Honam Mathematical J. **36** (2014), No. 4, pp. 707–722
http://dx.doi.org/10.5831/HMJ.2014.36.4.707

# GEOMETRY OF LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE KENMOTSU MANIFOLD

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**Abstract.** In this paper, we study the forms of the curvatures of lightlike submanifolds $M$ of an indefinite Kenmotsu manifold $\bar{M}$ subject to the conditions: $M$ is locally symmetric or $M$ is semi-symmetric.

## 1. Introduction

In the classical theory of Sasakian manifolds, the following result is well-known: If a Sasakian manifold is locally symmetric, then it is of constant positive curvature 1. In 2012, K.L. Duggal and D.H. Jin [4] studied the forms of curvatures of locally symmetric generic lightlike submanifolds of an indefinite Sasakian manifold. They obtained the following result: If a locally symmetric generic lightlike submanifold with flat lightlike transversal connection of an indefinite Sasakian manifold is totally umbilical, then it is of constant positive curvature 1.

Further in 1971, K. Kenmotsu proved the following result: If a Kenmotsu manifold is locally symmetric, then it is of constant negative curvature $-1$ [9]. D.H. Jin proved the lightlike version of this result for lightlike hypersurfaces [5, 7] and half lightlike submanifolds [6, 8].

The objective of this paper is to study the forms of the curvatures of general $r$-lightlike submanifolds of an indefinite Kenmotsu manifold subject to the conditions: (1) $M$ is locally symmetric, *i.e.*, the curvature tensor $R$ of $M$ satisfies $\nabla R = 0$, or (2) $M$ is semi-symmetric, *i.e.*, $R$ satisfies $R \cdot R = 0$. We prove the following theorems:

Received July 11, 2014. Accepted October 20, 2014.
2010 Mathematics Subject Classification. 53B25, 53C40, 53C50.
Key words and phrases. locally symmetric, semi-symmetric, lightlike submanifold, indefinite Kenmotsu manifold.

**Theorem 1.1.** *Let $M$ be a locally symmetric $r$-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ equipped with an almost contact metric structure $(J,\, \zeta,\, \theta,\, \bar{g})$.*

1. *If the structure vector field $\zeta$ of $\bar{M}$ is tangent to $M$, then $M$ is a space of constant negative curvature $-1$ and the induced connection on $M$ is a torsion-free metric connection of $M$.*
2. *If the transversal vector bundle $tr(TM)$ of $M$ is parallel and $M$ is irrotational, then $M$ is flat and $\zeta$ is a normal vector field of $M$.*

*In this two cases, the lightlike transversal connection of $M$ is flat and the Ricci type tensor is symmetric.*

**Theorem 1.2.** *Let $(M, g, S(TM), S(TM^{\perp}))$ be a semi-symmetric $r$-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$.*

1. *If $\zeta$ is tangent to $M$, then $M$ is a space of constant negative curvature $-1$, the induced connection on $M$ is a torsion-free metric connection of $M$ and the Ricci type tensor is symmetric.*
2. *If $tr(TM)$ is parallel on $T\bar{M}$, $M$ is irrotational and the tangential component $\omega$ of the structure vector field $\zeta$ of $\bar{M}$ satisfies $\theta(\omega) \neq 0$, then $\omega$ is a null vector field on $M$. Moreover, if $S(TM)$ is Riemannian vector bundle, then $R$ is given by*

$$R(X,Y)Z = \theta(\omega)^{-1}\,\theta(Z)\{\theta(X)Y - \theta(Y)X\},$$

*for any vector fields $X$, $Y$ and $Z$ of $M$.*

## 2. Lightlike submanifolds

An odd dimensional semi-Riemannian manifold $\bar{M}$ is said to be an *indefinite almost contact metric manifold* [9] if there exists a structure set $(J,\, \zeta,\, \theta,\, \bar{g})$, where $J$ is a $(1,1)$-type tensor field, $\zeta$ is a vector field which called the structure vector field, $\theta$ is a 1-form and $\bar{g}$ is the semi-Riemannian metric on $\bar{M}$ such that

$$J^2X = -X + \theta(X)\zeta, \quad J\zeta = 0, \quad \theta \circ J = 0, \quad \theta(\zeta) = 1,$$
$$\theta(X) = \bar{g}(\zeta, X), \quad \bar{g}(JX, JY) = \bar{g}(X,Y) - \theta(X)\theta(Y), \tag{2.1}$$

for any vector fields $X$, $Y$ on $\bar{M}$. An indefinite almost contact metric manifold $\bar{M}$ is called an *indefinite Kenmotsu manifold* [9] if

$$\bar{\nabla}_X\zeta = -X + \theta(X)\zeta, \tag{2.2}$$

$$(\bar{\nabla}_X J)Y = -\bar{g}(JX, Y)\zeta + \theta(Y)JX, \tag{2.3}$$

where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{M}$.

