

RADIAL OSCILLATION OF LINEAR DIFFERENTIAL EQUATION

ZHAOJUN WU

ABSTRACT. In this paper, the radial oscillation of the solutions of higher order homogeneous linear differential equation

$$f^{(k)} + A_{n-2}(z)f^{(k-2)} + \cdots + A_1(z)f' + A_0(z)f = 0$$

with transcendental entire function coefficients is studied. Results are obtained to extend some results in [Z. Wu and D. Sun, *Angular distribution of solutions of higher order linear differential equations*, J. Korean Math. Soc. **44** (2007), no. 6, 1329–1338].

1. Introduction and main results

In this paper, the meromorphic function always means a function being meromorphic in the whole complex plane \mathbb{C} . Assume that the basic definitions, theorems and standard notations of the Nevanlinna theory for meromorphic function (see [11], [22] or [24]) are known. There have appeared many papers on the global theory of complex differential equations which were studied from the point of view of Nevanlinna theory, since 1982 when the article by Bank and Laine [1] appeared in Trans. Amer. Math. Soc. We refer the reader to the books by Laine [12], and by Gao etc. [6]. The first general research on the radial oscillation theory of the solutions of

$$(1) \quad f'' - A(z)f = 0$$

is due to Wang [17] and Wu [19] respectively. Here, we recall some definitions by Wang [17] as follows (also see Rossi and Wang [14]).

Definition 1 (Sectorial exponent of convergence). For $a \in \mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$, define

$$\lambda_{\alpha,\beta}(f, a) = \limsup_{r \rightarrow \infty} \frac{\log n(r, \Omega(\alpha, \beta), f = a)}{\log r},$$

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where $\Omega(\alpha, \beta) = \{z \mid \alpha < \arg z < \beta\}$, $0 < \beta - \alpha \leq \pi$ and $n(r, \Omega(\alpha, \beta), f = a)$ is the number of the roots of $f(z) - a = 0$ in $\Omega(\alpha, \beta) \cap \{|z| < r\}$, counting multiplicities. Especially, when $a = 0$, we write $\lambda_{\alpha, \beta}(f) = \lambda_{\alpha, \beta}(f, 0)$.

Definition 2 (Radial exponent of convergence). For any $\theta \in [0, 2\pi)$ and $a \in \mathbb{C}_\infty$, we define

$$\lambda_\theta(f, a) = \lim_{\varepsilon \rightarrow 0^+} \lambda_{\theta - \varepsilon, \theta + \varepsilon}(a, f).$$

Especially, when $a = 0$, we write $\lambda_\theta(f) = \lambda_\theta(f, 0)$.

In 1994, Wu [19] had proved the following theorem.

Theorem A ([19]). *Let $A(z)$ be a transcendental entire function of finite order in the plane and let f_1, f_2 be two linearly independent solutions of (1). Set $E = f_1 f_2$. Then $\lambda_\theta(E) = +\infty$, if and only if $\limsup_{r \rightarrow \infty} \frac{\log \log M(r, \Omega_{\theta, \varepsilon}, E)}{\log r} = +\infty$ for any $\varepsilon > 0$, where $M(r, \Omega_{\theta, \varepsilon}, E) = \sup\{|E(te^{i\tau})| : \theta - \varepsilon \leq \tau \leq \theta + \varepsilon, 1 \leq t \leq r\}$.*

Recently, Wu [18] proved the following theorem on connection of the radial exponent of convergence of zeros with Borel direction of the product of a solution base of (1).

Theorem B ([18]). *Let $A(z)$ be a transcendental entire function of finite order in the plane and f_1, f_2 be two linearly independent solutions for (1). Let $E = f_1 f_2$. Suppose that the exponent of convergence of zero-sequence $\lambda(E)$ is ∞ . Then $L : \arg z = \theta_0$ is an infinity order Borel direction of E if and only if $\lambda_{\theta_0}(E) = \infty$.*

