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HARNACK INEQUALITY FOR A NONLINEAR PARABOLIC EQUATION UNDER GEOMETRIC FLOW

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Abstract. In this paper, we obtain some gradient estimates for positive solutions to the following nonlinear parabolic equation

$$\frac{\partial u}{\partial t} = \triangle u - b(x,t)u^{\sigma}$$

under general geometric flow on complete noncompact manifolds, where $0 < \sigma < 1$ is a real constant and $b(x,t)$ is a function which is C^2 in the x-variable and C^1 in the t-variable. As an application, we get an interesting Harnack inequality.

1. Introduction

In this paper, we shall study the following nonlinear parabolic equation

$$\frac{\partial u}{\partial t} = \triangle u - b(x,t)u^{\sigma} \tag{1.1}$$

under general geometric flow on complete noncompact manifolds M, where $0 < \sigma < 1$ is a real constant and $b(x,t)$ is a function which is C^2 in the x-variable and C^1 in the t-variable.

Gradient estimates have played an important role in the study of PDE, especially the Laplace equation and the heat equation. Zhang and Ma [9] considered gradient estimates for positive solutions to the following nonlinear equation

$$\triangle_f u + cu^{-\alpha} = 0, \ \alpha > 0 \tag{1.2}$$

on complete noncompact manifolds, where $\triangle_f = \triangle - \nabla f \cdot \nabla$. When N is finite and the N-Bakry-*Émery* Ricci tensor is bounded from below, the authors [9] got the following result.

Theorem 1.1 (Zhang and Ma [9])**.** *Let (M,g) be a complete noncompact n-dimensional Riemannian manifold with N-Bakry-Émery Ricci tensor bounded from below by the constant $-K =: -K(2R)$, where $R > 0$ and $K(2R) > 0$ in*

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the metric ball $B_p(2R)$ around $p \in M$. Let u be a positive solution of (1.2). *Then*

(1) *if $c > 0$, we have*

$$\frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} \leq \frac{(N+n)(N+n+2)c_1^2}{R^2} + \frac{(N+n)[(N+n-1)c_1 + c_2]}{R^2} + \frac{(N+n)\sqrt{(N+n)K}c_1}{R} + 2(N+n)K.$$

(2) *if $c < 0$, we have*

$$\begin{aligned}&\frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} \\ &\leq (A + \sqrt{A})|c|(\inf_{B_p(2R)} u)^{-\alpha-1} + \frac{(N+n)[(N+n-1)c_1 + c_2]}{R^2} \\ &\quad + \frac{(N+n)c_1^2}{R^2}(n + N + 2 + \frac{n+N}{2\sqrt{A}}) + \frac{(N+n)\sqrt{(N+n)K}c_1}{R} \\ &\quad + (2 + \frac{1}{\sqrt{A}})(n+N)K,\end{aligned}$$

where $A = (N+n)(\alpha+1)(\alpha+2)$ and c_1, c_2 are absolute positive constants.

For the parabolic equation, Li and Yau [5] developed the fundamental gradient estimate, which is now widely called the Li-Yau estimate, for any positive solution $u(x,t)$ of the heat equation on a Riemannian manifold M and showed how the classical Harnack inequality can be derived from their gradient estimate. Later, Hamiton [3] got the matrix Harnack estimate for the heat equation. Recently, Zhu [10] investigated the nonlinear parabolic equation

$$u_t = \triangle u + \lambda(x,t)u^\alpha(x,t), \tag{1.3}$$

where $0 < \alpha < 1$ and $\lambda(x,t)$ is a function defined on $M \times (-\infty, 0]$, which is C^1 in the first variable and C^0 in the second variable. The author got a Hamilton-type estimate and a Liouville-type theorem for positive solutions of (1.3).

Theorem 1.2 (Zhu [10])**.** *Let M be a Riemannian manifold of dimension $n \geq 2$ with $Ric(M) \geq -k$ for some $k \geq 0$. Suppose that u is a positive solution to the equation* (1.3) *in $Q_{R,T} \equiv B(x_0,R) \times [t_0 - T, t_0] \subset M \times (-\infty,\infty)$. Suppose also that $u \leq \widetilde{M}$ and $\frac{|\nabla\lambda|^2}{\lambda} \leq \theta$ in $Q_{R,T}$. Then there exists a constant $C = C(\alpha, \widetilde{M})$ such that*

$$\frac{|\nabla u|}{u} \leq C\widetilde{M}^{1-\alpha}(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k}) + C\theta^{\frac{1}{4}}\widetilde{M}^{\frac{2}{3}(1-\alpha)}$$

in $Q_{\frac{R}{2},\frac{T}{2}}$.