Let $(M, g)$ be an $m$-dimensional lightlike submanifold of an $(m+n)$-dimensional indefinite Kenmotsu manifold $\bar{M}$. We follow Duggal-Bejancu [2] and Duggal-Jin [3] for notations and structure equations used in this paper. Then the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ is a vector subbundle of the tangent bundle $TM$ and the normal bundle $TM^{\perp}$, of rank $r\,(1 \le r \le \min\{m, n\})$. In general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in $TM$ and $TM^{\perp}$ respectively, called the *screen distribution* and *co-screen distribution* of $M$, such that

$$TM = Rad(TM) \oplus_{orth} S(TM),$$
$$TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \tag{2.4}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike submanifold by $(M, g, S(TM), S(TM^{\perp}))$. Denote by $F(M)$ the algebra of smooth functions on $M$ and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle $E$ over $M$. We use the same notation for any other vector bundle. We use the following range of indices:

$$i, j, k, \dots \in \{1, \dots, r\}, \qquad \alpha, \beta, \gamma, \dots \in \{r+1, \dots, n\}.$$

Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to $TM$ in $T\bar{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$ respectively, which are called the *transversal vector bundle* and *lightlike transversal vector bundle* of $M$ with respect to $S(TM)$ respectively, where $S(TM)^{\perp}$ is the orthogonal complementary vector bundle to $S(TM)$ in $T\bar{M}_{|M}$. Let $\{N_1, \cdots, N_r\}$ be a lightlike frame fields of $ltr(TM)_{|\mathcal{U}}$ consisting of smooth sections of $S(TM)^{\perp}_{|\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of $M$, such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,$$

where $\{\xi_1, \cdots, \xi_r\}$ is a lightlike frame fields of $Rad(TM)$. Then we have

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned} \tag{2.5}$$

We say that a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is

(1) *r-lightlike* if $1 \le r < \min\{m, n\}$;
(2) *co-isotropic* if $1 \le r = n < m$;
(3) *isotropic* if $1 \le r = m < n$;
(4) *totally lightlike* if $1 \le r = m = n$.

The above three classes (2)∼(4) are particular cases of the class (1) as follows: $S(TM^{\perp}) = \{0\}$, $S(TM) = \{0\}$ and $S(TM) = S(TM^{\perp}) = \{0\}$ respectively. The geometry of $r$-lightlike submanifolds is more general form than that of the other three type submanifolds. For this reason, we consider only $r$-lightlike submanifolds $M \equiv (M, g, S(TM), S(TM^{\perp}))$ with the following local quasi-orthonormal field of frames of $\bar{M}$:

$$\{\xi_1, \cdots, \xi_r,\ N_1, \cdots, N_r,\ F_{r+1}, \cdots, F_m,\ W_{r+1}, \cdots, W_n\},$$

where $\{F_{r+1}, \cdots, F_m\}$ and $\{W_{r+1}, \cdots, W_n\}$ are orthonormal frame fields of $S(TM)$ and $S(TM^{\perp})$ respectively.

From now and in the sequel, let $X$, $Y$, $Z$ and $W$ be the vector fields on $M$, unless otherwise specified. Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and $P$ the projection morphism of $TM$ on $S(TM)$ with respect to the first equation of (2.4)[denote $(2.4)_1$]. For an $r$-lightlike submanifold, the local Gauss-Weingarten formulas for $M$ and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X, Y)N_i + \sum_{\alpha=r+1}^{n} h_{\alpha}^{s}(X, Y)W_{\alpha}, \tag{2.6}$$

$$\bar{\nabla}_X N_i = -A_{N_i}X + \sum_{j=1}^{r} \tau_{ij}(X)N_j + \sum_{\alpha=r+1}^{n} \rho_{i\alpha}(X)W_{\alpha}, \tag{2.7}$$

$$\bar{\nabla}_X W_{\alpha} = -A_{W_{\alpha}}X + \sum_{i=1}^{r} \phi_{\alpha i}(X)N_i + \sum_{\beta=r+1}^{n} \sigma_{\alpha\beta}(X)W_{\beta}, \tag{2.8}$$

$$\nabla_X PY = \nabla^*_X PY + \sum_{i=1}^{r} h_i^*(X, PY)\xi_i, \tag{2.9}$$

$$\nabla_X \xi_i = -A^*_{\xi_i}X - \sum_{j=1}^{r} \tau_{ji}(X)\xi_j, \quad \forall\, X,\, Y \in \Gamma(TM), \tag{2.10}$$

where $\nabla$ and $\nabla^*$ are induced linear connections on $TM$ and $S(TM)$ respectively, $h_i^{\ell}$ and $h_{\alpha}^{s}$ are called the *local lightlike* and *local screen second fundamental forms* on $TM$ respectively, $h_i^*$ are called the *local second fundamental forms* on $S(TM)$. $A_{N_i}$, $A_{W_{\alpha}}$ and $A^*_{\xi_i}$ are linear operators on $\Gamma(TM)$, which called the *shape operators* on $TM$ and $S(TM)$ respectively, and $\tau_{ij}$, $\rho_{i\alpha}$, $\phi_{\alpha i}$ and $\sigma_{\alpha\beta}$ are 1-forms on $TM$.

