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CO-CONTRACTIONS OF GRAPHS AND RIGHT-ANGLED COXETER GROUPS

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Abstract. We prove that if $\widehat{\Gamma}$ is a co-contraction of Γ, then the right-angled Coxeter group $C(\widehat{\Gamma})$ embeds into $C(\Gamma)$. Further, we provide a graph Γ without an induced long cycle while $C(\Gamma)$ does not contain a hyperbolic surface group.

1. Introduction

Let Γ be a finite simple graph with the vertex set $V(\Gamma) = \{v_1, \ldots, v_n\}$ and the edge set $E(\Gamma)$. Recall that a simple graph is a graph without loops and multiple edges. The *right-angled Coxeter group* $C(\Gamma)$ *on* Γ is the group with the presentation, in which the generators are $v_1, \ldots, v_n$ and relators are $v_1^2, \ldots, v_n^2$, and $[v_i, v_j]$ whenever $e_{\{v_i,v_j\}} \in E(\Gamma)$, where $e_{\{u,v\}}$ denotes the only edge connecting the vertices u and v. Considering that the *right-angled Artin group on* Γ is the group with the presentation, in which the generators are $v_1, \ldots, v_n$ and relators are $[v_i, v_j]$ whenever $e_{\{v_i,v_j\}} \in E(\Gamma)$, the presentation of the right-angled Coxeter group $C(\Gamma)$ has extra relators v^2 for all $v \in V(\Gamma)$ when it is compared with the presentation of the right-angled Artin group $A(\Gamma)$ ([3]). Γ is called the *defining graph of* $C(\Gamma)$. By a *graph*, we mean a finite simple graph throughout the paper.

Motivated by 3-manifold theory, it has been an intriguing question when right-angled Coxeter groups and right-angled Artin groups contain a closed hyperbolic surface group. Many right-angled Artin groups and right-angled Coxeter groups are known to be 3-manifold groups. A right-angled Coxeter group $C(\Gamma)$ contains a closed hyperbolic surface group if Γ contains an induced cycle $C_n (n \geq 5)$ ([5]). Similar results for right-angled Artin groups were obtained in [13] and [8]. In fact, a right-angled Artin group $A(\Gamma)$ contains a closed hyperbolic surface group if Γ contains an induced cycle $C_n (n \geq 5)$ ([13]) or an

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induced anti-cycle $\overline{C_n}(n \geq 5)$ ([8]). Gordon, Long and Reid raised a question whether the converse also holds ([5]).

Question 3.1 ([5]). *Does a right-angled Coxeter group $C(\Gamma)$ contain a hyperbolic surface subgroup if and only if Γ contains an induced n-cycle for some $n \geq 5$?*

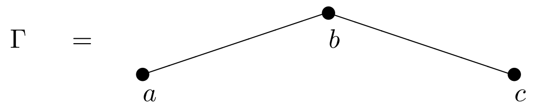

FIGURE 1. The defining graph of the right-angled Coxeter group with the presentation $\langle a,\ b,\ c \mid a^2,\ b^2,\ c^2,\ aba^{-1}b^{-1},\ bcb^{-1}c^{-1}\rangle$.

In order to answer the question for the right-angled Artin group case, Kim introduced an operation on a graph, called co-contraction. He proved that if Γ_2 is a co-contraction of a graph Γ_1, then there is a monomorphism from $A(\Gamma_2)$ to $A(\Gamma_1)$. Using this fact, he was able to show that $A(\overline{C_n})(n \geq 5)$ contains a closed hyperbolic surface group but $\overline{C_n}$ does not contain any cycle of length ≥ 5, thereby gave a negative answer to the question. On the other hand, Bell gave a new proof on Kim's co-contraction lemma on right-angled Artin groups using classical methods from combinatorial group theory ([2]). Bell's method enables us to prove co-contraction theorem for right-angled Coxeter groups as well (see Theorem 2.5). It should be noted that Kim proved Theorem 2.5 for graph products of the cyclic groups of order $m (0 < m \leq \infty)$, which includes the right-angled Artin groups and the right-angled Coxeter groups in his dissertation ([7]). But the proof we give is different from than his.

