Bull. Korean Math. Soc. **49** (2012), No. 5, pp. 1015–1025
http://dx.doi.org/10.4134/BKMS.2012.49.5.1015

REGULARITY AND GREEN'S RELATIONS ON SEMIGROUPS OF TRANSFORMATION PRESERVING ORDER AND COMPRESSION

Ping Zhao and Mei Yang

Abstract. Let $[n] = \{1, 2, \ldots, n\}$, and let PO_n be the partial order-preserving transformation semigroup on $[n]$. Let

$$CPO_n = \{\alpha \in PO_n : (\forall x, y \in \mathrm{dom}\alpha), |x\alpha - y\alpha| \leq |x - y|\}.$$

Then CPO_n is a subsemigroup of PO_n. In this paper, we characterize Green's relations and the regularity of elements for CPO_n.

1. Introduction

Let S be a semigroup, $a, b \in S$. If a and b generate the same left principal ideal, that is, $S^1a = S^1b$, then we say that a and b are $\mathcal{L}$ equivalent and write $a\mathcal{L}b$ or $(a, b) \in \mathcal{L}$. If a and b generate the same right principal ideal, that is, $aS^1 = bS^1$, then we say that a and b are $\mathcal{R}$ equivalent and write $a\mathcal{R}b$ or $(a, b) \in \mathcal{R}$. If a and b generate the same principal ideal, that is, $S^1aS^1 = S^1bS^1$, then we say that a and b are $\mathcal{J}$ equivalent and write $a\mathcal{J}b$ or $(a, b) \in \mathcal{J}$. Let $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$, then $\mathcal{H}, \mathcal{D}$ are equivalences on S, too. It is well known that in a finite semigroup $\mathcal{J} = \mathcal{D}$. These five equivalences are usually called Green's equivalences on S. They were introduced by J. A. Green in [2], and have a great importance in the study of the algebraic structure of semigroups.

Let $[n] = \{1, 2, \ldots, n\}$ ordered in the standard way. We denote by PT_n the semigroup of all partial transformations on $[n]$. We say $\alpha \in PT_n$ is order-preserving if, for all $x, y \in \mathrm{dom}\alpha$, $x \leq y$ implies $x\alpha \leq y\alpha$, and α is compressing-preserving if, for all $x, y \in \mathrm{dom}\alpha$, $|x\alpha - y\alpha| \leq |x - y|$. We denote by PO_n the subsemigroup of PT_n of all partial order-preserving transformations (excluding the identity mapping). Let

$$CPO_n = \{\alpha \in PO_n : (\forall x, y \in \mathrm{dom}\alpha), |x\alpha - y\alpha| \leq |x - y|\}$$

Received May 31, 2011; Revised January 25, 2012.
2010 *Mathematics Subject Classification.* 20M20.
Key words and phrases. order-preserving transformations, compression-preserving transformations, Green's relations.
This work is supported by Natural Science Fund of Guizhou(No. [2010] 3174).

be the set of PO_n consisting of all partial order-preserving and compressing-preserving mappings on $[n]$. It is easily verified that CPO_n is a subsemigroup of PO_n.

The Green's relations on various special subsemigroups of PT_n have been studied by many authors; see for example, [1], [3], [4], [5], [6], [7], [8], [9], [10], [11]. The purpose of this paper is to investigate regularity of elements and Green's relations for the new subsemigroup CPO_n of PT_n. Accordingly, in Section 2, the condition under which an element $\alpha \in CPO_n$ is regular is analyzed, and a necessary and sufficient condition is established. In Section 3, Green's equivalences on CPO_n are considered and the relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{D}$ are described for arbitrary elements.

2. Regular elements in CPO_n

In this section we investigate the condition under which an elements of CPO_n is regular and describe some properties of regular elements. We first need the following terminology and notation.

Let $x, y \in [n]$, $x < y$. The set $[x, y] = \{z \in [n] : x \le z \le y\}$ of $[n]$ is called a closed interval. Similarly, we can define the intervals of other kinds, such as $(x, y]$, (x, y) and $[x, y)$. Let P, Q be two subsets of $[n]$. If $a < b$ holds for arbitrary $a \in P$ and $b \in Q$, then we say that P is less than Q, and write $P < Q$. For any $\alpha \in PO_n$, it is obvious that $x\alpha^{-1} < y\alpha^{-1}$ if $x < y$ $(x, y \in im\alpha)$. Then every $\alpha \in CPO_n$ can be expressed as

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix}, \tag{2.1}$$

where $a_1 < a_2 < \cdots < a_s$, $A_1 < A_2 < \cdots < A_s$, $a_i - a_{i-1} \le \min A_i - \max A_{i-1}, i = 2, \ldots, s$ (if $s \ge 2$).

