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THE ALEKSANDROV PROBLEM AND THE MAZUR-ULAM THEOREM ON LINEAR n-NORMED SPACES

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Abstract. This paper generalizes the Aleksandrov problem and Mazur Ulam theorem to the case of n-normed spaces. For real n-normed spaces X and Y, we will prove that f is an affine isometry when the mapping satisfies the weaker assumptions that preserves unit distance, n-colinear and 2-colinear on same-order.

1. Introduction

Let X and Y be metric spaces. A mapping $f : X \to Y$ is called an isometry if f satisfies $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$, where $d_X(\cdot,\cdot)$ and $d_Y(\cdot,\cdot)$ denote the metrics in the spaces X and Y, respectively. For some fixed number $r > 0$, assume that f preserves the distance r, i.e., for all $x, y \in X$ with $d_X(x, y) = r$, it holds that $d_Y(f(x), f(y)) = r$. Then r is called a conservative (or preserved) distance for the mapping f.

In 1970, Aleksandrov [1] posed the following problem: *Examine whether the existence of a single conservative distance for some mapping T implies that T is an isometry.*

In 1932, Mazur and Ulam [8] proved a theorem that *every isometry of a real normed space onto a real normed space is a linear mapping up to a translation.*

In 1993, Rassias and Šemrl [10] proved a series of results on the DOPP (distance one preserving property) for the normed spaces (see also [6, 7, 9]). Since 2004, the Aleksandrov problem has been discussed in the n-normed spaces ($n \geq 2$) (see [2, 3, 4, 5]). Chu, Lee and Park proved:

Theorem 1.1 ([3]). *Let X and Y be two real linear n-normed spaces. If a mapping $f : X \to Y$ satisfies the following conditions*:

(1) *f has the n-DOPP*;
(2) *f is n-Lipschitz*;

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(3) *f preserves the m-collinearity for $m \in \{2, n\}$,*

then f is an n-isometry.

Theorem 1.2 ([3]). *Let X and Y be two real linear n-normed spaces. If a mapping $f : X \to Y$ satisfies the following conditions*:

(1) *f has the n-DOPP*;
(2) *f preserves n-colinear*;
(3) *f preserves 2-colinear on same order*;
(4) *f satisfies $(*)$,*

then f is an n-isometry.

Here, $(*)$: for every $x_0, x_1, \ldots, x_n \in X$ with $||x_1 - x_0, \ldots, x_n - x_0|| \neq 0$, there exists an $\omega \in X$ such that $||x_0 - \omega, x_1 - \omega, \ldots, x_{n-1} - \omega|| = 1$, and $||x_1 - \omega, x_2 - \omega, \ldots, x_n - \omega|| = 1$.

Theorem 1.3 ([5]). *Let X and Y are two real linear 2-normed spaces. If f satisfies 2-DOPP and preserves 2-colinear for x, y and $z \in X$ on same order, then f is a 2-isometry.*

In this paper, we generalize Theorems 1.1, 1.2 and 1.3 by providing that if a mapping f has the n-DOPP, preserves n-colinear and 2-colinear on same-order, then f is an n-isometry.

2. Terminologies

Definition 2.1 ([3]). Assume that X is a real linear space with $\dim X \geq n$ and $\|\cdot, \ldots, \cdot\| : X^n \to \mathbb{R}$ is a function. Then $(X, \|\cdot, \ldots, \cdot\|)$ is called a linear n-normed space provided that for any $\alpha \in \mathbb{R}$ and all $x_1, \ldots, x_n \in X$

(nN_1) $\|x_1, \ldots, x_n\| = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent;
(nN_2) $\|x_1, \ldots, x_n\| = \|x_{j_1}, \ldots, x_{j_n}\|$ for every permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$;
(nN_3) $\|\alpha x_1, \ldots, x_n\| = |\alpha| \|x_1, \ldots, x_n\|$;
(nN_4) $\|x + y, x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|y, x_2, \ldots, x_n\|$.