In Section 2, we define co-contractions on graphs and go over Bell's method. Applying Bell's method to right-angled Coxeter group, we prove Theorem 2.5. In Section 3, as an application of Theorem 2.5, we give a negative answer to Question 3.1:

Corollary 3.3. *There is a right-angled Coxeter group $C(\Gamma)$ containing a hyperbolic surface subgroup while its underlying graph Γ does not contain any induced n-cycle for $n \geq 5$.*

At the end of Section 3, using the results of Davis, Januszkiewicz [4] and Kim [7, 8], we provide examples of different types which justify Corollary 3.3, thereby answer negatively to Question 3.1 again.

2. Co-contractions of graphs and right-angled Coxeter groups

A *graph* is a 1-dimensional CW-complex. 0-cells are called *vertices* and closed 1-cells are called *edges.* We denote the vertex set of a graph Γ by $V(\Gamma)$

and the edge set of a graph Γ by $E(\Gamma)$. A *loop* of a graph is an edge that is homeomorphic to the circle. A graph *with multiple edges* is a graph that contains two distinct edges sharing their two end points. A *simple graph* is a graph without loops and multiple edges. In this paper, by a graph we always mean a finite simple graph. Note that every edge is uniquely determined by its two end points in a finite simple graph. We denote the unique edge connecting two adjacent vertices a, b in a graph by $e_{\{a,b\}}$.

Recall that a subgraph Γ' of a graph Γ is called an *induced subgraph* of Γ if for each pair of vertices a and b of Γ', $e_{\{a,b\}}$ is $E(\Gamma')$ whenever $e_{\{a,b\}}$ is in $E(\Gamma)$. We say that Γ_1 *has* Γ_2 *as an induced subgraph* (*or* Γ_1 *has induced* Γ_2) if Γ_1 has an induced subgraph that is an isomorphic copy of Γ_2. Note that for a subset S of the vertex set of a graph Γ, there is a unique induced subgraph of Γ having S as the vertex set. This induced subgraph is called *the subgraph generated by* S and denoted by Γ_S. It is known that if Γ has Γ' as an induced subgraph, then $C(\Gamma)$ $(A(\Gamma))$ has a subgroup which is isomorphic to $C(\Gamma')$ $(A(\Gamma'))$.

The *complement graph* $\overline{\Gamma}$ of a graph Γ is the graph with $V(\overline{\Gamma}) = V(\Gamma)$ and $E(\overline{\Gamma}) = \{e_{\{a,b\}} | \, e_{\{a,b\}} \notin E(\Gamma)\}$. The *n-cycle*, denoted by C_n, is the graph that is homeomorphic to the circle and has n edges ($n = 3, 4, 5, \ldots$). The complement graph $\overline{C_n}$ of C_n is called the *anti n-cycle*.

Let v be a vertex in a graph Γ. The *link* of v is defined as the set of vertices adjacent to v. The link of v is denoted by $link(v)$. Note that v itself is not in the link of v. The *star* of v, denoted by $star(v)$, is defined as $\{v\} \cup link(v)$.

For basic facts on graphs, see [6].

Definition ([7, 8]). Let Γ be a graph. Let a, b be two different non-adjacent vertices of Γ. The *co-contraction* $\widehat{\Gamma}$ *of* Γ *relative to* a, b is defined to be the graph with $V(\widehat{\Gamma}) = (V(\Gamma) - \{a, b\}) \cup \{v\}$ and $E(\widehat{\Gamma}) = E(\Gamma_{V(\Gamma - \{a,b\})}) \cup \{e_{\{v,q\}} \mid q \in V(\Gamma) - \{a, b\}$ and $\{a, b\} \subset link(q)\}$.