Since the connection $\bar{\nabla}$ is torsion-free, the induced connection $\nabla$ is also torsion-free and $h_i^{\ell}$ and $h_{\alpha}^{s}$ are symmetric. From the fact $h_i^{\ell}(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi_i)$, we know that $h_i^{\ell}$ are independent of the choice of the vector bundles $S(TM)$, $S(TM^{\perp})$ and $ltr(TM)$. The above three local second fundamental forms are related to their shape operators by

$$g(A^*_{\xi_i}X,\,Y) = h^\ell_i(X,\,Y) + \sum_{j=1}^{r} h^\ell_j(X,\,\xi_i)\eta_j(Y), \tag{2.11}$$
$$\bar g(A^*_{\xi_i}X, N_j) = 0,$$

$$g(A_{W_\alpha}X,\,Y) = \epsilon_\alpha h^s_\alpha(X,\,Y) + \sum_{i=1}^{r} \phi_{\alpha i}(X)\eta_i(Y), \tag{2.12}$$
$$\bar g(A_{W_\alpha}X, N_i) = \epsilon_\alpha \rho_{i\alpha}(X),$$

$$g(A_{N_i}X, PY) = h^*_i(X, PY),\ \ \epsilon_\beta\sigma_{\alpha\beta} = -\epsilon_\alpha\sigma_{\beta\alpha}, \tag{2.13}$$
$$\eta_j(A_{N_i}X) + \eta_i(A_{N_j}X) = 0,$$

where $\epsilon_\alpha(=\pm 1) = \bar g(W_\alpha, W_\alpha)$ and $\eta_i$ are the 1-forms defined by

$$\eta_i(X) = \bar g(X, N_i). \tag{2.14}$$

The induced connection $\nabla$ on $TM$ is not metric and satisfies

$$(\nabla_X g)(Y, Z) = \sum_{i=1}^{r}\{h^\ell_i(X, Y)\,\eta_i(Z) + h^\ell_i(X, Z)\,\eta_i(Y)\}. \tag{2.15}$$

Denote by $\bar R$ and $R$ the curvature tensors of $\bar\nabla$ and $\nabla$ respectively. Using the local Gauss-Weingarten formulas for $M$, we obtain

$$\begin{aligned}(2.16)\quad \bar R(X,Y)Z &= R(X,Y)Z \\ &\quad + \sum_{i=1}^{r}\{h^\ell_i(X,Z)A_{N_i}Y - h^\ell_i(Y,Z)A_{N_i}X\} \\ &\quad + \sum_{\alpha=r_1}^{n}\{h^s_\alpha(X,Z)A_{W_\alpha}Y - h^s_\alpha(Y,Z)A_{W_\alpha}X\} \\ &+ \sum_{i=1}^{r}\{(\nabla_X h^\ell_i)(Y,Z) - (\nabla_Y h^\ell_i)(X,Z) \\ &\quad + \sum_{j=1}^{r}[\tau_{ji}(X)h^\ell_j(Y,Z) - \tau_{ji}(Y)h^\ell_j(X,Z)] \\ &\quad + \sum_{\alpha=r+1}^{n}[\phi_{\alpha i}(X)h^s_\alpha(Y,Z) - \phi_{\alpha i}(Y)h^s_\alpha(X,Z)]\}N_i \\ &+ \sum_{\alpha=r+1}^{n}\{(\nabla_X h^s_\alpha)(Y,Z) - (\nabla_Y h^s_\alpha)(X,Z)\end{aligned}$$

$$
+\sum_{i=1}^{r}[\rho_{i\alpha}(X)h_i^{\ell}(Y,Z)-\rho_{i\alpha}(Y)h_\alpha^s(X,Z)]
+\sum_{\beta=r+1}^{n}[\sigma_{\beta\alpha}(X)h_\beta^s(Y,Z)-\sigma_{\beta\alpha}(Y)h_\beta^s(X,Z)]\}W_\alpha,
$$

$$
\begin{aligned}
(2.17)\quad \bar{R}(X,Y)N_i &= -\nabla_X(A_{N_i}Y)+\nabla_Y(A_{N_i}X)+A_{N_i}[X,Y] \\
&\quad +\sum_{j=1}^{r}[\tau_{ij}(X)A_{N_j}Y-\tau_{ij}(Y)A_{N_j}X] \\
&\quad +\sum_{\alpha=r+1}^{m}[\rho_{i\alpha}(X)A_{W_\alpha}Y-\rho_{i\alpha}(Y)A_{W_\alpha}X] \\
&+\sum_{j=1}^{r}\{h_j^{\ell}(Y,A_{N_i}X)-h_j^{\ell}(X,A_{N_i}Y)+2d\tau_{ij}(X,Y) \\
&\quad +\sum_{k=1}^{r}[\tau_{ik}(Y)\tau_{kj}(X)-\tau_{ik}(X)\tau_{kj}(Y)] \\
&\quad +\sum_{\alpha=r+1}^{m}[\rho_{i\alpha}(Y)\phi_{\alpha j}(X)-\rho_{i\alpha}(X)\phi_{\alpha j}(Y)]\}N_j \\
&+\sum_{\alpha=r+1}^{m}\{h_\alpha^s(Y,A_{N_i}X)-h_\alpha^s(X,A_{N_i}Y)+2d\rho_{i\alpha}(X,Y) \\
&\quad +\sum_{j=1}^{r}[\tau_{ij}(Y)\rho_{j\alpha}(X)-\tau_{ij}(X)\rho_{j\alpha}(Y)] \\
&\quad +\sum_{\beta=r+1}^{m}[\rho_{i\beta}(Y)\sigma_{\beta\alpha j}(X)-\rho_{i\beta}(X)\sigma_{\beta\alpha}(Y)]\}W_\alpha,
\end{aligned}
$$