The function $\|\cdot, \ldots, \cdot\|$ is called the n-norm on X.

Definition 2.2 ([3]). Let X and Y be two real linear n-normed spaces. A mapping $f : X \to Y$ is an n-isometry if

$$||f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0)|| = ||x_1 - x_0, \ldots, x_n - x_0||$$

for all $x_0, x_1, \ldots, x_n \in X$.

Definition 2.3 ([3]). A mapping $f : X \to Y$ is said to have the n-distance one preserving property (n-DOPP) if for all $x_1, \ldots, x_n, x_0 \in X$, $\|x_1 - x_0, \ldots, x_n - x_0\| = 1$ implies $\|f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0)\| = 1$.

Definition 2.4 ([3]). A mapping $f : X \to Y$ is said to be n-Lipschitz if

$$||f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0)|| \leq ||x_1 - x_0, \ldots, x_n - x_0||$$

for all $x_0, x_1, \ldots, x_n, y_1, \ldots, y_n \in X$.

Definition 2.5 ([3])**.** The points $x_0, x_1, \dots, x_n$ of X are called n-collinear if for every i, $\{x_i - x_j \,:\, 0 \le j \ne i \le n\}$ is linearly dependent.

• A mapping $f : X \to Y$ is said to preserve the n-collinearity if n-collinearity of $f(x_0), f(x_1), \dots, f(x_n)$ follows from the n-collinearity of $x_0, x_1, \dots, x_n$. A mapping $f : X \to Y$ is said to preserve the 2-collinearity if for all $x, y, z \in X$, the existence of $t \in \mathbb{R}$ with $z - x = t(y - x)$ implies the existence of $s \in \mathbb{R}$ with $f(z) - f(x) = s(f(y) - f(x))$.

• A mapping $f : X \to Y$ is called 2-colinear preserving property on same order if $y - x = t(z - x)$ for $t \in (0, 1]$ implies that $f(y) - f(x) = s(f(z) - f(x))$ for $s \in (0, 1]$.

3. Generalization of Theorems 1.1 and 1.2

Lemma 3.1 ([3])**.** *Let X and Y be two real n-normed spaces such that* $\dim X \ge n$*. Then an n-isometry $f : X \to Y$ is affine and 2-colinear.*

Lemma 3.2. *Let X and Y be two real n-normed spaces such that* $\dim X \ge n$*. Suppose that $f : X \to Y$ satisfies n-DOPP and preserves 2-colinear. Then f preserves $\frac{1}{k}$ distance for each $k \in \mathbb{N}$.*

Proof. Suppose there exist $x_0, x_1 \in X$, with $x_0 \ne x_1$ such that $f(x_0) = f(x_1)$. As $\dim X \ge n$, there are $x_2, \dots, x_n \in X$ such that $x_1 - x_0, \dots, x_n - x_0$ are linearly independent. Thus $||x_1 - x_0, \dots, x_n - x_0|| \ne 0$. Set

$$z_2 := x_0 + \frac{x_2 - x_0}{||x_1 - x_0, \dots, x_n - x_0||}.$$

Then we have

$$||x_1 - x_0, z_2 - x_0, x_3 - x_0, \dots, x_n - x_0|| = 1$$

and

$$||f(x_1) - f(x_0), f(z_2) - f(x_0), f(x_3) - f(x_0), \dots, f(x_n) - f(x_0)|| = 1.$$

It follows $f(x_0) \ne f(x_1)$ that is a contradiction with $f(x_0) = f(x_1)$. Hence f is injective.