If a and b are not adjacent in a graph Γ, then the co-contraction $\widehat{\Gamma}$ is obtained by collapsing $\{a, b\}$ onto one vertex v, removing all the edges connected to a or b, and adding new edges $e_{\{v,q\}}$ if $q \in V(\Gamma) - \{a, b\}$ and q is adjacent to both a and b in Γ. As a matter of fact, what Kim introduced was the co-contraction relative to an *anti-connected* subset S of $V(\Gamma)$. A subset S of $V(\Gamma)$ is called *anti-connected* if $\overline{\Gamma_S}$ is connected. He was able to prove the following proposition ([7, 8]).

Proposition 2.1 ([7, 8]). *For any co-contraction* $\widehat{\Gamma}$ *of* Γ*, there is a monomorphism from* $A(\widehat{\Gamma})$ *to* $A(\Gamma)$.

Note that the set of two non-adjacent vertices is anti-connected. Kim observed that for a given graph Γ and its anti-connected subset B, the co-contraction $\widehat{\Gamma}$ of Γ relative to B can be obtained from Γ by a finite successive sequence of co-contractions relative to two non-adjacent vertices. Therefore, it is enough to deal with the co-contractions relative two non-adjacent vertices to prove the above proposition. Likewise, in order to set up a similar result for Coxeter group case, we only need to handle co-contractions relative two non-adjacent vertices.

Recently, Bell made use of Reidemeister-Schreier method to give another proof of Kim's result on co-contractions ([2]). Reidemeister-Schreier method can be found in [1, 2, 9, 10]. Let G be a group given by the presentation $\langle X|R\rangle$, and suppose H is a subgroup of G. Then a useful presentation for H is given by Reidemeister-Schreier method.

Proposition 2.2 (Reidemeister-Schreier method)**.** *Let F be free with basis X, and let $\pi : F \longrightarrow G$ be the canonical map. Let $P = \pi^{-1}(H)$. And let $T \subset F$ be a right Schreier transversal for P in F, i.e., T is a complete set of right coset representatives such that for each element t in T any initial subword of t belongs to T again. Given $w \in F$, let $\overline{w}$ be the unique element of T such that $Pw = P\overline{w}$. For each $t \in T$ and $x \in X$, let $s(t,x) = tx(\overline{tx})^{-1}$. Define $S = \{s(t,x) \mid t \in T, x \in X, \text{ and } s(t,x) \neq 1 \text{ in } F\}$. Then S is a basis for the free group P. Define a rewriting process $\tau : F \longrightarrow P$ on freely reduced words over X by $\tau(y_1y_2\cdots y_n) = s(1,y_1)s(\overline{y_1},y_2)\cdots s(\overline{y_1\cdots y_{n-1}},y_n)$, where $y_i \in X \cup X^{-1}$. Then $\tau(w) = w\overline{w}^{-1}$ for every freely reduced word $w \in F$, and* $H = \langle S \mid \tau(trt^{-1}) = 1 \text{ for all } t \in T \text{ and } r \in R\rangle$.

Lemma 2.3 (Bell's lemma [2])**.** *Suppose $A(\Gamma)$ is a right-angled Artin group, and let n be a positive integer. Choose a vertex $z \in V(\Gamma)$, and define $\phi : A(\Gamma) \longrightarrow \langle x \mid x^n = 1\rangle$ by $\phi(z) = x$ and $\phi(v) = 1$ if $v \neq z$. Then* $\ker\phi$ *is the right-angled Artin group whose defining graph Γ' is obtained by gluing n copies of $\Gamma - star(z)$ to $star(z)$ along $link(z)$. Moreover, the vertices of Γ' naturally correspond to the following generating set.*

$$\{z^n\} \cup link(z) \cup \{u \mid u \notin star(z)\} \cup \{zuz^{-1} \mid u \notin star(z)\} \cup \cdots \cup \{z^{n-1}uz^{1-n} \mid u \notin star(z)\}.$$

Bell's method can also be applied to right-angled Coxeter groups to obtain the following proposition.

