Bull. Korean Math. Soc. **49** (2012), No. 5, pp. 997–1006
http://dx.doi.org/10.4134/BKMS.2012.49.5.997

CONVERGENCE OF MULTISPLITTING METHODS WITH PREWEIGHTING FOR AN H-MATRIX

Yu Du Han and Jae Heon Yun

Abstract. In this paper, we study convergence of multisplitting methods with preweighting for solving a linear system whose coefficient matrix is an H-matrix corresponding to both the AOR multisplitting and the SSOR multisplitting. Numerical results are also provided to confirm theoretical results for the convergence of multisplitting methods with preweighting.

1. Introduction

In this paper, we consider multisplitting methods with preweighting for solving a linear system of the form

$$Ax = b, \quad x, b \in \mathbb{R}^n, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a large sparse nonsingular matrix. Multisplitting method was first introduced by O'Leary and White [6] for parallel computation of the linear system (1).

For a vector $x \in \mathbb{R}^n$, $x \geq 0$ ($x > 0$) denotes that all components of x are nonnegative (positive), and $|x|$ denotes the vector whose components are the absolute values of the corresponding components of x. For two vectors x, $y \in \mathbb{R}^n$, $x \geq y$ ($x > y$) means that $x - y \geq 0$ ($x - y > 0$). These definitions carry immediately over to matrices. For a square matrix A, diag(A) denotes a diagonal matrix whose diagonal part coincides with the diagonal part of A. Let $\rho(A)$ denote the *spectral radius* of a square matrix A. Varga [8] showed that for any two square matrices A and B, $|A| \leq B$ implies $\rho(A) \leq \rho(B)$.

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called *monotone* if A is nonsingular with $A^{-1} \geq 0$. A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called an *M-matrix* if it is a monotone matrix with $a_{ij} \leq 0$ for $i \neq j$. The *comparison matrix* $\langle A\rangle = (\alpha_{ij})$ of a matrix $A = (a_{ij})$ is defined by

$$\alpha_{ij} = \begin{cases} |a_{ij}| & \text{if } i = j, \\ -|a_{ij}| & \text{if } i \neq j. \end{cases}$$

Received May 19, 2011; Revised November 15, 2011.
2010 *Mathematics Subject Classification.* 65F10, 65F15.
Key words and phrases. multisplitting method, preweighting, AOR multisplitting, SSOR multisplitting, H-matrix.

A matrix A is called an *H-matrix* if $\langle A\rangle$ is an M-matrix.

A representation $A = M - N$ is called a *splitting* of A if M is nonsingular. A splitting $A = M - N$ is called *regular* if $M^{-1} \geq 0$ and $N \geq 0$, and called *weak regular* if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$ [1]. It is well known that if $A = M - N$ is a weak regular splitting of A, then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geq 0$ [1, 8]. A collection of triples (M_k, N_k, E_k), $k = 1, 2, \ldots, \ell$, is called a *multisplitting* of A if $A = M_k - N_k$ is a splitting of A for $k = 1, 2, \ldots, \ell$, and E_k's, called weighting matrices, are nonnegative diagonal matrices such that $\sum_{k=1}^{\ell} E_k = I$.

The *multisplitting method with postweighting* which is usually called the multisplitting method has been extensively studied in the literature, see [2, 3, 4, 5, 6, 7, 9, 11, 12]. In 1989, White [10] proposed multisplitting method with different weighting schemes, and he showed that multisplitting method with preweighting yields the fastest method in certain situations. However, the *multisplitting method with preweighting* has not been studied extensively, see [4, 10]. This is the main motivation for studying convergence of multisplitting method with preweighting. The purpose of this paper is to study convergence of multisplitting methods with preweighting for solving a linear system whose coefficient matrix is an H-matrix corresponding to both the AOR multisplitting and the SSOR multisplitting. We also provide numerical results to confirm theoretical results for the convergence of multisplitting methods with preweighting.