Definition 2.1. For any two elements of CPO_n with the same rank:

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix}, \quad \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_s \\ b_1 & b_2 & \cdots & b_s \end{pmatrix},$$

where $a_1 < a_2 < \cdots < a_s$, $A_1 < A_2 < \cdots < A_s$, $a_i - a_{i-1} \le \min A_i - \max A_{i-1}$, $b_1 < b_2 < \cdots < b_s$, $B_1 < B_2 < \cdots < B_s$, $b_i - b_{i-1} \le \min B_i - \max B_{i-1}, i = 2, \ldots, s$, $s \ge 2$. Let $d = \max A_1 - \max B_1$. If $\min A_s - \min B_s = d$ and $A_i = B_i + d$ or $B_i = A_i - d$, $i \in [2, s-1]$ (if $s \ge 3$), then α, β are called same kernel-type, denoted $\alpha \underline{Ker} \beta$. If $|a_i - a_j| = |b_i - b_j|$ for all $i, j \in [1, s]$, then α, β are called same image-type, denoted $\alpha \underline{Im} \beta$.

Now we investigate the condition under which an element in CPO_n is regular.

Theorem 2.2. *Let $\alpha \in CPO_n$ be as defined in* (2.1) *and let* $d = \max A_1 - a_1$.

(1) *If $|im\alpha| = 1$, then α is regular.*

(2) *If $|im\alpha| = 2$, then α is regular if and only if* $\max A_1 - a_1 = \min A_2 - a_2$.

(3) *If* $|im\alpha| \geq 3$, *then* α *is regular if and only if* $\min A_s - a_s = d$ *and* $A_i = \{a_i + d\}$, $i \in [2, s-1]$.

Proof. (1) Note that $\alpha = \binom{A_1}{a_1}$. Let $x \in A_1$. Define β by

$$\beta = \begin{pmatrix} a_1 \\ x \end{pmatrix}.$$

Then $\beta \in CPO_n$ and $\alpha = \alpha\beta\alpha$.

(2) Note that $\alpha = \binom{A_1\ A_2}{a_1\ a_2}$. Suppose that α is regular. Then there exists $\beta \in CPO_n$ such that $\alpha = \alpha\beta\alpha$. Thus $a_i = A_i\alpha = (A_i\alpha)\beta\alpha = (a_i\beta)\alpha$ $(i = 1, 2)$ and so $a_i\beta \in A_i$ $(i = 1, 2)$. Note that $a_1 < a_2$, $A_1 < A_2$ and $\alpha, \beta \in CPO_n$. It follows that

$$\begin{aligned} a_2 - a_1 &= |a_2 - a_1| = |A_2\alpha - A_1\alpha| \\ &= |(\min A_2)\alpha - (\max A_1)\alpha| \\ &\leq |\min A_2 - \max A_1| = \min A_2 - \max A_1 \\ &\leq a_2\beta - a_1\beta = |a_2\beta - a_1\beta| \\ &\leq |a_2 - a_1| = a_2 - a_1. \end{aligned}$$

Thus $a_2 - a_1 = \min A_2 - \max A_1$ and so $\max A_1 - a_1 = \min A_2 - a_2$.

Conversely, suppose that $\max A_1 - a_1 = \min A_2 - a_2$. Define β by

$$\beta = \begin{pmatrix} a_1 & a_2 \\ \max A_1 & \min A_2 \end{pmatrix}.$$

Then $\beta \in CPO_n$ and $\alpha = \alpha\beta\alpha$.

(3) Suppose that α is regular. Then there exists $\beta \in CPO_n$ such that $\alpha = \alpha\beta\alpha$. Thus, for any $i, j \in [1, s]$,

$$a_i = A_i\alpha = (A_i\alpha)\beta\alpha = (a_i\beta)\alpha, \quad a_j = A_j\alpha = (A_j\alpha)\beta\alpha = (a_j\beta)\alpha,$$

and so

$$a_i\beta \in A_i, \quad a_j\beta \in A_j. \tag{2.2}$$

Take any $i, j \in [1, s]$ with $i < j$. Note that $a_i < a_j$, $A_i < A_j$ and (2.2). Since $\alpha, \beta \in CPO_n$, we have

$$\begin{aligned} a_j - a_i &= |a_j - a_i| = |A_j\alpha - A_i\alpha| \\ &= |(\min A_j)\alpha - (\max A_i)\alpha| \\ &\leq |\min A_j - \max A_i| = \min A_j - \max A_i \\ &\leq a_j\beta - a_i\beta = |a_j\beta - a_i\beta| \\ &\leq |a_j - a_i| = a_j - a_i. \end{aligned}$$