$$
\begin{aligned}
(2.18)\quad \bar{R}(X,Y)W_\alpha &= -\nabla_X(A_{W_\alpha}Y)+\nabla_Y(A_{W_\alpha}X)+A_{W_\alpha}[X,Y] \\
&\quad +\sum_{i=1}^{r}[\phi_{\alpha i}(X)A_{N_i}Y-\phi_{\alpha i}(Y)A_{N_i}X] \\
&\quad +\sum_{\alpha=r+1}^{m}[\sigma_{\alpha\beta}(X)A_{W_\beta}Y-\sigma_{\alpha\beta}(Y)A_{W_\beta}X]
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{i=1}^{r}\{h_i^{\ell}(Y, A_{W_\alpha}X) - h_i^{\ell}(X, A_{W_\alpha}Y) + 2d\phi_{\alpha i}(X,Y) \\
&\qquad + \sum_{j=1}^{r}[\phi_{\alpha j}(Y)\tau_{ji}(X) - \phi_{\alpha j}(X)\tau_{ji}(Y)] \\
&\qquad + \sum_{\alpha=r+1}^{m}[\sigma_{\alpha\beta}(Y)\phi_{\beta i}(X) - \sigma_{\alpha\beta}(X)\phi_{\beta i}(Y)]\}N_i \\
&+ \sum_{\alpha=r+1}^{m}\{h_\beta^{s}(Y, A_{W_\alpha}X) - h_\beta^{s}(X, A_{W_\alpha}Y) + 2d\sigma_{\alpha\beta}(X,Y) \\
&\qquad + \sum_{j=1}^{r}[\phi_{\alpha i}(Y)\rho_{i\beta}(X) - \phi_{\alpha i}(X)\rho_{i\beta}(Y)] \\
&\qquad + \sum_{\beta=r+1}^{m}[\sigma_{\alpha\gamma}(Y)\sigma_{\gamma\beta}(X) - \sigma_{\alpha\gamma}(X)\sigma_{\gamma\beta}(Y)]\}W_\beta.
\end{aligned}
$$

Using (2.9) and (2.10), for all $X$, $Y \in \Gamma(TM)$, we obtain

$$
\begin{aligned}
R(X,Y)\xi_i &= -\nabla^*_X(A^*_{\xi_i}Y) + \nabla^*_Y(A^*_{\xi_i}X) + A^*_{\xi}[X,Y] \\
&\quad + \tau_{ji}(Y)A^*_{\xi_j}X - \tau_{ji}(X)A^*_{\xi_j}Y \\
&\quad + \sum_{j=1}^{r}\{h_j^*(Y, A^*_{\xi_i}X) - h_j^*(X, A^*_{\xi_i}Y) - 2d\tau_{ji}(X,Y) \\
&\qquad + \sum_{k=1}^{r}[\tau_{jk}(X)\tau_{ki}(Y) - \tau_{jk}(Y)\tau_{ki}(X)]\}\xi_j.
\end{aligned} \tag{2.19}
$$

**Definition 1.** A lightlike submanifold $M$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be *irrotational* [10] if $\bar{\nabla}_X\xi_i \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi_i \in \Gamma(Rad(TM))$.

The above definition is equivalent to

$$
h_j^{\ell}(X, \xi_i) = 0, \quad \phi_{\alpha i}(X) = h_\alpha^{s}(X, \xi_i) = 0. \tag{2.20}
$$

A lightlike submanifold $M = (M, g, \nabla)$ equipped with a degenerate metric $g$ and a linear connection $\nabla$ is said to be of *constant curvature* $c$ if there exists a constant $c$ such that the curvature tensor $R$ of $\nabla$ satisfies

$$
R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}. \tag{2.21}
$$

The induced Ricci type tensor $R^{(0,2)}$ of $M = (M, g, \nabla)$ is defined by

$$R^{(0,2)}(X,Y) = trace\{Z \longmapsto R(Z,X)Y\}, \ \forall\, X,\, Y \in \Gamma(TM).$$

In general, the tensor field $R^{(0,2)}$ is not symmetric [3]. A tensor field $R^{(0,2)}$ of $M$ is called its *induced Ricci tensor*, denote *Ric*, of $M$ if it is symmetric. It is known [3] that $R^{(0,2)}$ is symmetric if and only if the 1-form $tr(\mathcal{T}_{ij})$ vanishes on any coordinate neighborhood $\mathcal{U} \subset M$, where $tr(\mathcal{T}_{ij})$ is the trace of the matrix $(\mathcal{T}_{ij})$, where

$$\mathcal{T}_{ij}(X,Y) = 2d\tau_{ij}(X,Y) + \sum_{k=1}^{r}\{\tau_{ik}(Y)\tau_{kj}(X) - \tau_{ik}(X)\tau_{kj}(Y)\}. \tag{2.22}$$

For any $X \in \Gamma(TM)$, let $\nabla^{\ell}_X N_i = Q(\bar{\nabla}_X N_i)$, where $Q$ is the projection morphism of $\Gamma(T\bar{M})$ on $\Gamma(ltr(TM))$ with respect to the decomposition (2.5). Then $\nabla^{\ell}$ is a linear connection on the lightlike transversal vector bundle $ltr(TM)$ of $M$. We say that $\nabla^{\ell}$ is the *lightlike transversal connection* of $M$. We define the curvature tensor $R^{\ell}$ of $tr(TM)$ by

$$R^{\ell}(X,Y)N_i = \nabla^{\ell}_X\nabla^{\ell}_Y N - \nabla^{\ell}_Y\nabla^{\ell}_X N - \nabla^{\ell}_{[X,\,Y]}N. \tag{2.23}$$

If $R^{\ell}$ vanishes identically, then the lightlike transversal connection $\nabla^{\ell}$ is said to be *flat* [5].