2. Convergence of multisplitting methods with preweighting

Let (M_k, N_k, E_k), $k = 1, 2, \ldots, \ell$, be a multisplitting of A. Then the corresponding multisplitting method with preweighting for solving $Ax = b$ [10] is given by

$$\begin{aligned} x_{i+1} &= H_0 x_i + G_0 b \\ &= x_i + G_0(b - Ax_i),\ i = 0, 1, 2, \ldots, \end{aligned} \tag{2}$$

where

$$G_0 = \sum_{k=1}^{\ell} {M_k}^{-1} E_k \ \text{ and } \ H_0 = I - G_0 A. \tag{3}$$

$H_0 = I - \sum_{k=1}^{\ell} {M_k}^{-1} E_k A$ is called an iteration matrix for the multisplitting method with preweighting. Notice that $H = I - \sum_{k=1}^{\ell} E_k {M_k}^{-1} A$ is called an iteration matrix for the multisplitting method. By simple calculation, one obtains

$${H_0}^T = A^T \left(I - \sum_{k=1}^{\ell} E_k ({M_k}^T)^{-1} A^T \right) (A^T)^{-1}.$$

Let $\hat{H} = I - \sum_{k=1}^{\ell} E_k ({M_k}^T)^{-1} A^T = \sum_{k=1}^{\ell} E_k ({M_k}^T)^{-1} {N_k}^T$. Then $\hat{H}$ is similar to ${H_0}^T$ and hence $\rho(H_0) = \rho(\hat{H})$. Notice that $\hat{H}$ is an iteration matrix for the multisplitting method corresponding to a multisplitting $({M_k}^T, {N_k}^T, E_k)$,

$k = 1, 2, \ldots, \ell$, of A^T. Hence, convergence result of *multisplitting method with preweighting* corresponding to a multisplitting of A can be obtained from that of *multisplitting method* corresponding to a multisplitting of A^T.

The multisplitting method with preweighting associated with a multisplitting (M_k, N_k, E_k), $k = 1, 2, \ldots, \ell$, of A for solving the linear system (1) is as follows:

ALGORITHM 1: MULTISPLITTING METHOD WITH PREWEIGHTING
Given an initial vector x_0
For $i = 0, 1, \ldots,$ until convergence
  For $k = 1$ to ℓ {parallel execution}
    $M_k y_k = E_k(b - Ax_i)$
  $x_{i+1} = x_i + \sum_{k=1}^{\ell} y_k$

From now on, it is assumed that $A = D - L_k - V_k$, where $D = \mathrm{diag}(A)$ is a nonsingular matrix, L_k is a strictly lower triangular matrix and V_k is a general matrix for $k = 1, 2, \ldots, \ell$. The AOR-multisplitting method with preweighting is defined by

$$x_{i+1} = H_0(\omega, \gamma)x_i + G_0(\omega, \gamma)b, \ i = 0, 1, 2, \ldots, \tag{4}$$

where

$$\begin{aligned} G_0(\omega, \gamma) &= \omega \sum_{k=1}^{\ell} (D - \gamma L_k)^{-1} E_k, \\ H_0(\omega, \gamma) &= I - \omega \sum_{k=1}^{\ell} (D - \gamma L_k)^{-1} E_k A. \end{aligned} \tag{5}$$

Notice that $\omega A = (D - \gamma L_k) - ((1 - \omega)D + (\omega - \gamma)L_k + \omega V_k)$ for $k = 1, 2, \ldots, \ell$ and

$$H_0(\omega, \gamma)^T = A^T \left(I - \omega \sum_{k=1}^{\ell} E_k (D - \gamma {L_k}^T)^{-1} A^T \right) A^{-T}.$$