Thus

$$a_j - a_i = \min A_j - \max A_i, \ i < j, \ i, j \in [1, s],$$

and so

$$\min A_j - a_j = \max A_i - a_i, \ i < j, \ i, j \in [1, s]. \tag{2.3}$$

Clearly, $\min A_s - a_s = \max A_1 - a_1 = d$. By (2.3), we have

$$\min A_s - a_s = \max A_i - a_i,\ \min A_i - a_i = \max A_1 - a_1, i \in [2, s-1].$$

Thus $\max A_i - a_i = \min A_i - a_i (= d)$ and so $\max A_i = \min A_i = a_i + d$. It follows that $A_i = \{a_i + d\}$.

Conversely, suppose that $\min A_s - a_s = d$ and $A_i = \{a_i + d\}$, $i \in [2, s-1]$. Define β by

$$\beta = \begin{pmatrix} a_1 & a_2 & \cdots & a_{s-1} & a_s \\ \max A_1 & a_2 + d & \cdots & a_{s-1} + d & \min A_s \end{pmatrix}.$$

Then $\beta \in CPO_n$ and $\alpha = \alpha\beta\alpha$. □

Remark 2.3. If $n \geq 5$, then the semigroup CPO_n is not regular. In fact, let

$$\alpha = \begin{pmatrix} 1 & \{2,3\} & 5 \\ 1 & 2 & 4 \end{pmatrix} \in CPO_n.$$

Then, by Theorem 2.2(3), α is not regular in CPO_n.

Next we observe two properties for regular elements in the semigroup CPO_n.

Theorem 2.4. *Let $\alpha, \beta \in CPO_n$ be regular elements with $im\alpha = im\beta$ ($|im\alpha| \geq 2$). Then $\alpha \underline{Ker} \beta$.*

Proof. Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix}, \quad \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix},$$

where $a_1 < a_2 < \cdots < a_s$, $A_1 < A_2 < \cdots < A_s$, $B_1 < B_2 < \cdots < B_s$. Let $d_1 = \max A_1 - a_1$, $d_2 = \max B_1 - a_1$ and $d = \max A_1 - \max B_1$. Then $d = d_1 - d_2$. Since α, β are regular, by Theorem 2.2, we have

$$\min A_s - a_s = \max A_1 - a_1 = d_1,\ \min B_s - a_s = \max B_1 - a_1 = d_2,$$

$$A_i = \{a_i + d_1\},\ B_i = \{a_i + d_2\},\ i \in [2, s-1] \text{ (if } s \geq 3).$$

It follows that

$$\min A_s - \min B_s = \max A_1 - \max B_1 = d,$$

$$A_i = \{a_i + d_2 + d\} = B_i + d,\ i \in [2, s-1] \text{ (if } s \geq 3).$$

Thus $\alpha \underline{Ker} \beta$. □

Theorem 2.5. *Let $\alpha, \beta \in CPO_n$ be regular elements with $Ker\alpha = Ker\beta$ ($|Ker\alpha| \geq 2$). Then $\alpha \underline{Im} \beta$.*

Proof. Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix}, \quad \beta = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ b_1 & b_2 & \cdots & b_s \end{pmatrix},$$

where $a_1 < a_2 < \cdots < a_s$, $b_1 < b_2 < \cdots < b_s$, $A_1 < A_2 < \cdots < A_s$. Let $d_1 = \max A_1 - a_1$ and $d_2 = \max A_1 - b_1$. Since α, β are regular, by Theorem 2.2, we have

$$\min A_s - a_s = \max A_1 - a_1 = d_1,\ \min A_s - b_s = \max A_1 - b_1 = d_2,$$
$$A_i = \{a_i + d_1\},\ A_i = \{b_i + d_2\},\ i \in [2, s-1] \text{ (if } s \geq 3).$$

Then

$$a_s - b_s = a_1 - b_1 = d_2 - d_1,$$
$$a_i - b_i = d_2 - d_1,\ i \in [2, s-1] \text{ (if } s \geq 3)$$

and so

$$a_i - b_i = d_2 - d_1,\ i \in [1, s].$$

It follows that for all $i, j \in [1, s]$,

$$|a_i - a_j| = |(b_i + d_2 - d_1) - (b_j + d_2 - d_1)| = |b_i - b_j|.$$

Thus $\alpha \underline{Im} \beta$. □

3. Green's relations on CPO_n

In this section, we describe Green's relations on CPO_n. Since it is well known that $\mathcal{D} = \mathcal{J}$ in every finite semigroup and $\mathcal{H} = \mathcal{L} \wedge \mathcal{R}$, we only consider the Green relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{D}$. We begin with the relation $\mathcal{L}$.