Let $x_0, x_1, \dots, x_n \in X$, and $k \in \mathbb{N}$ with

$$||x_1 - x_0, x_2 - x_0, \dots, x_n - x_0|| = \frac{1}{k}.$$

We take $i = 0, 1, 2, \dots, k$ and set

$$u_i = x_0 + i(x_1 - x_0) \tag{3.1}$$

and

$$v_i = x_0 + i(x_2 - x_0).$$

Clearly, we have

$$u_{i+1} - u_i = x_1 - x_0,\ u_i = \frac{u_{i-1} + u_{i+1}}{2} \ \text{ and } \ v_k - x_0 = k(x_2 - x_0).$$

Hence, it holds that

$$\begin{aligned} & ||u_{i+1}-u_i, v_k-u_i, x_3-u_i, \ldots, x_n-u_i|| \\ =\ & ||x_1-x_0, v_k-x_0, x_3-x_0, \ldots, x_n-x_0|| \\ =\ & ||x_1-x_0, k(x_2-x_0), x_3-x_0, \ldots, x_n-x_0|| \\ =\ & k\cdot\frac{1}{k}=1 \end{aligned}$$

and

$$\begin{aligned} & ||u_{i-1}-u_i, v_k-u_i, x_3-u_i, \ldots, x_n-u_i|| \\ =\ & ||x_1-x_0, v_k-x_0, x_3-x_0, \ldots, x_n-x_0|| \\ =\ & ||x_1-x_0, k(x_2-x_0), x_3-x_0, \ldots, x_n-x_0|| \\ =\ & k\cdot\frac{1}{k}=1. \end{aligned}$$

Since u_i, u_{i-1}, u_{i+1} are 2-colinear and f preserves 2-colinear, we have necessarily $\alpha \in \mathbb{R}$ with

$$f(u_{i+1})-f(u_i)=\alpha(f(u_i)-f(u_{i-1})). \tag{3.2}$$

f preserves n-DOPP implies that

$$||f(u_{i+1})-f(u_i), f(v_k)-f(u_i), f(x_3)-f(u_i), \ldots, f(x_n)-f(u_i)||=1,$$

$$||f(u_i)-f(u_{i-1}), f(v_k)-f(u_i), f(x_3)-f(u_i), \ldots, f(x_n)-f(u_i)||=1.$$

Clearly,

$$||\alpha(f(u_i)-f(u_{i-1})), f(v_k)-f(u_i), f(x_3)-f(u_i), \ldots, f(x_n)-f(u_i)||=1.$$

Thus $|\alpha|=1$ and $\alpha=1$ from f is injective. It follows from (3.2) that

$$f(u_k)-f(x_0)=k(f(x_1)-f(x_0)). \tag{3.3}$$

From (3.1)

$$||u_k-x_0, x_2-x_0, x_3-x_0, \ldots, x_n-x_0||=1$$

and f preserves n-DOPP yields

$$||f(u_k)-f(x_0)), f(x_2)-f(x_0), f(x_3)-f(x_0), \ldots, f(x_n)-f(x_0)||=1$$

and by (3.3)

$$||k(f(x_1)-f(x_0)), f(x_2)-f(x_0), f(x_3)-f(x_0), \ldots, f(x_n)-f(x_0)||=1.$$

Hence,

$$||f(x_1)-f(x_0), f(x_2)-f(x_0), f(x_3)-f(x_0), \ldots, f(x_n)-f(x_0)||=\frac{1}{k}. \quad \square$$

Theorem 3.3. *Let X and Y be two real n-normed spaces such that* $\dim X \geq n$. *Suppose that $f: X \to Y$ satisfies n-DOPP, n-colinear and 2-colinear on same order. Then f is an affine n-isometry.*

Proof. We firstly prove that f is an n-Lipschitz.

There are two cases for $x_0, x_1, \ldots, x_n \in X$ depending upon whether $||x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0|| = 0$ or not. In the case $||x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0|| = 0$, then $x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0$ are linear dependent, that is, n-colinear. Thus $f(x_1) - f(x_0), f(x_2) - f(x_0), \ldots, f(x_n) - f(x_0)$ are linear dependent. Hence $||f(x_1) - f(x_0), f(x_2) - f(x_0), \ldots, f(x_n) - f(x_0)|| = 0$.