Let $\tilde{H}(\omega, \gamma) = I - \omega \sum_{k=1}^{\ell} E_k (D - \gamma {L_k}^T)^{-1} A^T$. Then $\tilde{H}(\omega, \gamma)$ is similar to $H_0(\omega, \gamma)^T$ and $\tilde{H}(\omega, \gamma)$ can be written as

$$\tilde{H}(\omega, \gamma) = \sum_{k=1}^{\ell} E_k (D - \gamma {L_k}^T)^{-1} \left((1 - \omega)D + (\omega - \gamma){L_k}^T + \omega {V_k}^T \right).$$

Notice that $\tilde{H}(\omega, \gamma)$ is an iteration matrix of the multisplitting method corresponding to a multisplitting

$$\left(\frac{1}{\omega}(D - \gamma {L_k}^T), \frac{1}{\omega}((1 - \omega)D + (\omega - \gamma){L_k}^T + \omega {V_k}^T), E_k \right), \ k = 1, 2, \ldots, \ell$$

of A^T. The following lemma provides a convergence result of the multisplitting method corresponding to a multisplitting

$$\left(\frac{1}{\omega}(D-\gamma {L_k}^T), \frac{1}{\omega}((1-\omega)D+(\omega-\gamma){L_k}^T+\omega {V_k}^T), E_k\right),\ k=1,2,\ldots,\ell$$

of A^T when A is an H-matrix.

Lemma 2.1. *Let $A = D - B$ be an $n \times n$ H-matrix with $D = \operatorname{diag}(A)$. Let $A = D - L_k - V_k$, where L_k is a strictly lower triangular matrix and V_k is a general matrix for $k = 1, 2, \ldots, \ell$. Assume that $\langle A\rangle = |D| - |L_k| - |V_k|$ for $k = 1, 2, \ldots, \ell$. If $0 < \gamma \le \omega < \frac{2}{1+\alpha}$, then $\rho\left(\tilde{H}(\omega,\gamma)\right) < 1$, where $\alpha = \rho(|D|^{-1}|B|)$ and*

$$\tilde{H}(\omega,\gamma) = \sum_{k=1}^{\ell} E_k(D-\gamma {L_k}^T)^{-1}((1-\omega)D+(\omega-\gamma){L_k}^T+\omega {V_k}^T).$$

Proof. From the assumption, $\langle A^T\rangle = |D| - |{L_k}^T| - |{V_k}^T|$ for $k = 1, 2, \ldots, \ell$. Notice that for $k = 1, 2, \ldots, \ell$,

$$\omega\langle A^T\rangle = (|D|-\gamma|{L_k}^T|) - \left((1-\omega)|D|+(\omega-\gamma)|{L_k}^T|+\omega|{V_k}^T|\right). \tag{6}$$

Since $D-\gamma L_k$ is an H-matrix, $(D-\gamma L_k)^T$ is also an H-matrix. Consider the following inequality

$$\begin{aligned}|\tilde{H}(\omega,\gamma)| &\le \sum_{k=1}^{\ell} E_k\langle D-\gamma {L_k}^T\rangle^{-1}\left(|1-\omega||D|+(\omega-\gamma)|{L_k}^T|+\omega|{V_k}^T|\right)\\ &= \sum_{k=1}^{\ell} E_k(|D|-\gamma|{L_k}^T|)^{-1}\left(|1-\omega||D|+(\omega-\gamma)|{L_k}^T|+\omega|{V_k}^T|\right).\end{aligned}$$

Let we define

$$H^\star(\omega,\gamma) = \sum_{k=1}^{\ell} E_k(|D|-\gamma|{L_k}^T|)^{-1}\left(|1-\omega||D|+(\omega-\gamma)|{L_k}^T|+\omega|{V_k}^T|\right).$$