But in the above work, the authors considered the gradient estimates for positive solutions to nonlinear equation on noncompact manifolds with fixed metric, it is natural to ask how the gradient estimates vary if the metric on manifolds evolves with the time. Perelman [7] showed a gradient estimate for the fundamental solution of the conjugate heat equation

$$\triangle u - Ru + \partial_t u = 0,$$

under Ricci flow on a closed Riemannian manifold M, where R is the scalar curvature. Kuang and Zhang [4] established a point-wise gradient estimate for all positive solutions of the conjugate heat equation under Ricci flow. Liu [6] got first order gradient estimates for positive solutions of the heat equations on complete noncompact or closed Riemannian manifolds under Ricci flow and the author derived Harnack inequalities and second order gradient estimates for positive solutions of the heat equations under this flow. Later, Sun [8] extended these results to general geometric flow.

In this paper, we will study the interesting Li-Yau type estimate for the positive solutions of (1.1) on complete noncompact manifolds with evolving metric.

We present our main results about the equation (1.1) as follows.

Theorem 1.3. *Let $(M, g(t))$ be a smooth one-parameter family of complete Riemannian manifolds evolving by*

$$\frac{\partial}{\partial t} g = 2h \tag{1.4}$$

for t in some time interval $[0,T]$. Let M be complete under the initial metric $g(0)$. Given $x_0 \in M$, and $M_1, R > 0$, let u be a positive solution to the nonlinear equation (1.1) with $u \le M_1$ in the cube $Q_{2R,T} := \{(x,t) \mid d(x,x_0,t) \le 2R, 0 \le t \le T\}$. Suppose that there exist constants $K_1, K_2, K_3, K_4 \ge 0$ such that

$$Ric \ge -K_1 g, \ -K_2 g \le h \le K_3 g, \ |\nabla h| \le K_4$$

on $Q_{2R,T}$. Moreover, assume that $\triangle b \le \theta$ and $|\nabla b| \le \gamma$ in $Q_{2R,T}$ for some positive constants θ and γ. Then for any constant $0 < \beta < 1$ and $(x,t) \in Q_{R,T}$, if $\beta < \sigma < 1$, we have

$$\beta \frac{|\nabla u|^2}{u^2} - bu^{\sigma-1} - \frac{u_t}{u} \le H_1 + H_2 + \frac{n}{\beta}\frac{1}{t},$$

where

$$\begin{aligned} H_1 = {} & \frac{n}{\beta}\left(\frac{(n-1)(1+\sqrt{K_1}R)c_1^2 + c_2 + 2c_1^2}{R^2} + \sqrt{c_3}K_2 + |b|(1-\sigma)M_1^{\sigma-1} \right. \\ & \left. + \frac{nc_1^2}{2\beta(1-\beta)R^2}\right), \\ H_2 = {} & \left[\frac{n^2}{4\beta^2(1-\beta)^2}(|b|(\sigma-1)(\beta-\sigma)M_1^{\sigma-1} + 2(\sigma-\beta)M_1^{\sigma-1} + 2(1-\beta)K_3 \right. \\ & + 2\beta K_1 + \frac{3}{2}K_4) + \frac{n}{\beta}(M_1^{\sigma-1}\theta + n(\frac{1}{\beta}(K_2+K_3)^2 + \frac{3}{2}K_4) \\ & \left. + 2(\sigma-\beta)M_1^{\sigma-1}\gamma^2)\right]^{\frac{1}{2}}, \end{aligned}$$

and $0 < \beta < 1$, c_1, c_2, c_3 *are positive constants.*

Let $R \to \infty$, we can get the global Li-Yau type gradient estimates for the nonlinear parabolic equation (1.1).

Corollary 1.4. *Let* $(M, g(0))$ *be a complete noncompact Riemannian manifold without boundary, and let* $g(t)$ *evolve by* (1.4) *for* $t \in [0, T]$ *and satisfy*

$$Ric \geq -K_1 g, \ -K_2 g \leq h \leq K_3 g, \ |\nabla h| \leq K_4.$$

Moreover, assume that $\triangle b \leq \theta(M)$ *and* $|\nabla b| \leq \gamma(M)$ *in* $M \times [0, T)$ *for some constants* θ *and* γ. *Let* u *be a positive solution of* (1.1) *with* $u \leq M_1$. *Then for any constant* $0 < \beta < 1$, *if* $\beta < \sigma < 1$, *we have*

$$\beta \frac{|\nabla u|^2}{u^2} - bu^{\sigma-1} - \frac{u_t}{u} \leq \overline{H_1} + H_2 + \frac{n}{\beta}\frac{1}{t},$$

where

$$\overline{H_1} = \sqrt{c_3} K_2 + \frac{n}{\beta}|b|(1-\sigma) M_1^{\sigma-1},$$

As an application, we get the following Harnack inequality.