Theorem 3.1. *Let $\alpha, \beta \in CPO_n$.*
(1) *If $|im\alpha| = 1$, then $(\alpha, \beta) \in \mathcal{L}$ if and only if $im\alpha = im\beta$.*
(2) *If $|im\alpha| \geq 2$, then $(\alpha, \beta) \in \mathcal{L}$ if and only if $im\alpha = im\beta$ and $\alpha \underline{Ker} \beta$.*

Proof. (1) Suppose that $(\alpha, \beta) \in \mathcal{L}$. Then there exist $\delta, \gamma \in (CPO_n)^1$ such that $\alpha = \delta\beta$, $\beta = \gamma\alpha$. It follows easily that $im\alpha = im\beta$. Conversely, suppose that $im\alpha = im\beta$. Let

$$\alpha = \begin{pmatrix} A \\ a \end{pmatrix},\quad \beta = \begin{pmatrix} B \\ a \end{pmatrix}.$$

Let $c \in B$ and $d \in A$. Define δ, γ by

$$\delta = \begin{pmatrix} A \\ c \end{pmatrix},\ \gamma = \begin{pmatrix} B \\ d \end{pmatrix}.$$

Clearly, $\delta, \gamma \in CPO_n$ and $\alpha = \delta\beta$, $\beta = \gamma\alpha$. Thus $(\alpha, \beta) \in \mathcal{L}$.

(2) Suppose that $(\alpha, \beta) \in \mathcal{L}$. Then there exist $\delta, \gamma \in (CPO_n)^1$ such that $\alpha = \delta\beta$, $\beta = \gamma\alpha$. It follows easily that $im\alpha = im\beta$. Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix},\quad \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix},$$

where $a_1 < a_2 < \cdots < a_s$, $A_1 < A_2 < \cdots < A_s$, $B_1 < B_2 < \cdots < B_s$. Let $d = \max A_1 - \max B_1$. Note that

$$A_i\delta\beta = A_i\alpha = a_i = B_i\beta, B_i\gamma\alpha = B_i\beta = a_i = A_i\alpha,\ i \in [1, s].$$

We obtain that

$$A_i\delta \subseteq B_i, \quad B_i\gamma \subseteq A_i, \ i \in [1, s]. \tag{3.1}$$

Take any $i, j \in [1, s]$ with $i < j$. From (3.1) we know that

$$(\min A_j)\delta \in B_j, \ (\max A_i)\delta \in B_i, \ (\min B_j)\gamma \in A_j, \ (\max B_i)\gamma \in A_i. \tag{3.2}$$

Note that

$$A_i < A_j, B_i < B_j. \tag{3.3}$$

Since $\delta, \gamma \in (CPO_n)^1$, we have

$$\begin{aligned} \min B_j - \max B_i &\leq (\min A_j)\delta - (\max A_i)\delta = |(\min A_j)\delta - (\max A_i)\delta| \\ &\leq |\min A_j - \max A_i| = \min A_j - \max A_i, \\ \min A_j - \max A_i &\leq (\min B_j)\gamma - (\max B_i)\gamma = |(\min B_j)\gamma - (\max B_i)\gamma| \\ &\leq |\min B_j - \max B_i| = \min B_j - \max B_i. \end{aligned}$$

It follows from (3.2) that

$$\min A_j - \min B_j = \max A_i - \max B_i, \tag{3.4}$$

$$(\min A_j)\delta = \min B_j, \ (\max A_i)\delta = \max B_i, \tag{3.5}$$

$$(\min B_j)\gamma = \min A_j, \ (\max B_i)\gamma = \max A_i. \tag{3.6}$$

If $|im\alpha| = 2$, then, by (3.4), $\min A_2 - \min B_2 = \max A_1 - \max B_1 = d$. Thus $\alpha \overline{Ker} \beta$. If $|im\alpha| \geq 3$. We claim that

$$x\delta = x - d, \ \forall x \in A_i, \ i \in [2, s-1], \tag{3.7}$$

$$x\gamma = x + d, \ \forall x \in B_i, \ i \in [2, s-1]. \tag{3.8}$$

Note that (3.1) and (3.3). Since $\delta \in (CPO_n)^1$, we have

$$x\delta - (\max A_1)\delta = |x\delta - (\max A_1)\delta| \leq |x - \max A_1| = x - \max A_1,$$
$$(\min A_s)\delta - x\delta = |(\min A_s)\delta - x\delta| \leq |\min A_s - x| = \min A_s - x.$$