For $k \geq 2$, we consider the homogeneous linear differential equation

$$(2) \quad f^{(k)} + A_{k-2} f^{(k-2)} + \dots + A_0 f = 0,$$

where A_0, \dots, A_{k-2} are entire functions with $A_0 \not\equiv 0$. Bernal [3] studied the iterated p -order of solutions of (2). In this paper, we shall study the connection of the radial exponent of convergence of zeros with Borel direction of the product of a solution base of linear differential equation (2) with entire coefficients of finite iterated p -order. For the sake of convenience, we define inductively (see [3], [10], [11], [15]), for $r \in [0, +\infty)$, $\exp^{[1]} r = e^r$ and $\exp^{[n+1]} r = \exp(\exp^{[n]} r)$, $n \in \mathbb{N}$. For all r sufficiently large, we define $\log^{[1]} r = \log r$ and $\log^{[n+1]} r = \log(\log^{[n]} r)$, $n \in \mathbb{N}$. We also denote $\exp^{[0]} r = r = \log^{[0]} r$, $\log^{[-1]} r = \exp^{[1]} r$ and $\exp^{[-1]} r = \log^{[1]} r$. We recall the following definitions and remarks.

Definition 3 ([11, 15]). The iterated p -order $\sigma_p(f)$ of a meromorphic function $f(z)$ is defined by

$$\sigma_p(f) = \limsup_{r \rightarrow \infty} \frac{\log^{[p]} T(r, f)}{\log r} \quad (p \in \mathbb{N}).$$

Remark 1 ([4]). (1) If $p = 1$, then we denote $\sigma_1(f) = \sigma(f)$; (2) If $p = 2$, then we denote the so-called hyper order by $\sigma_2(f)$; (3) If $f(z)$ is an entire function, then

$$\sigma_p(f) = \limsup_{r \rightarrow \infty} \frac{\log^{[p+1]} M(r, f)}{\log r}.$$

Definition 4 ([3, 15]). The growth index of the iterated order of a meromorphic function $f(z)$ is defined by

$$i(f) = \begin{cases} 0 & \text{if } f \text{ is rational,} \\ \min\{n \in \mathbb{N} : \sigma_n(f) < \infty\} & \text{if } f \text{ is transcendental and} \\ & \sigma_n(f) < \infty \text{ for some } n \in \mathbb{N}, \\ \infty & \text{if } \sigma_n(f) = \infty \text{ for all } n \in \mathbb{N}. \end{cases}$$

In [3], Bernal prove the following theorem.

Theorem C ([3]). Assume that A_0, \dots, A_{n-1} are entire functions and $p = \max\{i(A_j), j = 0, \dots, k - 1\}$. Set

$$L(f) := f^{(n)} + A_{n-1}f^{(n-1)} + A_{n-2}(z)f^{(n-2)} + \dots + A_1(z)f' + A_0(z)f.$$

If $p > 0$, let $\sigma = \max\{\sigma_p(A_j), j = 0, \dots, k - 1\}$, then

- (i) $\delta \leq 1 + p$, where $\delta = \sup\{i(f) : L(f) = 0\}$.
- (ii) if $p < \infty$, then $\gamma_{p+1} \leq \sigma$, where $\gamma_{p+1} = \sup\{\sigma_{p+1}(f) : L(f) = 0\}$.

Definition 5 ([11, 15]). The iterated convergence exponent of the sequence of a -points ($a \in \mathbb{C}_\infty$) is defined by

$$\lambda_n(f - a) = \lambda_n(f, a) = \limsup_{r \rightarrow \infty} \frac{\log^{[n]} N(r, \frac{1}{f-a})}{\log r} \quad (n \in \mathbb{N}).$$

Remark 2 ([4]). $\lambda_1(f - a) = \lambda(f - a)$; $\lambda_1(f, 0) = \lambda(f)$; $\lambda_p(f, 0) = \lambda_p(f)$.

Definition 6. The sectorial iterated convergence exponent of the sequence of a -points ($a \in \mathbb{C}_\infty$) is defined by

$$\lambda_{n,\alpha,\beta}(f - a) = \lambda_{n,\alpha,\beta}(f, a) = \limsup_{r \rightarrow \infty} \frac{\log^{[n]} n(r, \Omega(\alpha, \beta), f = a)}{\log r} \quad (n \in \mathbb{N}).$$

The radial iterated convergence exponent of the sequence of a -points ($a \in \mathbb{C}_\infty$) is defined by

$$\lambda_{n,\theta}(f - a) = \lambda_{n,\theta}(f, a) = \lim_{\varepsilon \rightarrow 0^+} \lambda_{n,\theta-\varepsilon,\theta+\varepsilon}(f, a) \quad (n \in \mathbb{N}).$$

Remark 3. $\lambda_{1,\alpha,\beta}(f - a) = \lambda_{\alpha,\beta}(f - a)$; $\lambda_{1,\theta}(f - a) = \lambda_\theta(f - a)$.