Theorem 1.5. *Let* $(M, g(0))$ *be a complete noncompact Riemannian manifold without boundary, and let* $g(t)$ *evolve by* (1.4) *for* $t \in [0, T)$ *and satisfy*

$$Ric \geq -K_1 g, \ -K_2 g \leq h \leq K_3 g, \ |\nabla h| \leq K_4.$$

Assume that $\triangle b \leq \theta(M)$ *and* $|\nabla b| \leq \gamma(M)$ *in* $M \times [0, T)$ *for some constants* θ *and* γ. *Let* $u(x,t)$ *be a positive smooth solution to the equation*

$$u_t = \triangle u - bu^\sigma$$

on $M \times [0, T)$, *where* b *is a nonpositive constant. Then if* $\beta < \sigma < 1$, *for any points* (x_1, t_1) *and* (x_2, t_2) *on* $M \times [0, T)$ *with* $0 < t_1 < t_2$, *we have the following Harnack inequality*:

$$u(x_1, t_1) \leq u(x_2, t_2) (\frac{t_2}{t_1})^{\frac{n}{\beta}} e^{\phi(x_1, x_2, t_1, t_2) + \overline{H_1} + H_2 (t_2 - t_1)},$$

where $\phi(x_1, x_2, t_1, t_2) = \inf\limits_{\gamma} \int_{t_1}^{t_2} \frac{1}{4\beta} |\dot{\gamma}|^2 dt$, $\overline{H_1} = \sqrt{c_3} K_2 + \frac{n}{\beta}|b|(1-\sigma) M_1^{\sigma-1}$, *and*

$$\begin{aligned} H_2 = \ & \Big[\frac{n^2}{4\beta^2 (1-\beta)^2} (|b|(\sigma - 1)(\beta - \sigma) M_1^{\sigma-1} + 2(\sigma - \beta) M_1^{\sigma-1} + 2(1-\beta) K_3 \\ & + 2\beta K_1 + \frac{3}{2} K_4) + \frac{n}{\beta} (M_1^{\sigma-1} \theta + n (\frac{1}{\beta} (K_2 + K_3)^2 + \frac{3}{2} K_4) \\ & + 2(\sigma - \beta) M_1^{\sigma-1} \gamma^2) \Big]^{\frac{1}{2}}. \end{aligned}$$

2. Proof of Theorem 1.3

Let u be a positive solution to (1.1). Set $w = \ln u$, then w satisfies the equation

$$w_t = \triangle w + |\nabla w|^2 - be^{(\sigma-1)w}. \tag{2.1}$$

In order to prove the main Theorem 1.3, we need the following lemmas.

Lemma 2.1 (Sun [8])**.** *Suppose the metric evolves by* (1.4)*. Then, for any smooth function* w*, we have*

$$\frac{\partial}{\partial t}|\nabla w|^2 = -2h(\nabla w, \nabla w) + 2\nabla w \nabla w_t$$

and

$$\frac{\partial}{\partial t}\triangle w = \triangle \frac{\partial}{\partial t} w - 2h\nabla^2 w - 2\nabla w(\text{div}h - \frac{1}{2}\nabla(tr_g h)),$$

here div$h$ *is the divergence of* h*.*

Lemma 2.2. *Assume that* $(M, g(t))$ *satisfies the hypotheses of Theorem 1.3, we have*

$$\begin{aligned}
&(\triangle - \frac{\partial}{\partial t})F \\
\geq & -2\nabla w \nabla F + t\{\frac{\beta}{n}(|\nabla w|^2 - be^{(\sigma-1)w} - w_t)^2 \\
&+ [b(\sigma-1)(\beta-\sigma)e^{(\sigma-1)w} + 2(\beta-1)K_3 - 2\beta K_1 - \frac{3}{2}K_4]|\nabla w|^2 \\
&+ 2(\beta-\sigma)e^{(\sigma-1)w}\nabla w \nabla b - e^{(\sigma-1)w}\triangle b - n[\frac{1}{\beta}(K_2+K_3)^2 + \frac{3}{2}K_4]\} \\
&+ b(\sigma-1)e^{(\sigma-1)w}F - \frac{F}{t},
\end{aligned}$$

where

$$F = t(\beta|\nabla w|^2 - be^{(\sigma-1)w} - w_t),$$

and β *is any constant satisfying* $0 < \beta < 1$*.*

Proof. Define

$$F = t(\beta|\nabla w|^2 - be^{(\sigma-1)w} - w_t).$$

By the Bochner formula, we have

$$\triangle|\nabla w|^2 \geq 2|\nabla^2 w|^2 + 2\nabla w \nabla(\triangle w) - 2K_1|\nabla w|^2.$$