Thus

$$\max A_1 - (\max A_1)\delta \leq x - x\delta \leq \min A_s - (\min A_s)\delta.$$

It follows from (3.4) and (3.5) that

$$\begin{aligned} d &= \max A_1 - \max B_1 = \max A_1 - (\max A_1)\delta \leq x - x\delta \leq \min A_s - (\min A_s)\delta \\ &= \min A_s - \min B_s = \max A_1 - \max B_1 = d. \end{aligned}$$

Thus $x\delta = x - d$. Similarly, we can prove that (3.8) holds. Now, we shall prove that $\alpha \overline{Ker} \beta$. By (3.4), we have

$$\min A_s - \min B_s = \max A_1 - \max B_1 = d.$$

From (3.7) and (3.8) we know that $\delta|_{A_i}$ and $\gamma|_{B_i}$ are mutually inverse bijections from A_i onto $A_i\delta$ and B_i onto $B_i\gamma$, respectively. Then, by (3.1),

$$|A_i| = |A_i\delta| \leq |B_i|, \ |B_i| = |B_i\gamma| \leq |A_i|, \ i \in [2, s-1]$$

and so $|A_i| = |A_i\delta| = |B_i\gamma| = |B_i|$. Thus $A_i\delta = B_i$ and $B_i\gamma = A_i$ by (3.1). Moreover, by (3.7) and (3.8), we have

$$A_i = B_i + d,\ i \in [2, s-1],$$
$$B_i = A_i - d,\ i \in [2, s-1].$$

Thus $A \overline{\overset{Ker}{}} B$.

Conversely, suppose that $im\alpha = im\beta$ and $\alpha \overline{\overset{Ker}{}} \beta$. Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix},\quad \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix},$$

where $a_1 < a_2 < \cdots < a_s$, $A_1 < A_2 < \cdots < A_s$, $B_1 < B_2 < \cdots < B_s$. Let $d = \max A_1 - \max B_1$. If $|im\alpha| = 2$ $(s = 2)$. Define δ, γ by

$$\delta = \begin{pmatrix} A_1 & A_2 \\ \max B_1 & \min B_2 \end{pmatrix},\quad \gamma = \begin{pmatrix} B_1 & B_2 \\ \max A_1 & \min A_2 \end{pmatrix}.$$

Clearly, $\delta, \gamma \in PO_n$, $\alpha = \delta\beta$ and $\beta = \gamma\alpha$. Since $\alpha \overline{\overset{Ker}{}} \beta$, we have $\max A_1 - \max B_1 = \min A_2 - \min B_2 (= d)$ and so $\min A_2 - \max A_1 = \min B_2 - \max B_1$. It easily follows that $\delta, \gamma \in CPO_n$. Thus $(\alpha, \beta) \in \mathcal{L}$. If $|im\alpha| \geq 3$ $(s \geq 3)$, then since $\alpha \overline{\overset{Ker}{}} \beta$, we have

$$\min A_s - \min B_s = d,\quad A_i = B_i + d. \tag{3.9}$$

Define δ, γ by

$$x\delta = \begin{cases} \max B_1, & \text{if } x \in A_1, \\ x - d, & \text{if } x \in A_i, i \in [2, s-1], \\ \min B_s, & \text{if } x \in A_s, \end{cases}$$
$$x\gamma = \begin{cases} \max A_1, & \text{if } x \in B_1, \\ x + d, & \text{if } x \in B_i, i \in [2, s-1], \\ \min A_s, & \text{if } x \in B_s. \end{cases}$$

Clearly, $\delta, \gamma \in PO_n$, $\alpha = \delta\beta$ and $\beta = \gamma\alpha$. Take any $x, y \in \mathrm{dom}\delta$ with $x \leq y$. Note that $\mathrm{dom}\delta = A_1 \cup A_2 \cup \cdots \cup A_s$. We distinguish five cases.