Definition 7. Let $p \in \mathbb{N}$, and $f(z)$ be a meromorphic function of iterated p -order $\rho(0 < \rho \leq \infty)$. A ray $L : \arg z = \theta$ is called a Borel direction of iterated

p -order ρ of f , if no matter how small the positive number $0 < \varepsilon < \pi/2$ is, for each value $a \in \mathbb{C}_\infty$, holds

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p]} n(r, \Omega(\theta - \varepsilon, \theta + \varepsilon), f = a)}{\log r} = \rho,$$

with at most two exceptional values a .

Remark 4. When $p = 1$, Borel direction of iterated p -order ρ of f is called a Borel direction of order ρ of f . When $p = 2$, Borel direction of iterated p -order ρ of f is called a Borel direction of hyper order ρ of f .

In [23], the author prove the following theorem.

Theorem D ([23]). *Assume that some (or all) of A_0, \dots, A_{k-2} are transcendental entire functions of finite order growth and equation (2) possesses a solution base f_1, f_2, \dots, f_k . Set $E = f_1 \cdots f_k$. Then $\lambda_\theta(E) = +\infty$, if and only if*

$$\limsup_{r \rightarrow \infty} \frac{\log \log M(r, \Omega_{\theta, \varepsilon}, E)}{\log r} = +\infty$$

for any $\varepsilon > 0$.

This result and Theorem B motivate the present author to prove the following theorem in [20].

Theorem E. *Assume that some (or all) of A_0, \dots, A_{k-2} are transcendental entire functions of finite order growth and equation (2) possesses a solution base f_1, f_2, \dots, f_k . Set $E = f_1 \cdots f_k$. Suppose that the exponent of convergence of zero-sequence $\lambda(E) = \infty$ and E is an entire function of hyper order ρ ($0 < \rho < \infty$). Then $\lambda_{2, \theta}(E) = \rho$ if and only if $L : \arg z = \theta$ is a Borel direction of hyper order ρ of E .*

In this paper, we shall continue to research the radial oscillation of the solutions of (2). In fact, we shall prove the following theorem.

Theorem 1. *Assume that some (or all) of A_0, \dots, A_{k-2} are transcendental entire functions, and $p = \max\{i(A_j), j = 0, 1, \dots, k - 2\} < \infty$. If equation (2) possesses a solution base f_1, f_2, \dots, f_k and set $E = f_1 \cdots f_k$. Then $i(E) \leq (p + 1)$. If $\sigma_{(p+1)}(E) = \rho > 0$, then the following statements are equivalent:*

- (i) $L : \arg z = \theta$ is a Borel direction of iterated $(p + 1)$ -order ρ of E ;
- (ii) $\lambda_{(p+1), \theta}(E) = \rho$;
- (iii) $\limsup_{r \rightarrow \infty} \frac{\log^{[p+2]} M(r, \Omega_{\theta, \varepsilon}, E)}{\log r} = \rho$ for any $\varepsilon > 0$.

By Theorem C, we know that $i(E) \leq p + 1$. When $k = 2$ and $p = 1$, the equivalence of (i) and (ii) is a precise version of Theorem B in the case of $\sigma_2(E) = \rho > 0$. When $k \geq 2$ and $p = 1$, the equivalence of (i) and (ii) is Theorem E. When $k = 2$ and $p = 1$, the equivalence of (ii) and (iii) has been obtained by Huang and Chen [7] in an weakly form. When $k \geq 2$ and $p = 1$,

the equivalence of (ii) and (iii) has been obtained by Huang and Chen [8] in an weakly form. Most recently, Zheng [24] give a system research about the equivalence of (i) and (ii) by using the proximate order of Qinglai Hiong (see [5]). Here, the innovative point is the equivalence of (ii) and (iii).