Noting that

$$\begin{aligned}
\triangle w_t &= (\triangle w)_t + 2h\nabla^2 w + 2\nabla w(\text{div}h - \frac{1}{2}\nabla(tr_g h)) \\
&= -(|\nabla w|^2)_t + b_t e^{(\sigma-1)w} + b(\sigma-1)e^{(\sigma-1)w}w_t + w_{tt} \\
&\quad + 2h\nabla^2 w + 2\nabla w(\text{div}h - \frac{1}{2}\nabla(tr_g h)) \\
&= 2h(\nabla w, \nabla w) - 2\nabla w \nabla w_t + b_t e^{(\sigma-1)w} + b(\sigma-1)e^{(\sigma-1)w}w_t + w_{tt}
\end{aligned}$$

$$+ 2h\nabla^2 w + 2\nabla w(\mathrm{div} h - \frac{1}{2}\nabla(tr_g h))$$

and

$$\begin{aligned}\triangle w &= -|\nabla w|^2 + be^{(\sigma-1)w} + w_t \\ &= (1-\frac{1}{\beta})(be^{(\sigma-1)w} + w_t) - \frac{F}{\beta t} \\ &= (\beta-1)|\nabla w|^2 - \frac{F}{t},\end{aligned}$$

we know,

$$\triangle F = t(\beta\triangle|\nabla w|^2 - \triangle(be^{(\sigma-1)w}) - \triangle w_t).$$

By the above computation, we obtain

$$\begin{aligned}\beta\triangle|\nabla w|^2 \geq\ & 2\beta|\nabla^2 w|^2 + 2\beta\nabla w\nabla(\triangle w) - 2\beta K_1|\nabla w|^2 \\ =\ & 2\beta|\nabla^2 w|^2 + 2\beta\nabla w\nabla[(1-\frac{1}{\beta})(be^{(\sigma-1)w} + w_t) - \frac{F}{\beta t}] \\ & - 2\beta K_1|\nabla w|^2 \\ =\ & 2\beta|\nabla^2 w|^2 + 2(\beta-1)e^{(\sigma-1)w}\nabla w\nabla b \\ & + 2b(\beta-1)(\sigma-1)e^{(\sigma-1)w}|\nabla w|^2 + 2(\beta-1)\nabla w\nabla w_t \\ & - \frac{2}{t}\nabla w\nabla F - 2\beta K_1|\nabla w|^2,\end{aligned}$$

and

$$\begin{aligned}\triangle(be^{(\sigma-1)w}) =\ & e^{(\sigma-1)w}\triangle b + 2(\sigma-1)e^{(\sigma-1)w}\nabla w\nabla b + b(\sigma-1)^2 e^{(\sigma-1)w}|\nabla w|^2 \\ & + b(\sigma-1)e^{(\sigma-1)w}\triangle w \\ =\ & e^{(\sigma-1)w}\triangle b + 2(\sigma-1)e^{(\sigma-1)w}\nabla w\nabla b + b(\sigma-1)^2 e^{(\sigma-1)w}|\nabla w|^2 \\ & + b(\sigma-1)e^{(\sigma-1)w}[(\beta-1)|\nabla w|^2 - \frac{F}{t}].\end{aligned}$$

So, we have

$$\begin{aligned}\triangle F \geq t\{ & 2\beta|\nabla^2 w|^2 + 2(\beta-1)e^{(\sigma-1)w}\nabla w\nabla b \\ & + 2b(\beta-1)(\sigma-1)e^{(\sigma-1)w}|\nabla w|^2 + 2(\beta-1)\nabla w\nabla w_t \\ & - \frac{2}{t}\nabla w\nabla F - 2\beta K|\nabla w|^2 - e^{(\sigma-1)w}\triangle b - 2(\sigma-1)e^{(\sigma-1)w}\nabla w\nabla b \\ & - b(\sigma-1)^2 e^{(\sigma-1)w}|\nabla w|^2 - b(\sigma-1)e^{(\sigma-1)w}[(\beta-1)|\nabla w|^2 - \frac{F}{t}] \\ & - (-(|\nabla w|^2)_t + 2h\nabla^2 w + 2\nabla w(\mathrm{div} h - \frac{1}{2}\nabla(tr_g h))) + b_t e^{(\sigma-1)w} \\ & + b(\sigma-1)e^{(\sigma-1)w}w_t + w_{tt})\}\end{aligned}$$

and

$$\begin{aligned}
&F_t \\
=&\ \frac{F}{t}+t(\beta(|\nabla w|^2)_t-b_te^{(\sigma-1)w}-b(\sigma-1)e^{(\sigma-1)w}w_t-w_{tt}) \\
=&\ \frac{F}{t}+t(2\beta\nabla w\nabla w_t-2\beta h(\nabla w,\nabla w)-b_te^{(\sigma-1)w}-b(\sigma-1)e^{(\sigma-1)w}w_t-w_{tt}).
\end{aligned}$$