Proposition 2.4. *Suppose $C(\Gamma)$ is a right-angled Coxeter group. Choose a vertex $z \in V(\Gamma)$, and define $\phi : C(\Gamma) \longrightarrow \langle x \mid x^2 = 1\rangle \cong \mathbb{Z}_2$ by $\phi(z) = x$ and $\phi(v) = 1$ if $v \neq z$. Then* $\ker\phi$ *is the right-angled Coxeter group whose defining graph Γ'' is the double of $\Gamma - \{z\}$ along $link(z)$. Moreover, the vertices of Γ'' naturally correspond to the following generating set.*

$$link(z) \cup \{u \mid u \notin star(z)\} \cup \{zuz^{-1} \mid u \notin star(z)\}.$$

Proof. Let $G = C(\Gamma)$ and F be free on $X = V(\Gamma)$ and let R be the set of relators of $C(\Gamma)$. We apply Proposition 2.2 to get a proper presentation for $\ker\phi$. Let P be the inverse image of $\ker\phi$ in F under the canonical map $F \longrightarrow G$. It is easy to see that the set $T = \{1, z\}$ is a right Schreier transversal for P in F. In order to find the generating set in Reidemeister-Schreier method, we have to examine whether each of the following elements $s(1,v), s(1,z), s(z,v), s(z,z)$ with $v \neq z$ is trivial or not. Note that

$$\begin{aligned}
s(1,v) &= v\overline{v}^{-1} = v \text{ if } v \neq z, \\
s(z,v) &= zv\overline{zv}^{-1} = zvz^{-1} \text{ if } v \neq z, \\
s(1,z) &= z\overline{z}^{-1} = 1, \\
s(z,z) &= z^2\overline{z^2}^{-1} = z^2.
\end{aligned}$$

Thus the generating set of $\ker \phi$ we look for is the set $\{s(z,z), s(1,v), s(z,v) \mid v \in V(\Gamma) - \{z\}\}$. Next, to identify the relators, we need to compute $\tau(trt^{-1})$ for each $t \in T$ and $r \in R$. It is straightforward to compute the following:

$$\begin{aligned}
\tau(v^2) &= v^2\overline{v^2}^{-1} = v^2 = s(1,v)^2 \text{ if } v \neq z, \\
\tau(zv^2z^{-1}) &= zv^2z^{-1}\overline{zv^2z^{-1}}^{-1} = zv^2z^{-1} = s(z,v)^2 \text{ if } v \neq z, \\
\tau(z^2) &= z^2\overline{z^2}^{-1} = z^2 = s(z,z), \\
\tau(zz^2z^{-1}) &= z^2\overline{z^2}^{-1} = z^2 = s(z,z), \\
\tau([u,v]) &= [u,v]\overline{[u,v]}^{-1} = [u,v] = [s(1,u), s(1,v)] \\
&\quad \text{if } u, v \neq z \text{ and } u,\ v \text{ areadjacent in } \Gamma, \\
\tau(z[u,v]z^{-1}) &= z[u,v]z^{-1}\overline{z[u,v]z^{-1}}^{-1} = z[u,v]z^{-1} = [s(z,u), s(z,v)] \\
&\quad \text{if } u, v \neq z\text{and } u,\ v \text{ are adjacent in } \Gamma\ , \\
\tau([z,w]) &= [z,w]\overline{[z,w]}^{-1} = [z,w] = s(z,w)s(1,w)^{-1} \\
&\quad \text{if } w \in link(z) \text{ in } \Gamma, \\
\tau(z[z,w]z^{-1}) &= z[z,w]z^{-1}\overline{z[z,w]z^{-1}}^{-1} = z[z,w]z^{-1} = z^2wz^{-1}w^{-1}z^{-1} \\
&= z^2wz^{-2}zw^{-1}z^{-1} \\
&= s(z,z)s(1,w)s(z,z)^{-1}s(z,w)^{-1} \\
&\quad \text{if } w \in link(z) \text{ in } \Gamma.
\end{aligned}$$

Since $s(z,z) = \tau(z^2)$, $s(z,z) = 1$ in $\ker \phi$. Thus, the last equality implies that $s(1,w) = s(z,w)$ in $\ker \phi$ if $w \in link(z)$. Therefore, $\ker \phi$ has the presentation in which generators are $s(1,v), s(z,v)$ with $v \in V(\Gamma) - \{z\}\}$, and relators are $s(1,v)^2, s(z,v)^2$, $[s(1,u_1), s(1,u_2)]$, $[s(z,u_1), s(z,u_2)]$, $s(z,w)s(1,w)^{-1}$ where $v, u_1, u_2 \neq z$, and u_1, u_2 are adjacent, and $w \in link(z)$. The conclusion in the proposition follows. □

By using Lemma 2.3, Bell gave another proof of Proposition 2.1. Likewise, Proposition 2.4 gives rise to a similar result for right-angled Coxeter group.