2. Some lemmas

The proof of Theorem 1 requires the Nevanlinna theory in an angular domain. Let $f(z)$ be a meromorphic function and $\overline{\Omega}(\alpha, \beta) = \{z \mid \alpha \leq \arg z \leq \beta\}$ be an angular domain, where $0 < \beta - \alpha \leq 2\pi$. Nevanlinna defined the following notations (see e.g. [5], [13], [24]).

$$A_{\alpha,\beta}(r, f) = \frac{k}{\pi} \int_1^r \left(\frac{1}{t^k} - \frac{t^k}{r^{2k}}\right) \{\log^+ |f(te^{i\alpha})| + \log^+ |f(te^{i\beta})|\} \frac{dt}{t};$$

$$B_{\alpha,\beta}(r, f) = \frac{2k}{\pi r^k} \int_\alpha^\beta \log^+ |f(re^{i\theta})| \sin k(\theta - \alpha) d\theta;$$

$$C_{\alpha,\beta}(r, f) = 2 \sum_{b_v \in \Delta} \left(\frac{1}{|b_v|^k} - \frac{|b_v|^k}{r^{2k}}\right) \sin k(\beta_v - \alpha),$$

where $k = \frac{\pi}{\beta - \alpha}$, $1 \leq r < \infty$ and $b_v = |b_v|e^{i\theta_v}$ are the poles of $f(z)$ in the sector $\Delta := \{z : 1 < |z| < r, \alpha < \arg z < \beta\}$, each pole b_v occurs in the sum $\sum_{b_v \in \Delta}$ as many times as it's order. When pole b_v occurs in the sum $\sum_{b_v \in \Delta}$ only once, we denote it by $\overline{C}_{\alpha,\beta}(r, f)$. Moreover, for $r > 1$, we define

$$D_{\alpha,\beta}(r, f) = A_{\alpha,\beta}(r, f) + B_{\alpha,\beta}(r, f), \quad S_{\alpha,\beta}(r, f) = C_{\alpha,\beta}(r, f) + D_{\alpha,\beta}(r, f).$$

For the sake of simplicity, we omit the subscript of all the notations and use $A(r, f)$, $B(r, f)$, $C(r, f)$, $D(r, f)$ and $S(r, f)$ to replace $A_{\alpha,\beta}(r, f)$, $B_{\alpha,\beta}(r, f)$, $C_{\alpha,\beta}(r, f)$, $D_{\alpha,\beta}(r, f)$ and $S_{\alpha,\beta}(r, f)$ respectively. In the following, some properties of $S(r, f)$ are given.

Lemma 1 ([24]). *Let $f(z)$ be a nonconstant meromorphic function and $\overline{\Omega}(\alpha, \beta)$ be an angular domain, where $0 < \beta - \alpha \leq 2\pi$. Then for any value $a \in \mathbb{C}$, we have*

$$S\left(r, \frac{1}{f-a}\right) = S(r, f) + \varepsilon(r, a),$$

where $\varepsilon(r, a) = O(1)$ as $r \rightarrow \infty$.

Lemma 2 ([24]). *Let $f(z)$ be a meromorphic function and $\overline{\Omega}(\alpha, \beta)$ be an angular domain, where $0 < \beta - \alpha \leq 2\pi$. Then for arbitrary q distinct $a_j \in \mathbb{C}_\infty$, we have*

$$(3) \quad (q-2)S(r, f) \leq \sum_{j=1}^q \overline{C}\left(r, \frac{1}{f-a_j}\right) + O(\log r T(r, f)), r \notin F,$$

where F is a set of finite linear measure.

Lemma 3 ([19]). *Suppose that $f(z)$ is a nonconstant meromorphic function and $\overline{\Omega}(\alpha, \beta)$ is an angular domain, where $0 < \beta - \alpha \leq 2\pi$. Then for any $r < R$,*

$$A(r, \frac{f'}{f}) \leq K \left\{ \left(\frac{R}{r}\right)^k \int_1^R \frac{\log^+ T(t, f)}{t^{1+k}} dt + \log^+ \frac{r}{R-r} + \log \frac{R}{r} + 1 \right\},$$

$$B(r, \frac{f'}{f}) \leq \frac{4k}{r^k} m(r, \frac{f'}{f}).$$

where $k = \frac{\pi}{\beta - \alpha}$ and K is a positive constant not depending on r and R .

Lemma 4. *Let $p \in \mathbb{N}$ and $p > 1$ and $f(z)$ be a meromorphic function such that $\sigma_p(f) = \rho$ ($0 < \rho < \infty$). A ray $L : \arg z = \theta$ is a Borel direction of iterated p -order ρ of f if and only if for any positive number $0 < \varepsilon < \pi/2$, the equation*

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p]} S(r, f)}{\log r} = \rho,$$

holds in the angular domain $\Omega_\varepsilon := \{z : \theta - \varepsilon \leq \arg z \leq \theta + \varepsilon\}$.