This implies that

$$\begin{aligned}
&(\triangle-\frac{\partial}{\partial t})F \\
\geq&\ -2\nabla w\cdot\nabla F+t\{2\beta|\nabla^2 w|^2+2(\beta-1)h(\nabla w,\nabla w) \\
&+b(\sigma-1)(\beta-\sigma)e^{(\sigma-1)w}|\nabla w|^2+2(\beta-\sigma)e^{(\sigma-1)w}\nabla w\nabla b \\
&-e^{(\sigma-1)w}\triangle b-2\beta K_1|\nabla w|^2-2h\nabla^2 w-2\nabla w(\mathrm{div}h-\frac{1}{2}\nabla(tr_gh)))\} \\
&+b(\sigma-1)e^{(\sigma-1)w}F-\frac{F}{t}.
\end{aligned}$$

By our assumption, we have

$$-(K_2+K_3)g\leq h\leq(K_2+K_3)g,$$

which implies that

$$|h|^2\leq(K_2+K_3)^2|g|^2=n(K_2+K_3)^2.$$

Applying those bounds and Young's inequality yields

$$|h\nabla^2 w|\leq\frac{\beta}{2}|\nabla^2 w|^2+\frac{1}{2\beta}|h|^2\leq\frac{\beta}{2}|\nabla^2 w|^2+\frac{n}{2\beta}(K_2+K_3)^2.$$

On the other hand,

$$|\mathrm{div}h-\frac{1}{2}\nabla(tr_gh)|=|g^{ij}\nabla_ih_{jl}-\frac{1}{2}g^{ij}\nabla_lh_{ij}|\leq\frac{3}{2}|g||\nabla h|\leq\frac{3}{2}\sqrt{n}K_4.$$

Finally, with the help of the inequality

$$|\nabla^2 w|^2\geq\frac{1}{n}(tr\nabla^2 w)^2=\frac{1}{n}(\triangle w)^2=\frac{1}{n}(|\nabla w|^2-be^{(\sigma-1)w}-w_t)^2,$$

we get

$$\begin{aligned}
(\triangle-\frac{\partial}{\partial t})F\geq&\ -2\nabla w\nabla F+t\{\frac{\beta}{n}(|\nabla w|^2-be^{(\sigma-1)w}-w_t)^2 \\
&+b(\sigma-1)(\beta-\sigma)e^{(\sigma-1)w}|\nabla w|^2+2(\beta-\sigma)e^{(\sigma-1)w}\nabla w\nabla b \\
&-e^{(\sigma-1)w}\triangle b+2(\beta-1)K_3|\nabla w|^2-2K_1\beta|\nabla w|^2 \\
&-\frac{n}{\beta}(K_2+K_3)^2-3\sqrt{n}K_4|\nabla w|\}+b(\sigma-1)e^{(\sigma-1)w}F-\frac{F}{t}.
\end{aligned}$$

Since

$$3\sqrt{n}K_4|\nabla w|\leq 3K_4(\frac{n}{2}+\frac{|\nabla w|^2}{2}),$$

we get

$$
\begin{aligned}
&(\triangle-\frac{\partial}{\partial t})F\\
\geq\ & -2\nabla w\nabla F+t\{\frac{\beta}{n}(|\nabla w|^2-be^{(\sigma-1)w}-w_t)^2\\
&+[b(\sigma-1)(\beta-\sigma)e^{(\sigma-1)w}+2(\beta-1)K_3-2\beta K_1-\frac{3}{2}K_4]|\nabla w|^2\\
&+2(\beta-\sigma)e^{(\sigma-1)w}\nabla w\nabla b-e^{(\sigma-1)w}\triangle b-n[\frac{1}{\beta}(K_2+K_3)^2+\frac{3}{2}K_4]\}\\
&+b(\sigma-1)e^{(\sigma-1)w}F-\frac{F}{t}.
\end{aligned}
$$

We complete the proof of Lemma 2.2. □

We take a C^2 cut-off function $\tilde{\varphi}$ defined on $[0,\infty)$ such that $\tilde{\varphi}(r)=1$ for $r\in[0,1]$, $\tilde{\varphi}(r)=0$ for $r\in[2,\infty)$, and $0\leq\tilde{\varphi}(r)\leq 1$. Furthermore $\tilde{\varphi}$ satisfies

$$-\frac{\tilde{\varphi}'(r)}{\tilde{\varphi}^{\frac{1}{2}}(r)}\leq c_1$$

and

$$\tilde{\varphi}''(r)\geq -c_2$$

for some absolute constants $c_1,c_2>0$. Set

$$\varphi(x,t)=\tilde{\varphi}(\frac{r(x,t)}{R}),$$

where $r(x,t)=d(x,x_0,t)$. Using an argument of Calabi [2], we can assume $\varphi(x,t)\in C^2(M)$ with support in $Q_{2R,T}$. Direct calculation shows that on $Q_{2R,T}$

