homogeneous (prime) ideal of R is finitely generated. The *h-height* of a homogeneous prime ideal P (denoted by $\text{h-ht}\, P$) is defined to be the supremum of the lengths of chains of homogeneous prime ideals descending from P and the *h-dimension* of R (denoted by $\text{h-dim}\, R$) is defined to be $\sup\{\, \text{h-ht}\, P \mid P \text{ is a homogeneous prime ideal of } R\}$.

If R is a Γ-graded integral domain, then the set S of nonzero homogeneous elements of R is a multiplicatively closed set. The quotient ring $R_S = \bigoplus_{\gamma \in G} (R_S)_\gamma$ is a G-graded ring, where $(R_S)_\gamma = \{\frac{a}{b} \mid a \in R_\alpha, b \in R_\beta \setminus \{0\}, \text{ and } \alpha - \beta = \gamma\}$ for each $\gamma \in G$. It is called the *homogeneous quotient field*

Received October 20, 2010.
2010 *Mathematics Subject Classification.* 13A02, 13B25, 13E05, 13F05.
Key words and phrases. graded Noetherian ring, graded Krull domain, polynomial extension ring.
This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2008-331-C00011).

of R since each nonzero homogeneous element of R_S is a unit. An overring A of R contained in R_S is called a *homogeneous overring* if A is a G-graded ring with each $A_\gamma = A \cap (R_S)_\gamma$. Let $\{R_\lambda\}_{\lambda\in\Lambda}$ be a family of homogeneous overrings of R such that $R = \bigcap R_\lambda$. The intersection $R = \bigcap R_\lambda$ is said to be *homogeneously locally finite* if each nonzero homogeneous element of R is a unit in all R_λ but a finite number of the R_λ.

A graded domain R is called a *graded DVR* if R has the unique nonzero homogeneous prime ideal and it is principal. R is called a *graded Krull domain* if it is completely integrally closed and satisfies the ascending chain condition on homogeneous divisorial ideals, or equivalently, if $R = \bigcap V_\lambda$, where the intersection is homogeneously locally finite and each V_λ is a homogeneous overring of R which is a graded DVR.

Undefined terms and terminology are standard as in [2, 4, 5].

Theorem 1. *Let R be a graded Noetherian ring and let $R[X]$ be a graded polynomial extension ring of R in which X is a homogeneous element. Then $R[X]$ is also a graded Noetherian ring.*

Proof. Let J be a nonzero homogeneous ideal of $R[X]$ and let I_n be the ideal of R generated by the leading coefficients of homogeneous polynomials in J of degree n in X. Since $R[X]$ is a graded extension ring of R and X is a homogeneous element, each leading coefficient of a homogeneous polynomial in $R[X]$ is homogeneous. Therefore, I_n is a homogeneous ideal of R.

Let r be the leading coefficient of a homogeneous polynomial $f \in J$ of degree n in X. Then r is the leading coefficient of Xf and Xf is a homogeneous polynomial in J of degree $n+1$ in X. Hence $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$. Since R is a graded Noetherian ring, there exists an integer t such that $I_n = I_t$ for all $n \geq t$. Furthermore, each I_n is finitely generated, say $I_n = (r_{n1}, r_{n2}, \ldots, r_{ni_n})$, where each r_{nj} is the leading coefficient of a homogeneous polynomial f_{nj} in J of degree n. Observe that $f_{0j} = r_{0j} \in R$.

We claim that $J = (\{f_{nj} \mid 0 \leq n \leq t,\ 1 \leq j \leq i_n\})$.

Let $J' = (\{f_{nj} \mid 0 \leq n \leq t,\ 1 \leq j \leq i_n\})$. Clearly $J' \subseteq J$. For the opposite inclusion, let f be a nonzero homogeneous polynomial in J of degree k in X. Use induction on k. If $k = 0$, then $f \in J \cap R = I_0 \subseteq J'$. Assume that $k \geq 1$ and that every homogeneous polynomial in J of degree less than k is in J'. Let r be the leading coefficient of f.