Proof. Assume that L is a Borel direction of iterated p -order ρ of f , and that for some η ($0 < \eta < \frac{\pi}{2}$) in the angular domain Ω_η , we have

$$(4) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[p]} S(r, f)}{\log r} < \rho.$$

By Lemma 1, for any finite value a , in the angular domain Ω_η , we have $S(r, \frac{1}{f-a}) = S(r, f) + O(1)$. Since $C(r, a) \leq S(r, \frac{1}{f-a})$, then

$$(5) \quad C(r, a) \leq S(r, \frac{1}{f-a}) = S(r, f) + O(1).$$

On the other hand,

$$\begin{aligned} C(2r, a) &\geq C_{\theta - \frac{\eta}{2}, \theta + \frac{\eta}{2}}(2r, a) \\ &\geq 2 \sum_{1 < |b_v| < r, \theta - \frac{\eta}{2} < \beta_v < \theta + \frac{\eta}{2}} \left(\frac{1}{|b_v|^k} - \frac{|b_v|^k}{(2r)^{2k}} \right) \sin k(\beta_v - \theta + \frac{\eta}{2}) \\ &\geq 2 \sum_{1 < |b_v| < r, \theta - \frac{\eta}{3} < \beta_v < \theta + \frac{\eta}{3}} \left(\frac{1}{|b_v|^k} - \frac{|b_v|^k}{(2r)^{2k}} \right) \sin k(\beta_v - \theta + \frac{\eta}{2}), \end{aligned}$$

where $k = \frac{\pi}{\eta}$. In the sector $\Delta : 1 < |b| < r, \theta - \frac{\eta}{3} < \beta < \theta + \frac{\eta}{3}$, we have $0 < \frac{\eta}{6} < \beta_v - \theta + \frac{\eta}{2} < \frac{5\eta}{6} < \frac{\pi}{2}$. We write a sum of above expression as a Stieltjes-integral and the partial integration of the above Stieltjes-integrals

now results in

$$\begin{aligned}
 C(2r, a) &\geq \int_1^r \frac{1}{t^k} dn(t) + \frac{1}{(2r)^{2k}} \int_1^r t^k dn(t) \\
 &\geq k \int_1^r \frac{1}{t^{k+1}} n(t) dt + \frac{n(r)}{r^k} - \frac{r^k n(r)}{r^{2k}} \\
 (6) \quad &\quad + \frac{k}{(2r)^{2k}} \int_1^r t^{k-1} n(t) dt \\
 &\geq \frac{n(r)}{r^k} - \frac{r^k n(r)}{(2r)^{2k}} \\
 &\geq \left(1 - \frac{1}{2^{2k}}\right) \frac{n(r)}{r^k}
 \end{aligned}$$

where a short-hand notation $n(t) = n(t, \Omega(\theta - \frac{\eta}{3}, \theta + \frac{\eta}{3}), f = a)$ will be used. Substituting (6) to (5) and combining (4), we get

$$(7) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[p]} n(r, \Omega(\theta - \frac{\eta}{3}, \theta + \frac{\eta}{3}), f = a)}{\log r} < \rho.$$

Since a is arbitrary, the above expression is incompatible with the hypothesis that L is a Borel direction of iterated p -order ρ of f .

Conversely, assume that for any η ($0 < \eta < \frac{\pi}{2}$), in the angular domain Ω_η , we have

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p]} S(r, f)}{\log r} = \rho.$$

Suppose that L is not a Borel direction of iterated p -order ρ of f . Then there exist a η and three distinct values $a_j \in \mathbb{C}_\infty$ ($j = 1, 2, 3$), such that for sufficiently large r , we have

$$(8) \quad n(r, \Omega(\theta - \eta, \theta + \eta), f = a_j) < \exp^{[p-1]}(r^Q),$$

where $Q < \rho$ is a constant. For the three distinct value a_j , we have

$$(9) \quad C(r, a_j) \leq 2n(r, \Omega(\theta - \eta, \theta + \eta), f = a_j).$$

We deduce from (8), (9) and Lemma 2 that $S(r, f) < \exp^{[p-1]}(r^Q)$ for sufficiently large r . Hence, we get a contradiction and Lemma 4 follows. \square

Remark 5. In the proof of Lemma 4, we noted that for some η ($0 < \eta < \frac{\pi}{2}$) in the angular domain Ω_η , we have

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p]} S(r, f)}{\log r} \leq \rho.$$

In fact, if

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p]} S(r, f)}{\log r} > \rho.$$

By Lemma 2, we have for some $a \in \mathbb{C}_\infty$,

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p]} n(r, \Omega_\eta, f = a)}{\log r} > \rho.$$

This contradicts with $\sigma_p(f) = \rho$.