**Theorem 2.1.** *Let $M$ be a lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the lightlike transversal connection $\nabla^{\ell}$ is flat if and only if each 2-form $\mathcal{T}_{ij}$ vanishes identically on any coordinate neighborhood $\mathcal{U} \subset M$.*

*Proof.* From (2.7) and the definition of $\nabla^{\ell}$, we have

$$\nabla^{\ell}_X N_i = \sum_{j=1}^{r}\tau_{ij}(X)N_j, \quad \forall\, X \in \Gamma(TM).$$

Substituting this into the right side of (2.23) and using (2.22), we get

$$R^{\ell}(X,Y)N_i = \sum_{j=1}^{r}\mathcal{T}_{ij}(X,Y)N_j.$$

From this result we deduce our assertion.

## 3. Proof of Theorem 1.1

**(1) Step 1.** Assume that $\zeta$ is tangent to $M$. It is well known [1] that if $\zeta$ is tangent to $M$, then it belongs to $S(TM)$ which we assume this paper. Replacing $Y$ by $\zeta$ to (2.6) and using (2.2), we have

$$\nabla_X \zeta = -X + \theta(X)\zeta, \quad h_i^\ell(X,\zeta) = h_\alpha^s(X,\zeta) = 0. \tag{3.1}$$

Substituting $(3.1)_1$ into the right side of

$$R(X,Y)\zeta = \nabla_X\nabla_Y\zeta - \nabla_Y\nabla_X\zeta - \nabla_{[X,Y]}\zeta$$

and using $(3.1)_1$ and the fact that $\nabla$ is torsion-free, we have

$$R(X,Y)\zeta = \theta(X)Y - \theta(Y)X + 2d\theta(X,Y)\zeta. \tag{3.2}$$

Replacing $Z$ by $\zeta$ to (2.16) and using $(3.1)_2$ and the facts that

$$(\nabla_X h_i^\ell)(Y,\zeta) = h_i^\ell(X,Y), \qquad (\nabla_X h_\alpha^s)(Y,\zeta) = h_\alpha^s(X,Y),$$

we have $\bar{R}(X,Y)\zeta = R(X,Y)\zeta$. Thus we show that

$$\bar{R}(X,Y)\zeta = \theta(X)Y - \theta(Y)X + 2d\theta(X,Y)\zeta.$$

Taking the scalar product with $\zeta$ to this result and using $g(\bar{R}(X,Y)\zeta,\zeta) = 0$ and (2.1), we get $d\theta = 0$, *i.e.*, $\theta$ is closed on $TM$. Therefore the equation (3.2) is reduced to

$$R(X,Y)\zeta = \theta(X)Y - \theta(Y)X. \tag{3.3}$$

Applying $\bar{\nabla}_X$ to $\theta(Y) = g(Y,\zeta)$ and using (2.1), (2.2) and (2.6), we get

$$(\nabla_X\theta)(Y) = -g(X,Y) + \theta(X)\theta(Y). \tag{3.4}$$

**Step 2.** Assume that $M$ is locally symmetric. Applying $\nabla_Z$ to (3.3) and using $(3.1)_1$, (3.3) and (3.4), we have

$$R(X,Y)Z = g(X,Z)Y - g(Y,Z)X. \tag{3.5}$$

Thus, by (2.21), $M$ is a space of constant negative curvature $-1$.

Applying $\nabla_U$ to (3.5) and using (3.5) and the fact $\nabla_U R = 0$, we have

$$(\nabla_U g)(X,Z)Y = (\nabla_U g)(Y,Z)X.$$

Taking $Z = \xi_k$ and $Y = \xi_j$ to this and using (2.15) and $(2.20)_1$, we have

$$h_i^\ell(X,Y)\xi_j = 0.$$

This implies $h_i^\ell(X,Y) = 0$. From this result and (2.15), we show that $\nabla$ is a torsion-free metric connection on $M$.

As $h_i^\ell = 0$ for all $i$, we have $A^*_{\xi_i} = 0$ for all $i$ by (2.11). Thus, by (2.19), we get

$$R(X,Y)\xi_i = -\sum_{j=1}^{r} \mathcal{T}_{ji}(X,Y)\xi_j.$$

On the other hand, replacing $Z$ by $\xi_i$ to (3.5), we have $R(X,Y)\xi_i = 0$. Thus we have

$$\mathcal{T}_{ij}(X,Y) = 0.$$

From this result and Theorem 2.1, we show that the lightlike transversal connection $\nabla^\ell$ is flat and the Ricci type tensor $R^{(0,2)}$ on $M$ is a symmetric Ricci tensor of $M$.