Case 1: $x, y \in A_i$, $i \in \{1, s\}$. Clearly, $x\delta - y\delta = 0$. Thus $|x\delta - y\delta| \leq |x - y|$.

Case 2: $x \in A_1$, $y \in A_i$, $i \in [2, s-1]$. Since $A_1 < A_i$, we have

$$|x\delta - y\delta| = |\max B_1 - (y - d)| = |y - (d + \max B_1)| = |y - \max A_1| \leq |x - y|.$$

Case 3: $x \in A_1$, $y \in A_s$. Note that $A_1 < A_s$. By (3.9), we have

$$\begin{aligned} |x\delta - y\delta| &= |\max B_1 - \min B_s| \\ &= |(\max A_1 - d) - (\min A_s - d)| \\ &= |\max A_1 - \min A_s| \leq |x - y|. \end{aligned}$$

Case 4: $x \in A_i$, $y \in A_j$, $i, j \in [2, s-1]$, $i \neq j$. Clearly, $|x\delta - y\delta| = |(x - d) - (y - d)| = |x - y|$. Thus $|x\delta - y\delta| \leq |x - y|$.

Case 5: $x \in A_i$, $i \in [2, s-1]$, $y \in A_s$. Note that $A_i < A_s$. By (3.9), we have

$$|x\delta - y\delta| = |(x - d) - \min B_s| = |(\min B_s + d) - x| = |\min A_s - x| \leq |x - y|.$$

Form the discussion above, we have $\delta \in CPO_n$. Similarly, we can prove that $\gamma \in CPO_n$. Thus $(\alpha,\beta) \in \mathcal{L}$. □

Theorem 3.2. *Let $\alpha,\beta \in CPO_n$.*

(1) *If $|im\alpha| = 1$, then $(\alpha,\beta) \in \mathcal{R}$ if and only if $Ker\alpha = Ker\beta$.*

(2) *If $|im\alpha| \geq 2$, then $(\alpha,\beta) \in \mathcal{R}$ if and only if $Ker\alpha = Ker\beta$ and $\alpha \underline{Im} \beta$.*

Proof. (1) Suppose that $(\alpha,\beta) \in \mathcal{R}$. Then there exist $\delta,\gamma \in (CPO_n)^1$ such that $\alpha = \beta\delta$, $\beta = \alpha\gamma$. For each $(x,y) \in Ker\alpha$, we have

$$x\beta = x(\alpha\gamma) = (x\alpha)\gamma = (y\alpha)\gamma = y(\alpha\gamma) = y\beta.$$

Then $Ker\alpha \subseteq Ker\beta$. Similarly, we can prove that $Ker\beta \subseteq Ker\alpha$. Thus $Ker\alpha = Ker\beta$. Conversely, suppose that $Ker\alpha = Ker\beta$. Let

$$\alpha = \begin{pmatrix} A \\ a \end{pmatrix}, \quad \beta = \begin{pmatrix} A \\ b \end{pmatrix}.$$

Define δ, γ by

$$\delta = \begin{pmatrix} b \\ a \end{pmatrix}, \quad \gamma = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Clearly, $\delta,\gamma \in CPO_n$ and $\alpha = \beta\delta$, $\beta = \alpha\gamma$. Thus $(\alpha,\beta) \in \mathcal{R}$.

(2) Suppose that $(\alpha,\beta) \in \mathcal{R}$. Then there exist $\delta,\gamma \in (CPO_n)^1$ such that $\alpha = \beta\delta$, $\beta = \alpha\gamma$. Thus by the same proof as given for (1), we have $Ker\alpha = Ker\beta$. Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix}, \quad \beta = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ b_1 & b_2 & \cdots & b_s \end{pmatrix},$$

where $a_1 < a_2 < \cdots < a_s$, $b_1 < b_2 < \cdots < b_s$, $A_1 < A_2 < \cdots < A_s$. Note that for any $i,j \in [1,s]$,

$$a_i = A_i\alpha = A_i(\beta\delta) = (A_i\beta)\delta = b_i\delta, \quad a_j = A_j\alpha = A_j(\beta\delta) = (A_j\beta)\delta = b_j\delta,$$

$$b_i = A_i\beta = A_i(\alpha\gamma) = (A_i\alpha)\gamma = a_i\gamma, \quad b_j = A_j\beta = A_j(\alpha\gamma) = (A_j\alpha)\gamma = a_j\gamma.$$

Since $\delta,\gamma \in CPO_n$, we have

$$|a_i - a_j| = |b_i\delta - b_j\delta| \leq |b_i - b_j|,$$
$$|b_i - b_j| = |a_i\gamma - a_j\gamma| \leq |a_i - a_j|.$$

Thus $|a_i - a_j| = |b_i - b_j|$ and so $\alpha \underline{Im} \beta$.