Theorem 2.5. *For any co-contraction $\widehat{\Gamma}$ of Γ, there is a monomorphism from $C(\widehat{\Gamma})$ to $C(\Gamma)$.*

Proof. Let a and b be two non-adjacent vertices on Γ. And let $\widehat{\Gamma}$ be the co-contraction of Γ relative to a and b. Let $v_{a,b}$ be the new vertex in $\widehat{\Gamma}$. Consider the homomorphism $\phi : C(\Gamma) \longrightarrow \langle x \mid x^2 = 1\rangle \cong \mathbb{Z}_2$ by $\phi(a) = x$ and $\phi(v) = 1$ if $v \neq a$. By Proposition 2.4, $\ker\phi \leq C(\Gamma)$ is the right-angled Coxeter group $C(\Gamma'')$ whose defining graph Γ'' is the double of $\Gamma - \{a\}$ along $link(a)$. It is well-known that if Γ_1 is an induced subgraph of Γ_2, then $C(\Gamma_1)$ is isomorphic to a subgroup of $C(\Gamma_2)$. Therefore, it suffices to show that Γ'' has an induced subgraph which is isomorphic to $\widehat{\Gamma}$. Let Γ' be a copy of Γ and a' be the corresponding vertex to a in Γ'. Then Γ'' can be regarded as the graph amalgamation of the induced subgraph generated by $V(\Gamma) - \{a\}$ in Γ and the induced subgraph generated by $V(\Gamma') - \{a'\}$ in Γ' along $link(a) = link(a')$. For each $v \in V(\Gamma) - star(a) \subset V(\Gamma'')$, we denote the corresponding vertex of v in the copy $V(\Gamma') - star(a')$ by v'. Note that $b \in V(\Gamma) - star(a)$ and so $b' \in V(\Gamma') - star(a')$ in $V(\Gamma'')$. Consider the induced subgraph T of Γ'' generated by $V(T) := (V(\Gamma) - \{a, b\}) \cup \{b'\}$. Let $w \in V(T) - \{b'\}$. w is adjacent to b' in T if and only if $w \in link(a)$ and w is adjacent to b in Γ, that is, w is adjacent to both a and b in Γ. It is easy to see that T is isomorphic to $\widehat{\Gamma}$. In fact, sending b' to $v_{a,b}$ and w to w for $w \neq a, b$ defines an isomorphism $T \longrightarrow \widehat{\Gamma}$. □

3. Negative examples for Gordon, Long, and Reid's question

By a hyperbolic surface group, we mean the fundamental group of a closed orientable surface of genus n for some $n \geq 2$. Let $\mathrm{Isom}(H^2)$ denote the group of isometries of H^2. Consider a regular n-gon in H^2 with all its vertex angles equal to $\frac{\pi}{2}$. It is known that the subgroup of $\mathrm{Isom}(H^2)$ generated by reflections in the sides of the regular n-gon is isomorphic to the right-angled Coxeter group $C(C_n)$ ([5]). In [12], Scott showed that the subgroup of $\mathrm{Isom}(H^2)$ generated by reflections in the sides of the regular pentagon contains a subgroup of index four which is isomorphic to the fundamental group of the non-orientable closed surface with Euler number -1. Therefore, we see that $C(C_5)$ contains a subgroup of index eight which is isomorphic to the fundamental group of the orientable closed surface of genus 2 ([11]). Using co-contraction, it can be seen that for each $n \geq 5$ the right-angled Coxeter group $C(C_n)$ contains $C(C_5)$, and so $C(C_n)$ contains a hyperbolic surface subgroup. Therefore for any graph Γ containing an induced n-cycle, $C(\Gamma)$ contains a hyperbolic surface group. In [5], Gordon, Long, and Reid asked if the converse holds while they proved that a word-hyperbolic (not necessarily right-angled) Coxeter group either is torsion-free or contains a hyperbolic surface group.