Lemma 5 ([24]). *Let $f(z)$ be an analytic function on $\overline{\Omega}(\alpha, \beta)$. Then we have*

$$(10) \quad \log M(r, \overline{\Omega}(\alpha, \beta), f) \leq Kr^\omega \{S(2r, f) + 1\},$$

where $\omega = \frac{\pi}{\beta - \alpha}$, $M(r, \overline{\Omega}(\alpha, \beta), f) = \sup\{|f(te^{i\tau})| : \alpha \leq \tau \leq \beta, 1 \leq t \leq r\}$ and K is a positive constant.

3. Proof of Theorem 1

Proof. The Wronskian determinant $W(f_1, f_2, \dots, f_k)$ of fundamental system of solutions $\{f_1, f_2, \dots, f_k\}$ is given by

$$W = W(f_1, f_2, \dots, f_k) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{f'_1}{f_1} & \frac{f'_2}{f_2} & \cdots & \frac{f'_k}{f_k} \\ \dots & \dots & \dots & \dots \\ \frac{f_1^{(k-1)}}{f_1} & \frac{f_2^{(k-1)}}{f_2} & \cdots & \frac{f_k^{(k-1)}}{f_k} \end{vmatrix}.$$

Applying a Lemma [12, p. 16], we can derive that W is a positive constant and denote it by K . Hence

$$\frac{1}{E} = \frac{1}{K} \frac{W}{E} = \frac{1}{K} \sum_{1 \leq i_1 \neq i_2 \leq k} (-1)^{\tau} \prod_{l=1}^{k-1} \frac{f_{i_l}^{(l)}}{f_{i_l}}.$$

Let $f \neq 0$ be a solution of (2). It follows from Theorem C that the iterated p -order of $\log T(r, f)$ is at most σ , where $\sigma < \infty$ is a constant.

For any $\theta \in \mathbb{R}$, using Lemma 3 in which $R = 2r$ for sufficiently small ε , we have

$$(11) \quad \begin{aligned} A_{\theta - \varepsilon, \theta + \varepsilon}(r, \frac{f'_i}{f_i}) &= O\left(\int_1^{2r} \frac{\log^+ T(t, f_i)}{t^{1 + \frac{\pi}{2\varepsilon}}} dt\right) = O\left(\int_1^{2r} \frac{\exp^{[p-1]} t^{\sigma+1}}{t^{1 + \frac{\pi}{2\varepsilon}}} dt\right) \\ &= O(\exp^{[p-1]} r^{\sigma+1}) \end{aligned}$$

for any f_i , when $p \geq 2$. And when $p = 1$, we have $A_{\theta - \varepsilon, \theta + \varepsilon}(r, \frac{f'_i}{f_i}) = O(1)$.

Since

$$m(r, \frac{f'_i}{f_i}) = O(\log r T(r, f_i)) = O(\exp^{[p-1]} r^{\sigma+1}).$$

Then we can deduce the following equation from Lemma 3

$$(12) \quad B_{\theta - \varepsilon, \theta + \varepsilon}(r, \frac{f'_i}{f_i}) \leq \frac{4k}{r^k} m(r, \frac{f'_i}{f_i}) = O(r^{\sigma+1 - \frac{\pi}{2\varepsilon}}) = O(\exp^{[p-1]} r^{\sigma+1})$$

when $p \geq 2$, and when $p = 1$, we have $B_{\theta - \varepsilon, \theta + \varepsilon}(r, \frac{f'_i}{f_i}) = O(1)$.