$$\frac{|\nabla\varphi|^2}{\varphi}\leq\frac{c_1^2}{R^2}. \tag{2.2}$$

By the Laplace comparison theorem in [1],

$$\triangle\varphi\geq-\frac{(n-1)(1+\sqrt{K_1}R)c_1^2+c_2}{R^2}. \tag{2.3}$$

For any $0<T_1<T$, let (x_0,t_0) be a point in the cube Q_{2R,T_1}, at which φF attains its maximum value. we can assume that this value is positive (otherwise the proof is trivial). At the point (x_0,t_0), we have

$$\nabla(\varphi F)=0,\triangle(\varphi F)\leq 0,(\varphi F)_t\geq 0.$$

It follows that

$$0\geq(\triangle-\frac{\partial}{\partial t})(\varphi F)=(\triangle\varphi)F-\varphi_t F+\varphi(\triangle-\frac{\partial}{\partial t})F+2\nabla\varphi\nabla F.$$

By Sun [8] on p. 494, we know there exists a positive constant c_3 such that

$$-\varphi_t F\geq-\sqrt{c_3}K_2F.$$

So we obtain

$$\varphi(\triangle - \frac{\partial}{\partial t})F + F\triangle\varphi - \varphi_t F - 2F\varphi^{-1}|\nabla\varphi|^2 \le 0.$$

This inequality together with the inequalities (2.2) and (2.3) yield

$$\varphi(\triangle - \frac{\partial}{\partial t})F \le HF,$$

where

$$H = \frac{(n-1)(1+\sqrt{K_1}R)c_1^2 + c_2 + 2c_1^2}{R^2} + \sqrt{c_3}K_2.$$

At (x_0, t_0), by Lemma 2.2 and (2.4), we have

$$\begin{aligned}
0 \ge \varphi(\triangle - \frac{\partial}{\partial t})F - HF \ge -HF + \varphi\{-\frac{F}{t_0} + b(\sigma-1)e^{(\sigma-1)w}F \\
&+ \frac{\beta t_0}{n}(|\nabla w|^2 - be^{(\sigma-1)w} - w_t)^2 - 2\nabla w\cdot\nabla F \\
&+ t_0[b(\sigma-1)(\beta-\sigma)e^{(\sigma-1)w} + 2(\beta-1)K_3 - 2\beta K_1 - \frac{3}{2}K_4]|\nabla w|^2 \\
&+ 2t_0(\beta-\sigma)e^{(\sigma-1)w}\nabla w\nabla b - t_0 e^{(\sigma-1)w}\triangle b - nt_0[\frac{1}{\beta}(K_2+K_3)^2 + \frac{3}{2}K_4]\} \\
\ge\ & -HF - \varphi t_0^{-1}F + 2F\nabla w\nabla\varphi + \frac{\beta t_0}{n}\varphi(|\nabla w|^2 - be^{(\sigma-1)w} - w_t)^2 \\
&+ b(\sigma-1)e^{(\sigma-1)w}\varphi F + t_0\varphi[b(\sigma-1)(\beta-\sigma)e^{(\sigma-1)w} + 2(\beta-1)K_3 \\
&- 2\beta K_1 - \frac{3}{2}K_4]|\nabla w|^2 + 2t_0\varphi(\beta-\sigma)e^{(\sigma-1)w}\nabla w\nabla b - t_0\varphi e^{(\sigma-1)w}\triangle b \\
&- nt_0\varphi[\frac{1}{\beta}(K_2+K_3)^2 + \frac{3}{2}K_4] \\
\ge\ & -HF - \varphi t_0^{-1}F + 2F\nabla w\nabla\varphi + \frac{\beta t_0}{n}\varphi(|\nabla w|^2 - be^{(\sigma-1)w} - w_t)^2 \\
&+ |b|(\sigma-1)M_1^{\sigma-1}\varphi F - t_0\varphi[|b|(\sigma-1)(\beta-\sigma)M_1^{\sigma-1} + 2(1-\beta)K_3 \\
&+ 2\beta K_1 + \frac{3}{2}K_4]|\nabla w|^2 - t_0\varphi M_1^{\sigma-1}\theta - nt_0\varphi[\frac{1}{\beta}(K_2+K_3)^2 + \frac{3}{2}K_4] \\
&+ 2t_0\varphi(\beta-\sigma)M_1^{\sigma-1}|\nabla w|\gamma \\
\ge\ & -HF - \varphi t_0^{-1}F + 2F\nabla w\nabla\varphi + \frac{\beta t_0}{n}\varphi(|\nabla w|^2 - be^{(\sigma-1)w} - w_t)^2 \\
&+ |b|(\sigma-1)M_1^{\sigma-1}\varphi F - t_0\varphi[|b|(\sigma-1)(\beta-\sigma)M_1^{\sigma-1} + 2(\sigma-\beta)M_1^{\sigma-1} \\
&+ 2(1-\beta)K_3 + 2\beta K_1 + \frac{3}{2}K_4]|\nabla w|^2 - t_0\varphi[M_1^{\sigma-1}\theta \\
&+ n(\frac{1}{\beta}(K_2+K_3)^2 + \frac{3}{2}K_4) + 2(\sigma-\beta)M_1^{\sigma-1}\gamma^2].
\end{aligned}$$