Question 3.1 ([5]). *Does a right-angled Coxeter group $C(\Gamma)$ contain a hyperbolic surface subgroup if and only if Γ contains an induced n-cycle for some $n \geq 5$?*

In this section, we build up examples which provide a negative answer to this question using Theorem 2.5. First, note that for each $n \geq 3$, the anti n-cycle $\overline{C_n}$ is the co-contraction of $\overline{C_{n+1}}$ relative to a pair of non-adjacent vertices, which is adjacent in C_{n+1}, and so $C(\overline{C_n})$ is isomorphic to a subgroup of $C(\overline{C_{n+1}})$. Since the 5-cycle is equal to its complement graph, that is, $C_5 = \overline{C_5}$, $C(\overline{C_n})$ contains $C(C_5)$ as a subgroup for each $n \geq 5$. In fact, it is easy to see that $C(C_5) = C(\overline{C_5}) \leq C(\overline{C_6}) \leq C(\overline{C_7}) \leq \cdots$. Therefore, if Γ contains an induced C_n or $\overline{C_n}$ for some $n \geq 5$, then $C(\Gamma)$ contains a hyperbolic surface subgroup.

Proposition 3.2. *If Γ contains an induced C_n or $\overline{C_n}$ for some $n \geq 5$, then $C(\Gamma)$ contains a hyperbolic surface subgroup.*

For $n \geq 6$, $\overline{C_n}$ does not contain induced $C_m, m \geq 5$. In fact, C_n does not contain an induced $\overline{C_m}$ as every vertex of $\overline{C_m}$ has valence greater than 2 if $m \geq 5$. It can be seen easily that a graph Γ_1 is an induced subgraph of a graph Γ_2 if and only if $\overline{\Gamma_1}$ is an induced subgraph of $\overline{\Gamma_2}$. Hence $\overline{C_n}$ does not contain an induced C_m. It follows that $C(\overline{C_n})$ $(n \geq 6)$ is a right-angled Coxeter group containing a hyperbolic surface group, while its underlying graph does not contain any induced n-cycle for $n \geq 5$.

Corollary 3.3. *There is a right-angled Coxeter group $C(\Gamma)$ containing a hyperbolic surface subgroup, while its underlying graph Γ does not contain any induced n-cycle for $n \geq 5$.*

This corollary gives a negative answer to Question 3.1. It is known that for each $n \geq 5$ the right-angled Artin group $A(C_n)$ contains a hyperbolic surface subgroup ([13]). The above argument about anti n-cycles and their co-contractions was originally used by Kim [7, 8] to prove the following proposition.

Proposition 3.4 ([7, 8])**.** *If Γ contains an induced C_n or $\overline{C_n}$ for some $n \geq 5$, then $A(\Gamma)$ contains a hyperbolic surface subgroup.*

This proposition answered negatively to the right-angle Artin group version of Question 3.1, which was also raised by Gordon, Long, and Reid in [5]: Does a right-angled Artin group $A(\Gamma)$ contain a hyperbolic surface subgroup if and only if Γ contains an induced n-cycle for some $n \geq 5$?

It might be interesting to give different kinds of examples which justifies Corollary 3.3. We use the results of Kim [7, 8] and Davis, Januszkiewicz [4] to construct such examples. Given a graph Γ, construct a graph Γ'' with the vertex set $V(\Gamma'') = V(\Gamma) \times \{0, 1\}$ in the following way:

- Two vertices $(a, 1)$, $(b, 1)$ in $V(\Gamma) \times \{0, 1\}$ are connected by an edge in Γ'' if and only if $(a, b) \in E(\Gamma)$.
- $V(\Gamma) \times \{0\}$ forms a complete subgraph in Γ''.
- $(a, 0)$ and $(b, 1)$ are connected by an edge if and only if $a \neq b$.