Since

$$D_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{f_i^{(h)}}{f_i}) \leq \sum_{i=1}^h D_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{f_i^{(l)}}{f_i^{(l-1)}}) + O(1),$$

where $i = 1, 2, \dots, k; h = 2, 3, \dots, k - 1$, then we have

$$D_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{E}) = O(1);$$

when $p = 1$, and when $p \geq 2$,

$$D_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{E}) = O(\exp^{[p-1]} r^{\sigma+1}).$$

By the definition and Lemma 1, for any $\theta \in \mathbb{R}$ and any sufficiently small $\varepsilon > 0$ in angular domain $\Omega_\varepsilon = \{z \mid \theta - \varepsilon \leq \arg z \leq \theta + \varepsilon\}$, we can deduce

$$(13) \quad S(r, E) \leq P\{C(r, \frac{1}{E}) + O(\exp^{[p-1]} r^{\sigma+1})\},$$

where P is a constant.

Now, we are in the position to prove the statements those in Theorem 1 are equivalent.

3.1. Proof of the equivalence of (i) and (ii)

Suppose that $L : \arg z = \theta$ is a Borel direction of iterated $(p + 1)$ -order ρ of E . By Lemma 4, for any positive number $0 < \varepsilon < \pi/2$,

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p+1]} S(r, E)}{\log r} = \rho,$$

holds in the angular domain $\Omega_\varepsilon := \{z : \theta - \varepsilon \leq \arg z \leq \theta + \varepsilon\}$. Combining (13), we can obtain

$$(14) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[p+1]} C(r, \frac{1}{E})}{\log r} = \rho.$$

Noting $C(r, \frac{1}{E}) \leq 2n(r, \Omega_\varepsilon, E = 0)$, hence $\lambda_{(p+1), \theta}(E) = \rho$.

On the other hand, if $\lambda_{(p+1), \theta}(E) = \rho$, then for any $0 < \eta < \frac{\pi}{2}$, in the angular domain Ω_η , we can obtain the following equation as we did in the proof of (6)

$$(15) \quad S(2r, E) \geq (1 - \frac{1}{2^{2k}}) \frac{n(r)}{r^k} + O(1),$$

where $n(r) = n(t, \Omega(\theta - \frac{\eta}{3}, \theta + \frac{\eta}{3}), E = 0)$. Thus for any $0 < \eta < \frac{\pi}{2}$,

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p+1]} S(r, E)}{\log r} = \rho,$$

holds in the angular domain Ω_η . By Lemma 4, we know that $L : \arg z = \theta$ is a Borel direction of iterated $(p + 1)$ -order ρ of E .

3.2. Proof of the equivalence of (i) and (iii)

Suppose that $L : \arg z = \theta$ is a Borel direction of iterated $(p + 1)$ -order ρ of E . By Lemma 4, for any positive number $0 < \mu < \pi/2$,

$$(16) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[p+1]} S(r, E)}{\log r} = \rho$$

holds in the angular domain Ω_μ . If there exists $0 < \varepsilon < \pi/2$ such that

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p+2]} M(r, \Omega_{\theta, \varepsilon}, E)}{\log r} < T < \rho.$$

Then we have

$$\log |Ere^{i(\theta-\varepsilon)}| < \exp^{[p]} r^T, \quad \log |Ere^{i(\theta+\varepsilon)}| < \exp^{[p]} r^T$$

and $\log |Ere^{i\theta}| < \exp^{[p]} r^T$ for any large r and all $\theta \in [\theta - \varepsilon, \theta + \varepsilon]$. Noting that E is an entire function, we deduce from the definition of the Nevanlinna angular characteristic that

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p+1]} S(r, E)}{\log r} < \rho.$$

This contradicts with (16). Hence, for any $0 < \varepsilon < \pi/2$, we have

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p+2]} M(r, \Omega_{\theta, \varepsilon}, E)}{\log r} = \rho.$$

On the other hand, if

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p+2]} M(r, \Omega_{\theta, \varepsilon}, E)}{\log r} = \rho$$

for any $0 < \varepsilon < \pi/2$, by using Lemma 5 and Remark 5, we have

$$(17) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[p+1]} S(r, E)}{\log r} = \rho$$

holds in the angular domain Ω_ε . It follows Lemma 4 that $L : \arg z = \theta$ is a Borel direction of iterated $(p + 1)$ -order ρ of E . \square

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SCHOOL OF MATHEMATICS AND STATISTICS
 HUBEI UNIVERSITY OF SCIENCE AND TECHNOLOGY
 XIANNING, HUBEI PROVINCE, 437100, P. R. CHINA
 E-mail address: wuzj52@hotmail.com