Then we have

$$|\tilde{H}(\omega,\gamma)| \le H^\star(\omega,\gamma). \tag{7}$$

We first consider the case where $0<\gamma\le\omega\le 1$. Since $(|D|-\gamma|{L_k}^T|)^{-1}\ge 0$ and $(1-\omega)|D|+(\omega-\gamma)|{L_k}^T|+\omega|{V_k}^T|\ge 0$, (6) is a regular splitting of $\omega\langle A^T\rangle$ for each $k=1,2,\ldots,\ell$. Since $\langle A^T\rangle^{-1}\ge 0$, $\rho(H^\star(\omega,\gamma))<1$. From (7), $\rho(\tilde{H}(\omega,\gamma))<1$ for $0<\gamma\le\omega\le 1$. Next we consider the case where $1<\omega<\frac{2}{1+\alpha}$ and $\gamma\le\omega$. Let

$$\begin{aligned}\tilde{N}_k(\omega,\gamma) &= (\omega-1)|D|+(\omega-\gamma)|{L_k}^T|+\omega|{V_k}^T|\ (1\le k\le \ell),\\ \tilde{A} &= (2-\omega)|D|-\omega|B^T|.\end{aligned}$$

It is easy to show that $\tilde{A} = (|D| - \gamma|{L_k}^T|) - \tilde{N}_k(\omega, \gamma)$ for all $k = 1, 2, \ldots, \ell$. By simple calculation, one obtains

$$|\tilde{H}(\omega, \gamma)| \leq \sum_{k=1}^{\ell} E_k(|D| - \gamma|{L_k}^T|)^{-1}\tilde{N}_k(\omega, \gamma). \tag{8}$$

Since $\omega < \frac{2}{1+\alpha}$, $\frac{\omega\alpha}{2-\omega} < 1$ and thus

$$\frac{\omega}{2-\omega}\rho(|D|^{-1}|B^T|) = \frac{\omega}{2-\omega}\rho(|D|^{-1}|B|) = \frac{\omega\alpha}{2-\omega} < 1.$$

Since $\tilde{A} = (2-\omega)|D| - \omega|B^T|$ is a regular splitting of $\tilde{A}$, $\tilde{A}^{-1} \geq 0$. Since $\tilde{A} = (|D| - \gamma|{L_k}^T|) - \tilde{N}_k(\omega, \gamma)$ is a regular splitting of $\tilde{A}$ for each $k = 1, 2, \ldots, \ell$, $\rho\left(\sum_{k=1}^{\ell} E_k(|D| - \gamma|{L_k}^T|)^{-1}\tilde{N}_k(\omega, \gamma)\right) < 1$. From (8), $\rho(\tilde{H}(\omega, \gamma)) < 1$ for $1 < \omega < \frac{2}{1+\alpha}$ and $\gamma \leq \omega$. Therefore, $\rho(\tilde{H}(\omega, \gamma)) < 1$ for $0 < \gamma \leq \omega < \frac{2}{1+\alpha}$. $\square$

The following theorem provides a convergence result of the AOR-multisplitting method with preweighting when A is an H-matrix.

Theorem 2.2. *Let $A = D - B$ be an $n \times n$ H-matrix with $D = \operatorname{diag}(A)$. Let $A = D - L_k - V_k$, where L_k is a strictly lower triangular matrix and V_k is a general matrix for $k = 1, 2, \ldots, \ell$. Assume that $\langle A \rangle = |D| - |L_k| - |V_k|$ for $k = 1, 2, \ldots, \ell$. If $0 < \gamma \leq \omega < \frac{2}{1+\alpha}$, then*

$$\rho\left(H_0(\omega, \gamma)\right) < 1,$$

where $H_0(\omega, \gamma) = I - \omega\sum_{k=1}^{\ell}(D - \gamma L_k)^{-1}E_k A$ and $\alpha = \rho(|D|^{-1}|B|)$.