Davis and Januszkiewicz [4] showed that $A(\Gamma)$ is a normal subgroup of index $2^{|V(\Gamma)|}$ in $C(\Gamma'')$. So if $A(\Gamma)$ contains a hyperbolic surface subgroup, then $C(\Gamma'')$ contains a hyperbolic surface subgroup.

Let $n \geq 6$. We claim that Γ'' does not contain an induced m-cycle for all $m \geq 5$, when $\Gamma = \overline{C_n}$. Suppose not. Let C_m be an induced m-cycle for some m in Γ''. Since $\overline{C_n}$ with $n \geq 6$ does not contain an induced m-cycle for any $m \geq 5$, there must be at least one vertex of C_m in $V(\Gamma) \times \{0\}$. But since $V(\Gamma) \times \{0\}$ forms a complete subgraph in Γ'', there can be at most two vertices of C_m in $V(\Gamma) \times \{0\}$.

First, suppose there are two vertices of C_m in $V(\Gamma) \times \{0\}$. Let $(A, 0)$ and $(B, 0)$ be such points. Since $m \geq 5$, there are at least three vertices of C_m in $V(\Gamma) \times \{1\}$. It follows that there is a vertex $(P, 1)$ of C_m in $V(\Gamma) \times \{1\}$ with $P \neq A, B$. The way we construct Γ'' forces the induced cycle C_m to contain a 3-cycle whose vertices are $(A, 0)$, $(B, 0)$ and $(P, 1)$. For the case where there is only one vertex $(A, 0)$ of C_m in $V(\Gamma) \times \{0\}$, note that there should be more than three vertices of C_m in $V(\Gamma) \times \{1\}$, since $m \geq 5$. Among these vertices, we can choose three vertices $(P, 1), (Q, 1), (R, 1)$ with $A \neq P, Q, R$, which means the vertex $(A, 0)$ has valence greater than 2 in C_m. This is a contradiction. This proves our claim.

Proposition 3.4 says that the right-angled Artin group $A(\overline{C_n})$ $(n \geq 5)$ contains a hyperbolic surface subgroup, thereby $C(\overline{C_n}'')$ has a hyperbolic surface subgroup although $\overline{C_n}''$ does not contain any induced m-cycle for $m \geq 5$. This justifies Corollary 3.3 again.

Now we describe another example which ensures Corollary 3.3 is true. Given a graph Γ, we define a graph Γ' as follows: The vertex set of Γ' is $V(\Gamma) \times \{-1, 1\}$. The vertices $(a, -1)$, $(b, 1)$ in $V(\Gamma) \times \{-1, 1\}$ are connected by an edge in Γ' if and only if $a \neq b$ and $(a, b) \in E(\Gamma)$. Davis and Januszkiewicz [4] showed that $C(\Gamma')$ is a normal subgroup of index $2^{|V(\Gamma)|}$ in $C(\Gamma'')$.

Proposition 3.5. $A(\Gamma)$ *contains a hyperbolic surface subgroup if and only if* $C(\Gamma')$ *contains a hyperbolic surface subgroup.*

Proof. It is known that any finite index subgroup of a hyperbolic surface group is isomorphic to a hyperbolic surface group. Suppose that $A(\Gamma)$ contains a hyperbolic surface subgroup Q. Then $C(\Gamma'')$ contains a hyperbolic surface subgroup Q. Since $C(\Gamma')$ is a finite index subgroup of $C(\Gamma'')$, $C(\Gamma') \cap Q$ is a finite index subgroup of Q, so is isomorphic to a hyperbolic surface subgroup. Hence, $C(\Gamma')$ contains a hyperbolic surface subgroup. The same argument holds for the converse, as $A(\Gamma)$ is a finite index normal subgroup of $C(\Gamma'')$ □

An argument similar to the case of $\overline{C_n}''$ shows that $\overline{C_n}'$ with $n \geq 6$ does not contain an induced m-cycle for all $m \geq 5$. But $C(\overline{C_n}')$ contains a hyperbolic surface subgroup.

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