In the case $||x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0|| > 0$, let

$$\frac{m-1}{k} < ||x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0|| \le \frac{m}{k}, \tag{3.4}$$

where m, k are positive integers with $m \ge 2$. We only to show that

$$||f(x_1) - f(x_0), f(x_2) - f(x_0), \ldots, f(x_n) - f(x_0)|| \le \frac{m}{k}.$$

Set

$$p_i = x_0 + \frac{i}{k} \cdot \frac{x_1 - x_0}{||x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0||}, \quad i = 0, 1, \ldots, m.$$

Then

$$\begin{aligned} &||p_i - p_{i-1}, x_2 - p_i, \ldots, x_n - p_i|| \\ &= ||\frac{1}{k} \cdot \frac{x_1 - x_0}{||x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0||}, x_2 - x_0, \ldots, x_n - x_0|| = \frac{1}{k}. \end{aligned}$$

It follows from Lemma 3.2 that

$$||f(p_i) - f(p_{i-1}), f(x_2) - f(p_i), \ldots, f(x_n) - f(p_i)|| = \frac{1}{k}. \tag{3.5}$$

Clearly, $p_m - p_{m-1} = \frac{1}{k} \cdot \frac{x_1 - x_0}{||x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0||}$, and

$$\begin{aligned} x_1 - p_{m-1} &= x_1 - x_0 - \frac{m-1}{k} \cdot \frac{x_1 - x_0}{||x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0||} \\ &= (k||x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0|| - (m-1))(p_m - p_{m-1}). \end{aligned}$$

From (3.4)

$$0 < k||x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0|| - (m-1) \le 1.$$

and f preserves 2-colinear on same order, we obtain that

$$f(x_1) - f(p_{m-1}) = t(f(p_m) - f(p_{m-1}))$$

for some $t \in (0, 1]$. Again by (3.5) and Lemma 3.2

$$\begin{aligned} &||f(x_1) - f(p_{m-1}), f(x_2) - f(x_0), \ldots, f(x_n) - f(x_0)|| \\ &= ||t(f(p_m) - f(p_{m-1}), f(x_2) - f(p_m), \ldots, f(x_n) - f(p_m)|| \\ &= |t|\frac{1}{k} \\ &\le \frac{1}{k}. \end{aligned}$$

Hence

$$\begin{aligned}
&||f(x_1) - f(x_0), f(x_2) - f(x_0) \ldots, f(x_n) - f(x_0)|| \\
\leq &\sum_{i=1}^{m-1} ||f(p_i) - f(p_{i-1}), f(x_2) - f(x_0), \ldots, f(x_n) - f(x_0)|| \\
&+ ||f(x_1) - f(p_{m-1}), f(x_2) - f(x_0) \ldots, f(x_n) - f(x_0)|| \\
= &\sum_{i=1}^{m-1} ||f(p_i) - f(p_{i-1}), f(x_2) - f(p_i), \ldots, f(x_n) - f(p_i)|| \\
&+ ||f(x_1) - f(p_{m-1}), f(x_2) - f(x_0), \ldots, f(x_n) - f(x_0)|| \\
\leq &\frac{m}{k}.
\end{aligned}$$

Thus

$$||f(x_1)-f(x_0), f(x_2)-f(x_0), \ldots, f(x_n)-f(x_0)|| \leq ||x_1-x_0, x_2-x_0, \ldots, x_n-x_0||.$$

That means f is n-Lipschitz, and f is an n-isometry by Theorem 1.1, f is affine by Lemma 3.1. □

Corollary 3.4 ([5])**.** *Assume that X and Y are linear 2-normed pace, $f : X \to Y$ preserves 2-DOPP and 2-colinear on same order. Then f is an affine 2-isometry.*

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