Proof. Let $\tilde{H}(\omega, \gamma) = I - \omega\sum_{k=1}^{\ell} E_k(D - \gamma{L_k}^T)^{-1}A^T$. Since $\tilde{H}(\omega, \gamma)$ is similar to $H_0(\omega, \gamma)^T$, $\rho(\tilde{H}(\omega, \gamma)) = \rho(H_0(\omega, \gamma))$. From Lemma 2.1, $\rho(\tilde{H}(\omega, \gamma)) < 1$ for $0 < \gamma \leq \omega < \frac{2}{1+\alpha}$. Therefore, $\rho(H_0(\omega, \gamma)) < 1$ for $0 < \gamma \leq \omega < \frac{2}{1+\alpha}$. $\square$

Notice that if A is an M-matrix, then A is an H-matrix. We easily obtain the following corollary which is a convergence results of the AOR-multisplitting method with preweighting when A is an M-matrix.

Corollary 2.3. *Let $A = D - B$ be an $n \times n$ M-matrix with $D = \operatorname{diag}(A)$. Let $A = D - L_k - V_k$, where $L_k \geq 0$ is a strictly lower triangular matrix and $V_k \geq 0$ is a general matrix for $k = 1, 2, \ldots, \ell$. If $0 < \gamma \leq \omega < \frac{2}{1+\alpha}$, then*

$$\rho\left(H_0(\omega, \gamma)\right) < 1,$$

where $H_0(\omega, \gamma) = I - \omega\sum_{k=1}^{\ell}(D - \gamma L_k)^{-1}E_k A$ and $\alpha = \rho(D^{-1}B)$.

The SSOR-multisplitting method with preweighting is defined by

$$x_{i+1} = H_0(\omega)x_i + G_0(\omega)b, \;\; i = 0, 1, 2, \ldots, \tag{9}$$

where

$$
\begin{aligned}
G_0(\omega) &= \omega(2-\omega)\sum_{k=1}^{\ell}\left((D-\omega L_k)D^{-1}(D-\omega V_k)\right)^{-1}E_k,\\
H_0(\omega) &= I-\omega(2-\omega)\sum_{k=1}^{\ell}\left((D-\omega L_k)D^{-1}(D-\omega V_k)\right)^{-1}E_kA.
\end{aligned} \tag{10}
$$

Notice that for $k = 1, 2, \ldots, \ell$,

$$
\begin{aligned}
\omega(2-\omega)A =&(D-\omega L_k)D^{-1}(D-\omega V_k)\\
&-\left((1-\omega)D+\omega L_k\right)D^{-1}\left((1-\omega)D+\omega V_k\right)
\end{aligned}
$$

and

$$
H_0(\omega)^T = A^T\left(I-\omega(2-\omega)\sum_{k=1}^{\ell}E_k\left((D-\omega L_k)D^{-1}(D-\omega V_k)\right)^{-T}A^T\right)A^{-T}.
$$

Let $\tilde{H}(\omega) = I-\omega(2-\omega)\sum_{k=1}^{\ell}E_k\left((D-\omega L_k)D^{-1}(D-\omega V_k)\right)^{-T}A^T$. Then $\tilde{H}(\omega)$ is similar to $H_0(\omega)^T$ and $\tilde{H}(\omega)$ can be written as

$$
\widetilde{H}(\omega) = \sum_{k=1}^{\ell}E_kM_k(\omega)^{-T}N_k(\omega)^T,
$$

where $M_k(\omega) = (D-\omega L_k)D^{-1}(D-\omega V_k)$ and $N_k(\omega) = ((1-\omega)D+\omega L_k)D^{-1}((1-\omega)D+\omega V_k)$. Note that $\tilde{H}(\omega)$ is an iteration matrix of multisplitting method corresponding to a multisplitting

$$
\left(\frac{1}{\omega(2-\omega)}M_k(\omega)^T, \frac{1}{\omega(2-\omega)}N_k(\omega)^T, E_k\right),\ k = 1, 2, \ldots, \ell
$$

of A^T. The following theorem provides a convergence result of the SSOR-multisplitting method with preweighting when A is an H-matrix.