**(2) Step 1.** From the decomposition of $T\bar{M}$, $\zeta$ is decomposed by

$$\zeta = \omega + \sum_{i=1}^{r} a_i\, N_i + \sum_{\alpha=r+1}^{n} b_\alpha\, W_\alpha, \tag{3.6}$$

where $\omega$ is a smooth vector field on $M$ and $a_i = \theta(\xi_i)$ and $b_\alpha = \epsilon_\alpha\theta(W_\alpha)$ are smooth functions. Applying $\bar{\nabla}_X$ to (3.6) and using (2.2), (2.7), (2.8) and (3.6), we obtain

$$\nabla_X\omega = -X + \theta(X)\omega + \sum_{i=1}^{r} a_i A_{N_i}X + \sum_{\alpha=r+1}^{m} b_\alpha A_{W_\alpha}X, \tag{3.7}$$

$$Xa_i + \sum_{j=1}^{r} a_j\tau_{ji}(X) + \sum_{\alpha=r+1}^{m} b_\alpha\phi_{\alpha i}(X) + h_i^\ell(X,\,\omega) = a_i\theta(X), \tag{3.8}$$

$$Xb_\alpha + \sum_{i=1}^{r} a_i\rho_{i\alpha}(X) + \sum_{\beta=r+1}^{m} b_\beta\sigma_{\beta\alpha}(X) + h_\alpha^s(X,\,\omega) = b_\alpha\theta(X). \tag{3.9}$$

Substituting (3.8) and (3.9) into the following equations

$$[X,Y]a_i = X(Ya_i) - Y(Xa_i), \quad [X,Y]b_\alpha = X(Yb_\alpha) - Y(Xb_\alpha)$$

and using (2.16)∼(2.18), (3.6)∼(3.9), we have respectively

$$\begin{aligned} 2a_i\, d\theta(X,Y) &= \bar{g}(\bar{R}(X,Y)\zeta,\, \xi_i), \\ 2b_\alpha\, d\theta(X,Y) &= \bar{g}(\bar{R}(X,Y)\zeta,\, W_\alpha). \end{aligned} \tag{3.10}$$

Substituting (3.7) into the right side of

$$R(X,Y)\omega = \nabla_X\nabla_Y\omega - \nabla_Y\nabla_X\omega - \nabla_{[X,Y]}\omega$$

and using (2.16)∼(2.18), (3.6)∼(3.9) and (3.10), we have

$$\bar{R}(X,Y)\zeta = \theta(X)Y - \theta(Y)X + 2d\theta(X,Y)\zeta. \tag{3.11}$$

Taking the scalar product with $\zeta$ to (3.11) and using (2.1) and

$$g(\bar{R}(X,Y)\zeta,\zeta)=0,$$

we show that $d\theta=0$ on $TM$, *i.e.*, $\theta$ is closed on $TM$.

**Step 2.** Assume that $tr(TM)$ is parallel on $T\bar{M}$ and $M$ is irrotational. As $tr(TM)$ is parallel, we have $A_{N_i}=A_{W_\alpha}=0$ due to (2.7) and (2.8). Let $d_i=\bar{g}(\zeta,N_i)=0$. Applying $\bar{\nabla}_X$ to this and using (2.2) and (2.7), we have $\eta_i(X)=0$ for all $X\in\Gamma(TM)$. It is a contradiction to $\eta_i(\xi_i)=1$. Thus $d_i\neq 0$ for each $i$ and $\zeta$ is not tangent to $M$. As $M$ is irrotational, by (2.20) we have $h^\ell_j(X,\xi_i)=0$ and $\phi_{\alpha i}(X)=h^s_\alpha(X,\xi_i)=0$ for all $X\in\Gamma(TM)$. Thus, from (2.12) and (2.13) we get $h^s_\alpha=h^*_i=\rho_{i\alpha}=0$. Then we have

$$\bar{R}(X,Y)N_i=\sum_{j=1}^r \mathcal{T}_{ij}(X,Y)N_j,\quad \bar{R}(X,Y)W_\alpha=0. \tag{3.12}$$

Substituting (3.6) into (3.11) with $d\theta=0$ and using (3.12), we have

$$\bar{R}(X,Y)\omega=\theta(X)Y-\theta(Y)X-\sum_{i,j=1}^r a_i\mathcal{T}_{ij}(X,Y)N_j. \tag{3.13}$$

Substituting (3.6) into $(3.10)_1$ with $d\theta=0$ and using (3.12), we have

$$\bar{g}(\bar{R}(X,Y)\omega,\,\xi_i)=-\sum_{j=1}^r a_j\mathcal{T}_{ji}(X,Y).$$

Replacing $Z$ by $\omega$ to (2.16) and using the last equation, we get

$$\begin{aligned}\bar{R}(X,Y)\omega&=R(X,Y)\omega+\sum_{i=1}^r\bar{g}(\bar{R}(X,Y)\omega,\,\xi_i)N_i\\&=R(X,Y)\omega-\sum_{i,j=1}^r a_i\mathcal{T}_{ij}(X,Y)N_j.\end{aligned}$$

Substituting this equation into (3.13), we get

$$R(X,Y)\omega=\theta(X)Y-\theta(Y)X. \tag{3.14}$$

Applying $\bar{\nabla}_X$ to $\theta(Y)=g(Y,\zeta)$ and using (2.2) and (2.6), we have

$$(\nabla_X\theta)(Y)=\theta(X)\theta(Y)-g(X,Y)+\sum_{i=1}^r d_ih^\ell_i(X,Y). \tag{3.15}$$

**Step 3.** Assume that $M$ is locally symmetric. Applying $\nabla_Z$ to (3.14) and using (3.7), (3.14) and (3.15), we have

$$R(X,Y)Z = \{g(X,Z) - \sum_{i=1}^{r} d_i h_i^{\ell}(X,Z)\}Y - \{g(Y,Z) - \sum_{i=1}^{r} d_i h_i^{\ell}(Y,Z)\}X. \tag{3.16}$$