If $k \leq t$, then $r \in I_k$ and hence $r = s_1 r_{k1} + s_2 r_{k2} + \cdots + s_{i_k} r_{ki_k}$, where each s_j is a homogeneous element of R and each $s_j r_{kj}$ belongs to the same homogeneous component of R. Therefore, the polynomial $\sum_{j=1}^{i_k} s_j f_{kj}$ is a homogeneous polynomial in J'. Note that f and $\sum_{j=1}^{i_k} s_j f_{kj}$ belong to the same homogeneous component of $R[X]$ and that they have the same leading coefficient r and the same degree k in X. Consequently, $f - \sum_{j=1}^{i_k} s_j f_{kj}$ is a homogeneous polynomial in J of degree less than k. By the induction hypothesis, $f - \sum_{j=1}^{i_k} s_j f_{kj} \in J'$, whence $f \in J'$.

If $k > t$, then $r \in I_k = I_t$ and $r = \sum_{j=1}^{i_t} s_j r_{tj}$, where each s_j is a homogeneous element of R and each $s_j r_{tj}$ belongs to the same homogeneous component of R. Note that $(\sum_{j=1}^{i_t} s_j f_{tj})X^{k-t}$ is a homogeneous polynomial in J' of degree k and that it has leading coefficient r. Thus $f - \sum_{j=1}^{i_t} s_j X^{k-t} f_{tj}$ is a homogeneous polynomial in J of degree less than k. By the induction hypothesis, $f - \sum_{j=1}^{i_t} s_j X^{k-t} f_{tj} \in J'$. Consequently, $f \in J'$.

Therefore, $J = J'$. Thus since each homogeneous ideal of $R[X]$ is finitely generated, $R[X]$ is a graded Noetherian ring. □

Remark 2. Let k be a field, let X, Y be indeterminates over k, and let $R = k[Y] = \oplus_{i \geq 0} kY^i$. Then R is an $\mathbb{N}_0$-graded Noetherian ring. Consider the polynomial extension ring $R[X]$ over R. It can be made into an $\mathbb{N}_0^2$-graded ring by defining $R[X]_{(i,j)} = kY^iX^j$ for each $(i,j) \in \mathbb{N}_0^2$, i.e., $R[X] = k[Y,X] = \oplus_{(i,j)\in\mathbb{N}_0^2} kY^iX^j$. It also can be made into an $\mathbb{N}_0$-graded ring by defining $R[X]_m = \oplus_{i+j=m} kY^iX^j$ for each $m \geq 0$, i.e.,

$$R[X] = k[Y,X] = \oplus_{m\geq 0}(\oplus_{i+j=m} kY^iX^j).$$

Both graded rings are graded extension rings of R in which X is a homogeneous element. Note that $X + Y$ is not a homogeneous element in the first grading, while it is a homogeneous element in the second grading. Thus $R[X]$ has many distinct gradings which extend the grading of R and in which X is a homogeneous element.

A graded integral domain R is called a *graded PID* if each homogeneous ideal of R is principal (generated by a homogeneous element).

Lemma 3 ([8, Theorem 3.5])**.** *Let x be a nonunit homogeneous element in a graded Noetherian ring and let P be a prime ideal minimal over (x). Then* $\text{h-ht}\, P \leq 1$.

Theorem 4. *Let R be a graded integral domain in which each nonzero homogeneous element is unit, and let $R[X]$ be a graded polynomial extension ring of R in which X is a homogeneous element. Then $R[X]$ is a graded PID and* $\text{h-dim}\, R[X] = 1$.