Theorem 2.4. *Let $A = D - B$ be an $n \times n$ H-matrix with $D = \operatorname{diag}(A)$. Let $A = D - L_k - V_k$, where L_k is a strictly lower triangular matrix and V_k is a general matrix for $k = 1, 2, \ldots, \ell$. Assume that $\langle A\rangle = |D| - |L_k| - |V_k|$ for $k = 1, 2, \ldots, \ell$. If $0 < \omega < \frac{2}{1+\alpha}$, then*

$$
\rho\left(H_0(\omega)\right) < 1,
$$

where $H_0(\omega) = I-\omega(2-\omega)\sum_{k=1}^{\ell}\left((D-\omega L_k)D^{-1}(D-\omega V_k)\right)^{-1}E_kA$ and $\alpha = \rho(|D|^{-1}|B|)$.

Proof. Let $\tilde{H}(\omega) = I-\omega(2-\omega)\sum_{k=1}^{\ell}E_k\left((D-\omega L_k)D^{-1}(D-\omega V_k)\right)^{-T}A^T$. Then, $\tilde{H}(\omega)$ is similar to $H_0(\omega)^T$ and thus $\rho(\tilde{H}(\omega)) = \rho(H_0(\omega))$. Hence, it is sufficient to show that $\rho(\tilde{H}(\omega)) < 1$. Notice that $(D - \omega L_k)^T$ and $(D - \omega V_k)^T$ are H-matrices since $0 < \omega < \frac{2}{1+\alpha}$ and $\alpha < 1$. For $k = 1, 2, \ldots, \ell$, let

$$\begin{aligned}
\tilde{M}_k(\omega) &= (|D| - \omega|U_k^T|)\ |D^{-1}|\ (|D| - \omega|L_k^T|),\\
\tilde{N}_k(\omega) &= (|1-\omega||D| + \omega|U_k^T|)\ |D^{-1}|\ (|1-\omega||D| + \omega|L_k^T|),\\
H^*(\omega) &= \sum_{k=1}^{\ell} E_k \tilde{M}_k(\omega)^{-1}\tilde{N}_k(\omega),\\
\tilde{A}(\omega) &= \tilde{M}_k(\omega) - \tilde{N}_k(\omega).
\end{aligned}$$

Then it can be easily shown that

$$|\tilde{H}(\omega)| \le H^*(\omega). \tag{11}$$

We first consider the case where $0 < \omega \le 1$. By simple calculation, $\tilde{A}(\omega) = \omega(2-\omega)\langle A\rangle^T$. Since $\tilde{A}(\omega) = \tilde{M}_k(\omega) - \tilde{N}_k(\omega)$ is a regular splitting of $\tilde{A}(\omega)$ and $\tilde{A}(\omega)^{-1} \ge 0$, $\rho(H^*(\omega)) < 1$. From (11), $\rho(\tilde{H}(\omega)) < 1$ for $0 < \omega \le 1$. We next consider the case where $1 < \omega < \frac{2}{1+\alpha}$. By simple calculation, $\tilde{A}(\omega) = \omega(2-\omega)|D| - \omega^2|B^T|$. Since $\omega < \frac{2}{1+\alpha}$, $\frac{\omega\alpha}{2-\omega} < 1$ and thus

$$\frac{\omega^2}{\omega(2-\omega)}\rho(|D|^{-1}|B^T|) = \frac{\omega}{2-\omega}\rho(|D|^{-1}|B|) < 1.$$

Hence $\tilde{A}(\omega)^{-1} \ge 0$. Since $\tilde{A}(\omega) = \tilde{M}_k(\omega) - \tilde{N}_k(\omega)$ is a regular splitting of $\tilde{A}(\omega)$, $\rho(H^*(\omega)) < 1$. From (11), $\rho(\tilde{H}(\omega)) < 1$ for $1 < \omega < \frac{2}{1+\alpha}$. Therefore, $\rho(\tilde{H}(\omega)) < 1$ for $0 < \omega < \frac{2}{1+\alpha}$. □