Set

$$\widetilde{C_1} = |b|(\sigma-1)(\beta-\sigma)M_1^{\sigma-1} + 2(\sigma-\beta)M_1^{\sigma-1} + 2(1-\beta)K_3 + 2\beta K_1 + \frac{3}{2}K_4,$$

and

$$\widetilde{C_2} = M_1^{\sigma-1}\theta + n(\frac{1}{\beta}(K_2+K_3)^2 + \frac{3}{2}K_4) + 2(\sigma-\beta)M_1^{\sigma-1}\gamma^2.$$

Multiplying both sides of the above inequality by $t_0\varphi$, noting the fact that $0 < \varphi < 1$, we have

$$\begin{aligned} 0 \geq & -Ht_0\varphi F - \varphi F + 2t_0\varphi F\nabla w\nabla\varphi + |b|(\sigma-1)M_1^{\sigma-1}t_0\varphi F \\ & + \frac{\beta t_0^2}{n}\varphi^2(|\nabla w|^2 - be^{(\sigma-1)w} - w_t)^2 - \widetilde{C_1}t_0^2\varphi^2|\nabla w|^2 - \widetilde{C_2}t_0^2\varphi^2 \\ \geq & -Ht_0\varphi F - \varphi F - \frac{2c_1}{R}t_0\varphi F|\nabla w|\varphi^{\frac{3}{2}} + |b|(\sigma-1)M_1^{\sigma-1}t_0\varphi F \\ & + \frac{\beta t_0^2}{n}\varphi^2[(|\nabla w|^2 - be^{(\sigma-1)w} - w_t)^2 - \frac{n}{\beta}\widetilde{C_1}|\nabla w|^2] - \widetilde{C_2}t_0^2\varphi^2, \end{aligned}$$

where the last inequality used

$$-2\varphi\nabla w\nabla F = 2F\nabla w\nabla\varphi \geq -2F|\nabla w||\nabla\varphi| \geq -\frac{2c_1}{R}\varphi^{\frac{1}{2}}F|\nabla w|.$$

Let

$$y = \varphi|\nabla w|^2,\ z = \varphi(be^{(\sigma-1)w} + w_t).$$

It follows that

$$\begin{aligned} 0 \geq & \varphi F(-Ht_0 + |b|(\sigma-1)M_1^{\sigma-1}t_0 - 1) - \frac{2c_1}{R}t_0F|\nabla w|\varphi^{\frac{3}{2}} \\ & + \frac{\beta t_0^2}{n}\varphi^2[|\nabla w|^2 - \frac{n}{\beta}\widetilde{C_1}|\nabla w|^2] - \widetilde{C_2}t_0^2\varphi^2 \\ \geq & \varphi F(-Ht_0 + |b|(\sigma-1)M_1^{\sigma-1}t_0 - 1) \\ & + \frac{\beta t_0^2}{n}\{(y-z)^2 - \frac{n}{\beta}\widetilde{C_1}y - 2nc_1R^{-1}y^{\frac{1}{2}}(y - \frac{1}{\beta}z)\} - \widetilde{C_2}t_0^2. \end{aligned}$$

Using the inequality $ax^2 - bx \geq -\frac{b^2}{4a}$, valid for $a, b > 0$, one obtains

$$\begin{aligned} & \frac{\beta t_0^2}{n}\{(y-z)^2 - \frac{n}{\beta}\widetilde{C_1}y - 2\frac{nc_1}{R}y^{\frac{1}{2}}(y - \frac{z}{\beta})\} \\ = & \frac{\beta t_0^2}{n}\{\beta^2(y - \frac{z}{\beta})^2 + (1-\beta)^2y^2 - \frac{n}{\beta}\widetilde{C_1}y + (2(\beta-\beta^2)y - \frac{nc_1}{R}y^{\frac{1}{2}})(y - \frac{z}{\beta})\} \\ \geq & \frac{\beta t_0^2}{n}\{\beta^2(y - \frac{z}{\beta})^2 - \frac{n^2\widetilde{C_1}^2}{4\beta^2(1-\beta)^2} - \frac{n^2c_1^2}{2R^2(\beta-\beta^2)}(y - \frac{z}{\beta})\} \\ = & \frac{\beta}{n}(\varphi F)^2 - \frac{n\widetilde{C_1}^2t_0^2}{4\beta(1-\beta)^2} - \frac{nc_1^2t_0}{2R^2(\beta-\beta^2)}(\varphi F). \end{aligned}$$