Proof. Let I be a nonzero homogeneous ideal of $R[X]$. Choose a nonzero homogeneous polynomial g in I of minimal degree in X. Since its leading coefficient is a nonzero homogeneous element of R, it is a unit element. Therefore, we may assume that g is a monic polynomial. We claim that $I = (g)$. Let f be a nonzero homogeneous polynomial in I. By division algorithm, there exist $q, r \in R[X]$ such that $f = gq + r$, where $r = 0$ or the degree of r in X is less than the degree of g in X. Note that q and r are homogeneous polynomials. Since $r = f - gq \in I$, by choice of g, r must be zero. Thus $f = gq \in (g)$.

Thus $R[X]$ is a graded PID. It follows from Lemma 3 that $\text{h-dim}\, R[X] = 1$. □

Corollary 5. *Let R be a graded ring and let $R[X]$ be a graded polynomial extension ring of R in which X is a homogeneous element. Then there does not exist in $R[X]$ a chain of three distinct homogeneous prime ideals with the same contraction in R.*

Proof. There are two easy observations: If Q is a homogeneous prime ideal of $R[X]$, then $Q \cap R$ is a homogeneous prime ideal of R and $Q \supseteq (Q \cap R)R[X]$; If R is a graded integral domain with homogeneous quotient field R_S, then there is a one-to-one correspondence between the homogeneous prime ideals of $R[X]$ that contract to (0) in R and the homogeneous prime ideals of $R_S[X]$.

Together with the above theorem, these two statements prove the corollary. □

Corollary 6. *Let R be a graded Krull domain and let $R[X]$ be a graded polynomial extension ring of R in which X is a homogeneous element. Then $R[X]$ is also a graded Krull domain.*

Proof. Let S be the set of nonzero homogeneous elements of R. By Theorem 4, $R_S[X]$ is a graded PID. Then, obviously, $R_S[X]$ is graded Noetherian and hence satisfies the ascending chain condition on homogeneous (divisorial) ideals. Also, since each nonzero homogeneous ideal I of $R_S[X]$ is principal, we have $(I : I) = R_S[X]$, and hence by [1, Proposition 5.2], $R_S[X]$ is completely integrally closed. Thus it follows that $R_S[X]$ is a graded Krull domain.

Let V be a graded DVR with a single nonzero homogeneous principal prime ideal M and let $V[X]$ be a graded polynomial extension ring of V in which X is a homogeneous element. Then, by Theorem 1, $V^* := V[X]_{T \setminus M[X]}$, where T is the set of nonzero homogeneous elements of $V[X]$, is a graded Noetherian domain with the unique maximal homogeneous ideal MV^*. Since MV^* is principal, h-ht $MV^* = 1$ and hence MV^* is the unique nonzero homogeneous prime ideal of V^*. Moreover, it is principal. Therefore, V^* is a graded DVR.

Since R is a graded Krull domain, by [1, Theorem 5.15] we can write $R = \bigcap V_\alpha$, where the intersection is homogeneously locally finite and each V_α is a homogeneous overring of R which is a graded DVR. Then $R[X] = \bigcap V_\alpha[X] = \bigcap (V_\alpha^* \cap R_S[X]) = (\bigcap V_\alpha^*) \cap R_S[X]$. Since each V_α^* and $R_S[X]$ are graded Krull domains, by [1, Theorem 5.15] again, it suffices to show that the intersection $R[X] = (\bigcap V_\alpha^*) \cap R_S[X]$ is homogeneously locally finite.

Consider a nonzero homogeneous element $f = a_0 + a_1X + \cdots + a_nX^n$ of $R[X]$. We may assume that each a_i is a homogeneous element of R. If $a_i \neq 0$, then a_i is a nonunit in only finitely many V_α's, say $V_1, V_2, \ldots, V_m$. Then for each $\alpha \neq 1, 2, \ldots, m$, f is a unit in V_α^*. Thus the intersection $R[X] = (\bigcap V_\alpha^*) \cap R_S[X]$ is homogeneously locally finite. □

Lemma 7. *Let R be a graded Noetherian domain and let $R[X]$ be a graded polynomial extension ring of R in which X is a homogeneous element. If P is a nonzero minimal homogeneous prime ideal of R, then $P[X]$ is a nonzero minimal homogeneous prime ideal of $R[X]$.*