Corollary 2.5. *Let $A = D - B$ be an $n \times n$ M-matrix with $D = \operatorname{diag}(A)$. Let $A = D - L_k - V_k$, where $L_k \ge 0$ is a strictly lower triangular matrix and $V_k \ge 0$ is a general matrix for $k = 1, 2, \ldots, \ell$. If $0 < \omega < \frac{2}{1+\alpha}$, then*

$$\rho\left(H_0(\omega)\right) < 1,$$

where $H_0(\omega) = I - \omega(2-\omega)\sum_{k=1}^{\ell}\left((D-\omega L_k)D^{-1}(D-\omega V_k)\right)^{-1} E_k A$ and $\alpha = \rho(D^{-1}B)$.

We now provide numerical results to illustrate the convergence of the multisplitting methods with preweighting. All numerical values are computed using MATLAB.

Example 2.6. Suppose that $\ell = 3$. Consider an H-matrix A of the form

$$A = \begin{pmatrix} F & I & 0\\ -I & F & I\\ 0 & -I & F\end{pmatrix},\ \text{where } F = \begin{pmatrix} 4 & -1 & 0\\ 1 & 4 & -1\\ 0 & 1 & 4\end{pmatrix} \text{ and } I = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

TABLE 1. Numerical values of $\rho(H_0(\omega,\gamma))$ for Example 2.6

ω	γ	$\rho(H_0(\omega,\gamma))$	ω	γ	$\rho(H_0(\omega,\gamma))$
1.17	1.17	0.8470	0.9	0.9	**0.2647**
	1.0	0.6542		0.8	0.2880
	0.8	**0.5716**		0.7	0.3384
	0.7	0.5889	0.8	0.8	**0.2540**
	0.6	0.6215		0.7	0.2946
1.1	1.1	0.6214		0.6	0.3517
	1.0	0.5574	0.7	0.7	**0.3306**
	0.9	0.5043		0.6	0.3539
	0.8	**0.4904**		0.5	0.4002
	0.7	0.5143	0.6	0.6	**0.4159**
	0.6	0.5516		0.5	0.4344
1.0	1.0	0.4204		0.4	0.4683
	0.9	**0.3800**	0.5	0.5	**0.5076**
	0.8	0.3812		0.4	0.5232
	0.7	0.4168		0.3	0.5472

TABLE 2. Numerical values of $\rho(H_0(\omega))$ for Example 2.6

ω	$\rho(H_0(\omega))$	ω	$\rho(H_0(\omega))$
1.17	0.1603	0.7	0.1353
1.1	0.1279	0.6	0.1939
1.0	0.1014	0.5	0.2728
0.9	0.1172	0.4	0.3731
0.8	**0.1000**	0.3	0.4961

Let $F = D_* - L_* - U_*$, where $D_* = \text{diag}(F)$,

$$L_* = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \text{ and } U_* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $D = \text{diag}(A)$, $B = D - A$,

$$L_1 = \begin{pmatrix} L_* & 0 & 0 \\ I & L_* & 0 \\ 0 & 0 & L_* \end{pmatrix}, \quad L_2 = \begin{pmatrix} L_* & 0 & 0 \\ 0 & L_* & 0 \\ 0 & I & L_* \end{pmatrix}, \quad L_3 = \begin{pmatrix} L_* & 0 & 0 \\ 0 & L_* & 0 \\ 0 & 0 & L_* \end{pmatrix},$$

$$V_1 = \begin{pmatrix} U_* & -I & 0 \\ 0 & U_* & -I \\ 0 & I & U_* \end{pmatrix}, \quad V_2 = \begin{pmatrix} U_* & -I & 0 \\ I & U_* & -I \\ 0 & 0 & U_* \end{pmatrix}, \quad V_3 = \begin{pmatrix} U_* & -I & 0 \\ I & U_* & -I \\ 0 & I & U_* \end{pmatrix},$$