Conversely, suppose that $Ker\alpha = Ker\beta$ and $\alpha \underline{Im} \beta$. Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix}, \quad \beta = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ b_1 & b_2 & \cdots & b_s \end{pmatrix},$$

where $a_1 < a_2 < \cdots < a_s$, $b_1 < b_2 < \cdots < b_s$, $A_1 < A_2 < \cdots < A_s$. Define δ, γ by

$$b_i\delta = a_i, i \in [1,s],$$
$$a_i\gamma = b_i, i \in [1,s].$$

Clearly, $\delta,\gamma \in PO_n$, $\alpha = \beta\delta$ and $\beta = \alpha\gamma$. Since $\alpha \underline{Im} \beta$, we have $|a_i - a_j| = |b_i - b_j|$. Thus $\delta,\gamma \in CPO_n$ and so $(\alpha,\beta) \in \mathcal{R}$. □

Theorem 3.3. *Let* α, $\beta \in CPO_n$.

(1) *If* $|im\alpha| = 1$, *then* $(\alpha, \beta) \in \mathcal{D}$ *if and only if* $|im\alpha| = |im\beta|$.

(2) *If* $|im\alpha| \geq 2$, *then* $(\alpha, \beta) \in \mathcal{D}$ *if and only if* $|im\alpha| = |im\beta|$, $\alpha \overset{Ker}{\longrightarrow} \beta$ *and* $\alpha \overset{Im}{\longrightarrow} \beta$.

Proof. (1) Suppose that $(\alpha, \beta) \in \mathcal{D}$. Then there exists $\gamma \in (CPO_n)^1$ such that $\alpha \mathcal{L} \gamma \mathcal{R} \beta$. By Theorems 3.1 and 3.2, we have $im\alpha = im\gamma$ and $Ker\gamma = Ker\beta$. It follows easily that $|im\alpha| = |im\beta|$. Conversely, suppose that $|im\alpha| = |im\beta|$. Let

$$\alpha = \begin{pmatrix} A \\ a \end{pmatrix}, \quad \beta = \begin{pmatrix} B \\ b \end{pmatrix}.$$

Define γ by

$$\gamma = \begin{pmatrix} B \\ a \end{pmatrix}.$$

Clearly, $\gamma \in CPO_n$ and $im\alpha = im\gamma$, $Ker\gamma = Ker\beta$. Then, by Theorems 3.1 and 3.2, $\alpha \mathcal{L} \gamma$ and $\gamma \mathcal{R} \beta$. Thus $(\alpha, \beta) \in \mathcal{D}$.

(2) Suppose that $(\alpha, \beta) \in \mathcal{D}$. Then there exists $\gamma \in (CPO_n)^1$ such that $\alpha \mathcal{L} \gamma \mathcal{R} \beta$. By Theorems 3.1 and 3.2, we have $im\alpha = im\gamma$, $\alpha \overset{Ker}{\longrightarrow} \gamma$ and $Ker\gamma = Ker\beta$, $\gamma \overset{Im}{\longrightarrow} \beta$. It follows easily that $|im\alpha| = |im\beta|$ and $\alpha \overset{Ker}{\longrightarrow} \beta$, $\alpha \overset{Im}{\longrightarrow} \beta$.

Conversely, suppose that $|im\alpha| = |im\beta|$, $\alpha \overset{Ker}{\longrightarrow} \beta$ and $\alpha \overset{Im}{\longrightarrow} \beta$. Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix}, \quad \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_s \\ b_1 & b_2 & \cdots & b_s \end{pmatrix},$$

where $a_1 < a_2 \cdots < a_s$, $A_1 < A_2 < \cdots < A_s$, $b_1 < b_2 \cdots < b_s$, $B_1 < B_2 < \cdots < B_s$. Define γ by

$$\gamma = \begin{pmatrix} B_1 & B_2 & \cdots & B_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix}.$$

Clearly, $im\alpha = im\gamma$ and $Ker\gamma = Ker\beta$. Since $\alpha \overset{Ker}{\longrightarrow} \beta$ and $\alpha \overset{Im}{\longrightarrow} \beta$, we have that $\alpha \overset{Ker}{\longrightarrow} \gamma$ and $\gamma \overset{Im}{\longrightarrow} \beta$. Then, by Theorems 3.1 and 3.2, $\alpha \mathcal{L} \gamma$ and $\gamma \mathcal{R} \beta$. Thus $(\alpha, \beta) \in \mathcal{D}$. □

As a consequence of Theorems 2.3, 2.4, 3.1 and 3.2, the following result follows readily.