Hence,

$$\frac{\beta}{n}(\varphi F)^2 + \left[-Ht_0 + |b|(\sigma-1)M_1^{\sigma-1}t_0 - 1 - \frac{nc_1^2 t_0}{2R^2(\beta-\beta^2)}\right](\varphi F)$$
$$- \frac{n\widetilde{C_1}^2 t_0^2}{4\beta(1-\beta)^2} - \widetilde{C_2}t_0^2 \le 0.$$

From the inequality $Ax^2 - 2Bx \le C$, we have $x \le \frac{2B}{A} + \sqrt{\frac{C}{A}}$.
We can get

$$\varphi F \le \frac{n}{\beta}\left(Ht_0 + |b|(1-\sigma)M_1^{\sigma-1}t_0 + 1 + \frac{nc_1^2 t_0}{2R^2(\beta-\beta^2)}\right)$$
$$+ \left[\frac{n}{\beta}\left(\frac{n\widetilde{C_1}^2}{4\beta(1-\beta)^2} + \widetilde{C_2}\right)\right]^{\frac{1}{2}} t_0.$$

If $d(x, x_0, T_1) < R$, we have $\varphi(x, T_1) = 1$. Then

$$\begin{aligned} F(x, T_1) &= T_1(\beta|\nabla w|^2 + ce^{-w(\alpha+1)} - w_t) \\ &\le \varphi F(x_0, t_0) \\ &\le \frac{n}{\beta}\left(Ht_0 + |b|(1-\sigma)M_1^{\sigma-1}t_0 + 1 + \frac{nc_1^2 t_0}{2R^2(\beta-\beta^2)}\right) \\ &\quad + [\frac{n}{\beta}(\frac{n\widetilde{C_1}^2}{4\beta(1-\beta)^2} + \widetilde{C_2})]^{\frac{1}{2}} t_0. \end{aligned}$$

As T_1 is arbitrary, we can get the case (1) of Theorem 1.3.

Proof of Theorem 1.5. For any points (x_1, t_1) and (x_2, t_2) on $M \times [0, T)$ with $0 < t_1 < t_2$, we take a curve $\gamma(t)$ parameterized with $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$. One gets from Corollary 1.4 that

$$\begin{aligned} &\log u(x_2, t_2) - \log u(x_1, t_1) \\ = &\int_{t_1}^{t_2} ((\log u)_t + \langle \nabla \log u, \dot{\gamma}\rangle)dt \\ \ge &\int_{t_1}^{t_2} (\beta|\nabla \log u|^2 - \frac{n}{\beta t} - bu^{\sigma-1} - \overline{H_1} - H_2 - |\nabla \log u||\dot{\gamma}|)dt \\ \ge &-\int_{t_1}^{t_2} (\frac{1}{4\beta}|\dot{\gamma}|^2 + \frac{n}{\beta t} + bu^{\sigma-1} + \overline{H_1} + H_2)dt \\ = &-\left(\int_{t_1}^{t_2} \frac{1}{4\beta}|\dot{\gamma}|^2 dt + \log(\frac{t_2}{t_1})^{\frac{n}{\beta}} + \overline{H_1} + H_2(t_2 - t_1)\right), \end{aligned}$$

which means that

$$\log \frac{u(x_1, t_1)}{u(x_2, t_2)} \le \int_{t_1}^{t_2} \frac{1}{4\beta}|\dot{\gamma}|^2 dt + \log(\frac{t_2}{t_1})^{\frac{n}{\beta}} + \overline{H_1} + H_2(t_2 - t_1).$$

Therefore,

$$u(x_1,t_1) \le u(x_2,t_2)(\frac{t_2}{t_1})^{\frac{n}{\beta}} e^{\phi(x_1,x_2,t_1,t_2)+\overline{H_1}+H_2(t_2-t_1)},$$

where $\phi(x_1,x_2,t_1,t_2) = \inf_{\gamma} \int_{t_1}^{t_2} \frac{1}{4\beta}|\dot{\gamma}|^2 dt$, $\overline{H_1} = \sqrt{c_3}K_2 + \frac{n}{\beta}|b|(1-\sigma)M_1^{\sigma-1}$, and

$$\begin{aligned} H_2 = \Big[&\frac{n^2}{4\beta^2(1-\beta)^2}(|b|(\sigma-1)(\beta-\sigma)M_1^{\sigma-1} + 2(\sigma-\beta)M_1^{\sigma-1} + 2(1-\beta)K_3 \\ &+ 2\beta K_1 + \frac{3}{2}K_4) + \frac{n}{\beta}(M_1^{\sigma-1}\theta + n(\frac{1}{\beta}(K_2+K_3)^2 + \frac{3}{2}K_4) \\ &+ 2(\sigma-\beta)M_1^{\sigma-1}\gamma^2)\Big]^{\frac{1}{2}}. \end{aligned}$$

□

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