$$E_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Then, $A = D - L_k - U_k$ and $\langle A \rangle = |D| - |L_k| - |U_k|$ for $k = 1, 2, 3$. The iteration matrix for the AOR-multisplitting method with preweighting is $H_0(\omega, \gamma) = I - \omega \sum_{k=1}^{\ell} (D - \gamma L_k)^{-1} E_k A$, and the iteration matrix for the SSOR-multisplitting method with preweighting is given by

$$H_0(\omega) = I - \omega(2 - \omega) \sum_{k=1}^{\ell} \left((D - \omega L_k) D^{-1} (D - \omega V_k) \right)^{-1} E_k A.$$

Note that $\alpha = \rho(|D|^{-1}|B|) \approx 0.7071$ and $\frac{2}{1+\alpha} \approx 1.1716$. For various values of ω and γ, the numerical values of $\rho(H_0(\omega, \gamma))$ are listed in Table 1. For various values of ω, the numerical values of $\rho(H_0(\omega))$ are listed in Table 2.

From Tables 1 and 2, it can be seen that numerical results are in good agreement with the theoretical results obtained in this section. For Example 2.6, the AOR-multisplitting method with preweighting performs best for about $\omega = \gamma = 0.8$, and the SSOR-multisplitting method with preweighting performs best for about $\omega = 0.8$. Notice that the optimal values of ω and γ vary depending upon the problem. Future work will include theoretical study for finding optimal values of ω and γ for which these multisplitting methods perform best.

Acknowledgements. The authors would like to thank anonymous referees for their useful suggestions which improved the quality of this paper.

References

[1] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, 1979.

[2] Z. H. Cao and Z. Y. Liu, *Convergence of relaxed parallel multisplitting methods with different weighting schemes*, Appl. Math. Comp. **106** (1999), no. 2-3, 181–196.

[3] A. Frommer and G. Mayer, *Convergence of relaxed parallel multisplitting methods*, Linear Algebra Appl. **119** (1989), 141–152.

[4] ———, *On the theory and practice of multisplitting methods in parallel computation*, Computing **49** (1992), no. 1, 63–74.

[5] M. Neumann and R. J. Plemmons, *Convergence of parallel multisplitting iterative methods for M-matrices*, Linear Algebra Appl. **88/89** (1987), 559–573.

[6] D. P. O'Leary and R. E. White, *Multisplittings of matrices and parallel solution of linear systems*, SIAM J. Algebraic Discrete Methods **6** (1985), no. 4, 630–640.

[7] D. B. Szyld and M. T. Jones, *Two-stage and multisplitting methods for the parallel solution of linear systems*, SIAM J. Matrix Anal. Appl. **13** (1992), no. 2, 671–679.

[8] R. S. Varga, *Matrix Iterative Analysis*, Springer, Berlin, 2000.

[9] C. L. Wang and J. H. Zhao, *Further results on regular splittings and multisplittings*, Int. J. Comput. Math. **82** (2005), no. 4, 421–431.

[10] R. E. White, *Multisplitting with different weighting schemes*, SIAM J. Matrix Anal. Appl. **10** (1989), no. 4, 481–493.

[11] ———, *Multisplitting of a symmetric positive definite matrix*, SIAM J. Matrix Anal. Appl. **11** (1990), no. 1, 69–82.

[12] J. H. Yun, *Performance of ILU factorization preconditioners based on multisplittings*, Numer. Math. **95** (2003), 761–779.

Yu Du Han
Department of Mathematics
College of Natural Sciences
Chungbuk National University
Cheongju 361-763, Korea
E-mail address: hanyd@hanmail.net

Jae Heon Yun
Department of Mathematics
College of Natural Sciences
Chungbuk National University
Cheongju 361-763, Korea
E-mail address: gmjae@chungbuk.ac.kr