Proof. Let a be a nonzero homogeneous element of P. Then P is minimal over aR. We claim that $P[X]$ is minimal over $aR[X]$. Suppose that there exists a homogeneous prime ideal Q of $R[X]$ such that $aR[X] \subseteq Q \subseteq P[X]$. Then $a \in Q \cap R \subseteq P$. Therefore, $Q \cap R = P$ and $Q \supseteq P[X]$, whence $Q = P[X]$. Thus $P[X]$ is minimal over the principal homogeneous ideal $aR[X]$, and hence by Theorem 1 and Lemma 3, $\text{h-ht}\, P[X] = 1$. □

Theorem 8. *Let R be a graded Noetherian ring and let $R[X]$ be a graded polynomial extension ring of R in which X is a homogeneous element. Let P be a homogeneous prime ideal of R with* $\text{h-ht}\, P = n$ *and let Q be a homogeneous prime ideal of $R[X]$ such that $Q \cap R = P$ and $Q \supsetneq P[X]$. Then* $\text{h-ht}\, P[X] = n$ *and* $\text{h-ht}\, Q = n + 1$.

Proof. It is obvious that $\text{h-ht}\, P[X] \geq n$ and $\text{h-ht}\, Q \geq n + 1$. So we only need to prove the inequalities $\text{h-ht}\, P[X] \leq n$, $\text{h-ht}\, Q \leq n + 1$. We will use induction on n.

For $n = 0$, let Q' be a homogeneous prime ideal of $R[X]$ contained in $P[X]$. Then $Q' \cap R \subseteq P$. By minimality of P, $Q' \cap R = P$ and $Q' \supseteq P[X]$. Thus we have $Q' = P[X]$, i.e., $P[X]$ is a minimal homogeneous prime ideal of $R[X]$. The statement $\text{h-ht}\, Q = 1$ follows from Corollary 5.

Let $n \geq 1$. Suppose that $\text{h-ht}\, P[X] > n$. Then there exists a chain of homogeneous prime ideals $Q_0 \subsetneq Q_1 \subsetneq Q_2 \subsetneq \cdots \subsetneq Q_{n+1} = P[X]$. Let $P' = Q_n \cap R$. Then P' is a homogeneous prime ideal of R properly contained in P and hence $\text{h-ht}\, P' \leq n - 1$. By the induction hypothesis, $\text{h-ht}\, P'[X] = \text{h-ht}\, P'$ and $\text{h-ht}\, Q_n \leq \text{h-ht}\, P' + 1 \leq n$. But since $\text{h-ht}\, Q_n \geq n$, we have $\text{h-ht}\, Q_n = n$ and $\text{h-ht}\, P' = n - 1$. Therefore, $P'[X] \subsetneq Q_n \subsetneq Q_{n+1} = P[X]$. This implies that, in the graded Noetherian domain $(R/P')[X]$, $\text{h-ht}\, (P/P')[X] \geq 2$. But since P/P' is a nonzero minimal homogeneous prime ideal of R/P', $(P/P')[X]$ is also a nonzero minimal homogeneous prime ideal of $(R/P')[X]$ by the above lemma. This contradiction proves that $\text{h-ht}\, P[X] \leq n$.

Now we will prove $\text{h-ht}\, Q \leq n + 1$. Let Q' be a homogeneous prime ideal of $R[X]$ properly contained in Q. If $Q' \cap R = P$, then by Corollary 5, we must have $Q' = P[X]$ and hence $\text{h-ht}\, Q' \leq n$. If $Q' \cap R \subsetneq P$, then $\text{h-ht}\, (Q' \cap R) \leq n - 1$. By the induction hypothesis, $\text{h-ht}\, Q' \leq n$. Therefore, $\text{h-ht}\, Q \leq n + 1$. □

Corollary 9. *Let R be a graded Noetherian ring and let $R[X]$ be a graded polynomial extension ring of R in which X is a homogeneous element. Then* $\text{h-dim}\, R[X] = \text{h-dim}\, R + 1$.