Replacing $Z$ by $\xi_i$ to (3.16) and using $(2.20)_1$, we have $R(X,Y)\xi_i = 0$. Comparing this and (2.19), we have $\mathcal{T}_{ij} = 0$. Thus the lightlike transversal connection $\nabla^{\ell}$ is flat and $R^{(0,2)}$ is a symmetric Ricci tensor of $M$. Using (2.16), (2.17), (3.16) and the fact $\mathcal{T}_{ij} = 0$, we have

$$\begin{aligned} 0 &= \bar{g}(\bar{R}(X,Y)N_i, Z) = -\bar{g}(\bar{R}(X,Y)Z, N_i) \\ &= -\bar{g}(R(X,Y)Z, N_i) \\ &= \{g(Y,Z) - \sum_{k=1}^{r} d_k h_k^{\ell}(Y,Z)\}\eta_i(X) \\ &\quad - \{g(X,Z) - \sum_{k=1}^{r} d_k h_k^{\ell}(X,Z)\}\eta_i(Y). \end{aligned}$$

Replacing $Y$ by $\xi_i$ to this equation and using $(2.20)_1$, we get

$$g(X,Y) = \sum_{k=1}^{r} d_k h_k^{\ell}(X,Y).$$

From this equation and (3.16), we obtain $R = 0$. Thus $M$ is flat. From this result and (3.14), we have

$$\theta(X)Y = \theta(Y)X.$$

Replacing $Y$ by $\xi_i$ to this and using $X = PX + \sum_{j=1}^{r} \eta_j(X)\xi_j$, we have

$$a_i PX = \theta(X)\xi_i - a_i \sum_{j=1}^{r} \eta_j(X)\xi_j.$$

As the left term of this equation belongs to $S(TM)$ and the right term belongs to $Rad(TM)$, we have $a_i\, PX = 0$ and $\theta(X)\xi_i = a_i \sum_{j=1}^{r} \eta_j(X)\xi_j$ for all $X \in \Gamma(TM)$ and for all $i$. This results imply $a_i = 0$ for all $i$ and $\theta = 0$ on $TM$. As $\theta(X) = g(X,\omega) + \sum_{i=1}^{r} \eta_i(X)$, we have $g(X,\omega) = 0$ for all $X \in \Gamma(TM)$. This implies $\omega = \sum_{i=1}^{r} d_i\xi_i$. Thus $\zeta$ is decomposed by

$$\zeta = \sum_{i=1}^{r} d_i \xi_i + \sum_{\alpha=r+1}^{n} b_\alpha W_\alpha.$$

Thus $\zeta$ is a normal vector field of $M$, *i.e.*, we have $\zeta \in \Gamma(TM^\perp)$.

**Corollary 1.** *Let $M$ be a lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then the structure 1-form $\theta$ is closed, i.e., $d\theta = 0$.*

## 4. Proof of Theorem 1.2

**(1)** Assume that $\zeta$ is tangent to $M$. Then we can use all equations and results of Step 1 in (1) of Theorem 1.1. Applying $\nabla_Z$ to (3.3) and using $(3.1)_1$ and (3.4), we have

$$(\nabla_Z R)(X,Y)\zeta = R(X,Y)Z - g(X,Z)Y + g(Y,Z)X. \tag{4.1}$$

Substituting (4.1) into $(R(U,Z)R)(X,Y)\zeta = 0$ and using (2.15), $(3.1)_1$ and (4.1), we have

$$\begin{aligned} \sum_{k=1}^{r} \Big\{ & [h_k^\ell(U,Y)\eta_k(Z) - h_k^\ell(Z,Y)\eta_k(U)]X \\ & - [h_k^\ell(U,X)\eta_k(Z) - h_k^\ell(Z,X)\eta_k(U)]Y \Big\} \\ = \;& \theta(U)(\nabla_Z R)(X,Y)\zeta - \theta(Z)(\nabla_U R)(X,Y)\zeta. \end{aligned} \tag{4.2}$$

Replacing $U$ by $\zeta$ to (4.2) and using the fact that $(\nabla_\zeta R)(X,Y)\zeta = 0$ due to (3.3) and (4.1), we have $(\nabla_Z R)(X,Y)\zeta = 0$. From this result and (4.1), we show that

$$R(X,Y)Z = g(X,Z)Y - g(Y,Z)X. \tag{4.3}$$

Thus, by (2.21), $M$ is a space of constant negative curvature $-1$.

Replacing $U$ by $\xi_i$ to (4.2) and using (2.14), $(2.20)_1$ and the fact that

$$(\nabla_Z R)(X,Y)\zeta = 0,$$

we have

$$h_i^\ell(Y,Z)X = h_i^\ell(X,Z)Y.$$

Replacing $Y$ by $\xi_j$ to this equation and using $(2.20)_1$, we have

$$h_i^\ell(X,Y) = 0.$$

Thus we show that $\nabla$ is a torsion-free metric connection on $M$ by (2.15). Using (2.19), (4.3) and the method of (1) in Theorem 1.1, we see that the lightlike transversal connection $\nabla^{\ell}$ is flat and the tensor field $R^{(0,2)}$ is a symmetric Ricci tensor on $M$.