Corollary 3.4. *Let* $\alpha, \beta \in CPO_n$ *be regular elements. Then the following statements hold*:

(1) $(\alpha, \beta) \in \mathcal{L}$ *if and only if* $im\alpha = im\beta$.

(2) $(\alpha, \beta) \in \mathcal{R}$ *if and only if* $Ker\alpha = Ker\beta$.

For the condition under which two regular elements in CPO_n are $\mathcal{D}$ equivalent, we have:

Theorem 3.5. *Let* $\alpha, \beta \in CPO_n$ *be regular elements.*

(1) *If* $|im\alpha| = 1$, *then* $(\alpha, \beta) \in \mathcal{D}$ *if and only if* $|im\alpha| = |im\beta|$.

(2) *If* $|im\alpha| \geq 2$, *then* $(\alpha, \beta) \in \mathcal{D}$ *if and only if* $|im\alpha| = |im\beta|$ *and* $\alpha \overset{Im}{\longrightarrow} \beta$.

Proof. (1) It is an immediate consequence of Theorem 3.3.

(2) Suppose that $(\alpha,\beta)\in\mathcal{D}$. Then, by Theorem 3.3, $|im\alpha| = |im\beta|$ and $\alpha\underline{Im}\beta$.

Conversely, suppose that $|im\alpha| = |im\beta|$ and $\alpha\underline{Im}\beta$. Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix}, \quad \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_s \\ b_1 & b_2 & \cdots & b_s \end{pmatrix},$$

where $a_1 < a_2 \cdots < a_s$, $A_1 < A_2 < \cdots < A_s$, $b_1 < b_2 \cdots < b_s$, $B_1 < B_2 < \cdots < B_s$. Since $\alpha\underline{Im}\beta$, we have

$$|a_i - a_j| = |b_i - b_j|, i, j \in [1, s]. \tag{3.10}$$

Note that $a_i \geq a_j$, $b_i \geq b_j$ if $i \geq j$; $a_i \leq a_j$, $b_i \leq b_j$ if $i \leq j$. It follows easily from (3.10) that

$$a_i - b_i = a_j - b_j, i, j \in [1, s]. \tag{3.11}$$

Let $d_1 = \max A_1 - a_1$, $d_2 = \max B_1 - b_1$ and $d = \max A_1 - \max B_1$. Then

$$a_1 - b_1 = (\max A_1 - \max B_1) + d_2 - d_1 = d + d_2 - d_1.$$

By (3.11), we have

$$a_i - b_i = a_1 - b_1 = d + d_2 - d_1, i \in [1, s]. \tag{3.12}$$

Since α,β are regular, by Theorem 3.1, we have

$$\min A_s - a_s = \max A_1 - a_1 = d_1,\ \min B_s - b_s = \max B_1 - b_1 = d_2,$$

$$A_i = \{a_i + d_1\},\ B_i = \{b_i + d_2\},\ i \in [2, s-1] \text{ (if } s \geq 3).$$

Then, by (3.12) that

$$\min A_s - \min B_s = (d_1 + a_s) - (d_2 + b_s) = (a_s - b_s) + (d_1 - d_2) = d,$$

$$A_i = \{a_i + d_1\} = \{(d + d_2 - d_1 + b_i) + d_1\} = \{(b_i + d_2) + d\} = B_i + d.$$

Thus $\alpha\underline{Ker}\beta$ and so, by Theorem 3.3, $(\alpha,\beta)\in\mathcal{D}$. □

Acknowledgments. The authors wish to express their appreciation to the referees for some valuable comments and suggestions that helped to improve the presentation of this paper.

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Ping Zhao
School of Mathematics and Computer Science
GuiZhou Normal University
Guiyang, GuiZhou Province 550001, P. R. China
and
Mathematics Teaching & Research Section
Guiyang Medical College
Guiyang, GuiZhou Province 550004, P. R. China
E-mail address: zhaoping731108@hotmail.com, peaceguorong@yahoo.com

Mei Yang
Department of Petroleum Engineering
Texas A&M University
College Station, Texas, 77840, USA
E-mail address: mei.yang@tamu.edu