Proof. By the above theorem, it suffices to show that for each homogeneous prime ideal P of R, there exists a homogeneous prime ideal Q of $R[X]$ such that $Q \cap R = P$, $Q \supsetneq P[X]$. Note that $P + XR[X]$ is our desired homogeneous prime ideal of $R[X]$. □

Lemma 10 ([8, Theorem 4.2])**.** *Let $R \subseteq A$ be graded integral domains with homogeneous quotient fields $K \subseteq L$, respectively. Assume that R is graded Noetherian with* $\text{h-dim}\, R = 1$ *and L is finite over K. Then A is graded Noetherian with* $\text{h-dim}\, A \leq 1$*. Moreover, if J is a nonzero homogeneous ideal of A, then A/J is a finitely generated R-module.*

Theorem 11. *Let R be a graded Noetherian domain with* $\text{h-dim}\, R \leq 2$ *and let A be a graded extension ring of R which is a graded Krull domain. If the homogeneous quotient field of A is a module finite over that of R, then A is a graded Noetherian domain with* $\text{h-dim}\, A \leq 2$*.*

Proof. By [8, Theorem 6.7], it suffices to show that, for every homogeneous prime ideal Q of A of height 1, A/Q is a graded Noetherian domain with $\text{h-dim}(A/Q) \leq 1$.

Let Q be a homogeneous prime ideal of A with $\text{ht}\, Q = 1$. If Q is a maximal homogeneous ideal, then there is nothing to prove. Assume that Q is not a maximal homogeneous ideal and let Q' be a homogeneous prime ideal of A such that $Q' \supsetneq Q$. Take a homogeneous element $x \in Q' \setminus Q$.

Make the polynomial ring $R[X]$ into a graded extension ring of R by defining the degree of X to be the degree of x in the graded ring A. Consider the map $\varphi : R[X] \to R[x]$ given by $f(X) \mapsto f(x)$. Then φ is a graded ring epimorphism and hence $R[X]/\text{Ker}\, \varphi \cong R[x]$ as graded rings. Therefore, $R[x]$ is a graded Noetherian domain. Since the (homogeneous) quotient field of A is finite over the (homogeneous) quotient field of R, x is algebraic over R and hence $\text{Ker}\, \varphi \neq (0)$. Therefore, $\text{h-dim}\, R[x] \leq 2$.

Thus by replacing R with $R[x]$, we may assume that $x \in R$. Then $Q' \cap R \supsetneq P := Q \cap R \supsetneq (0)$. Therefore, $\text{h-ht}\, P = 1$.

Let S, T be the sets of nonzero homogeneous elements of R, A, respectively. Then $R_{S \setminus P}$ is a graded Noetherian domain with $\text{h-dim}\, R_{S\setminus P} = \text{h-ht}\, P = 1$, $A_{T\setminus Q}$ is a graded extension ring of $R_{S\setminus P}$, and the homogeneous quotient field A_T of $A_{T\setminus Q}$ is finite over the homogeneous quotient field R_S of $R_{S\setminus P}$ (by assumption). Therefore, by the above lemma, $A_{T\setminus Q}/QA_{T\setminus Q}$ is finite over $R_{S\setminus P}/PR_{S\setminus P}$. Note that $A_{T\setminus Q}/QA_{T\setminus Q}$, $R_{S\setminus P}/PR_{S\setminus P}$ are the homogeneous quotient fields of A/Q, R/P, respectively. Since A/Q is a graded extension ring of R/P and R/P is a graded Noetherian domain with $\text{h-dim}\,(R/P) = 1$, by the above lemma again, A/Q is a graded Noetherian domain with $\text{h-dim}\,(A/Q) \leq 1$. □

Finally, as a corollary, a graded version of the Heinzer Theorem can be obtained.