**(2)** Assume that $tr(TM)$ is parallel and $M$ is irrotational. Then we can use all equations and results of Step 1 and Step 2 in (2) of Theorem 1.1. Thus each $d_i$ does not vanishes and $\zeta$ is not tangent to $M$. Taking the scalar product with $N_i$ to (3.13) and using (3.12) and the fact $g(\bar{R}(X,Y)\omega,\, N_i) = -g(\bar{R}(X,Y)N_i,\, \omega)$, we have

$$\sum_{j=1}^{r} d_j \mathcal{T}_{ij}(X,Y) = \theta(Y)\eta_i(X) - \theta(X)\eta_i(Y). \tag{4.4}$$

Applying $\nabla_Z$ to (3.14) and using (3.7), (3.14) and (3.15), we have

$$\begin{aligned}(\nabla_Z R)(X,Y)\omega = R(X,Y)Z &+ \{g(Y,Z) - \sum_{i=1}^{r} d_i h_i^{\ell}(Y,Z)\}X \\ &- \{g(X,Z) - \sum_{i=1}^{r} d_i h_i^{\ell}(X,Z)\}Y.\end{aligned} \tag{4.5}$$

Applying $\bar{\nabla}_X$ to $d_i = \bar{g}(\zeta, N_i)$ and using (2.2) and (2.7), we have

$$Xd_i = d_i\theta(X) + \sum_{j=1}^{r} d_j \tau_{ij}(X) - \eta_i(X). \tag{4.6}$$

Substituting (4.5) into $(R(U,Z)R)(X,Y)\omega = 0$ and using (2.15), (3.7), (4.5), (4.6) and the equation $\bar{R}(U,Z)X = \bar{R}(X,Z)U + \bar{R}(U,X)Z$, we get

$$\begin{aligned}&\theta(Z)\{R(X,Y)U + g(Y,U)X - g(X,U)Y\} \\ &- \theta(U)\{R(X,Y)Z + g(Y,Z)X - g(X,Z)Y\} \\ &+ \sum_{i=1}^{r} d_i\{\bar{g}(\bar{R}(X,Z)U + \bar{R}(U,X)Z,\, \xi_i)Y \\ &\qquad - \bar{g}(\bar{R}(Y,Z)U + \bar{R}(U,Y)Z,\, \xi_i)X\} = 0.\end{aligned} \tag{4.7}$$

Taking $U = \xi_j$ and $Z = \omega$ to (4.7) and using (3.13) and the fact that $\bar{g}(\bar{R}(X,Y)\xi_j,\, \xi_i) = 0$, we have

$$\begin{aligned}&\theta(\omega)R(X,Y)\xi_j + \sum_{k=1}^{r} a_k\{a_j\eta_k(Y) + \sum_{i=1}^{r} d_i \mathcal{T}_{ki}(\xi_j, Y)\}X \\ &= \sum_{k=1}^{r} a_k\{a_j\eta_k(X) + \sum_{i=1}^{r} d_i \mathcal{T}_{ki}(\xi_j, X)\}Y.\end{aligned}$$

Taking the scalar product with $N_i$ and using (2.19) and (4.4), we have

$$\mathcal{T}_{ij}(X,Y) = (\theta(\omega))^{-1}a_j\{\theta(Y)\eta_i(X) - \theta(X)\eta_i(Y)\}. \tag{4.8}$$

Substituting (4.8) into (4.4), we have $(\theta(\omega))^{-1}\sum_{j=1}^{r} a_j d_j = 1$, *i.e.*, $\theta(\omega) = \sum_{i=1}^{r} a_i d_i$. From (2.1) and (3.6), we have

$$\theta(\omega) = \bar{g}(\zeta,\omega) = g(\omega,\omega) + \sum_{i=1}^{r} a_i d_i,$$

$$g(\omega,\omega) = 1 - 2\sum_{i=1}^{r} a_i d_i - \sum_{\alpha=r+1}^{n} b_\alpha^2 = 0.$$

Thus the projection of $\zeta$ on $M$ is a null vector field.

If $S(TM)$ is Riemannian vector bundle, then $\omega$ must be $\omega = \sum_{i=1}^{r} d_i\xi_i$. Consequently $\zeta$ is decomposed by

$$\zeta = \sum_{i=1}^{r} d_i\xi_i + \sum_{i=1}^{r} a_i N_i + \sum_{\alpha=r+1}^{n} b_\alpha W_\alpha.$$

From this results, $(2.20)_1$ and $\omega = \sum_{i=1}^{r} d_i\xi_i$, we have

$$g(X,\omega) = 0, \quad h_i^\ell(X,\omega) = 0.$$

Applying $\bar{\nabla}_X$ to $g(Y,\omega) = 0$ and using (2.6) and (3.7), we have

$$\sum_{i=1}^{r} d_i h_i^\ell(X,Y) = g(X,Y). \tag{4.9}$$

Using this, (3.15), (4.5) and (4.7) are reduced respectively to

$$(\nabla_X\theta)(Y) = \theta(X)\theta(Y), \quad (\nabla_Z R)(X,Y)\omega = R(X,Y)Z,$$

$$(R(U,Z)R)(X,Y)\omega = \theta(Z)R(X,Y)U - \theta(U)R(X,Y)Z = 0. \tag{4.10}$$

Replacing $U$ by $\omega$ to (4.10) and using (3.14), we have

$$R(X,Y)Z = \theta(\omega)^{-1}\,\theta(Z)\{\theta(X)Y - \theta(Y)X\}.$$

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