Corollary 12. *Let R be a graded Noetherian domain with* $\text{h-dim}\, R \leq 2$ *and let A be a graded Krull overring of R. Then A is a graded Noetherian domain with* $\text{h-dim}\, A \leq 2$*.*

As another corollary, the graded version [8, Theorem 6.8] of the Nagata Theorem [6] can be recovered.

Corollary 13. *Let R be a graded Noetherian domain with* $\text{h-dim}\, R \leq 2$*. Then the integral closure R' of R is also a graded Noetherian domain with* $\text{h-dim}\, R' \leq 2$*.*

Proof. By [7, Theorem 2.10] or [8, Theorem 5.3], R' is a graded Krull domain. Hence it follows from Corollary 12 that R' is a graded Noetherian domain with $\text{h-dim}\, R' \leq 2$. □

Remark 14. There exists an example of a graded Noetherian domain R such that $\text{h-dim}\, R = 2$, $\dim R > 2$, but R is not Noetherian. For instance, let k be a field, let Γ be the additive monoid $\mathbb{Q} \oplus \mathbb{N}_0 \oplus \mathbb{N}_0$, and let R be the Γ-monoid ring $k[\Gamma]$. Then $R \cong k[\mathbb{Q}][Y, Z] = \bigoplus_{(i,j) \in \mathbb{N}_0^2} k[\mathbb{Q}]Y^i Z^j$, where Y, Z are indeterminates over $k[\mathbb{Q}]$. Since the $\mathbb{Q}$-group ring $k[\mathbb{Q}]$ is equal to its homogeneous quotient field, it is a graded Noetherian domain with $\text{h-dim}\, k[\mathbb{Q}] = 0$. By Theorem 1 and Corollary 9, R is a graded Noetherian domain with $\text{h-dim}\, R = 2$. Meanwhile, since $\mathbb{Q}$ is not a finitely generated abelian group, $k[\mathbb{Q}]$ is not Noetherian ([9, Theorem 2.2]) and hence R is not Noetherian, either. Since $k[\mathbb{Q}]$ is integral over $k[\mathbb{Z}]$, $k[\mathbb{Q}][Y, Z]$ is integral over $k[\mathbb{Z}][Y, Z]$. Therefore, $\dim R = \dim k[\mathbb{Q}][Y, Z] = \dim k[\mathbb{Z}][Y, Z] = \dim k[X, X^{-1}][Y, Z] = 3$.

References

[1] D. D. Anderson and D. F. Anderson, *Divisibility properties of graded domains*, Canad. J. Math. **34** (1982), no. 1, 196–215.

[2] R. Gilmer, *Commutative Semigroup Rings*, University of Chicago Press, Chicago, 1984.

[3] W. Heinzer, *On Krull overrings of a Noetherian domain*, Proc. Amer. Math. Soc. **22** (1969), 217–222.

[4] I. Kaplansky, *Commutative Rings*, Allyn and Bacon, Boston, 1970.

[5] H. Matsumura, *Commutative Ring Theory*, Cambridge Univ. Press, Cambridge, 1986.

[6] J. Nishimura, *Note on integral closures of a Noetherian integral domain*, J. Math. Kyoto Univ. **16** (1976), no. 1, 117–122.

[7] M. H. Park, *Integral closure of graded integral domains*, Comm. Algebra **35** (2007), no. 12, 3965–3978.

[8] C. H. Park and M. H. Park, *Integral closure of a graded Noetherian domain*, J. Korean Math. Soc. **48** (2011), no. 3, 449–464.

[9] D. E. Rush, *Noetherian properties in monoid rings*, J. Pure Appl. Algebra **185** (2003), no. 1-3, 259–278.

Eun Kyung Lee
Department of Mathematics
Chung-Ang University
Seoul 156-756, Korea
E-mail address: kkamzzyc@gmail.com

Mi Hee Park
Department of Mathematics
Chung-Ang University
Seoul 156-756, Korea
E-mail address: mhpark@cau.ac.kr