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A METRIC INDUCED BY THE BERGMAN KERNEL

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Abstract. In this paper, we define a metric induced by the Bergman Kernel and prove a property that the metric has under any biholomorphic mapping.

1. Introduction and notations

H. L. Royden published the infinitesimal form in [9], which is called the Kobayashi or Kobayashi-Royden metric is studied in [3], [7], [9] etc. The higher order Kobayashi metric was introduced in [11] and also investigated in [6], [8].

The Carathéodory metric and Kobayashi-Royden metric have the decreasing property under holomorphic mappings([3], [6], [9]). In particular, they have invariance properties under biholomorphic mappings. It is well-known that the Bergman metric does not have the decreasing property under holomorphic mappings, but it has the invariant property under biholomorphic mappings([2]). Moreover, the Bergman metric is Kähler, whereas both the Carathéodory metric and Kobayashi-Royden metric are only Finsler.

The Bergman kernel function has a reproducing property which was investigated in [1]. Given a Bergman kernel, we can obtain a positive definite Hermitian matrix $(g_{j,k})$. Then the matrix $(g_{j,k})$ is invertible and hence we have the inverse Hermitian matrix $(g^{j,k})$ of the Hermitian matrix $(g_{j,k})$. This Hermitian matrix $(g^{j,k})$ is also positive definite.

In this article, we will deal with the metric which is induced by the Bergman kernel. To do this, we introduce some notations which are dealt in this article. By $\mathbb{N}$ and $\mathbb{C}$ we denote the set of natural numbers and the set of complex numbers, respectively. We use the usual inner

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product $\langle\cdot,\cdot\rangle$ and the usual norm $||\cdot||$ on $\mathbb{C}^n$. By $\mathrm{Aut}(\Omega)$ we mean the group of all biholomorphic mappings of Ω onto itself.

If A is a matrix with complex entries, then the conjugate transpose of A is denoted by A^*. That is, $A^* := {}^t\bar{A}$, where $\bar{A}$ is the matrix whose entries are the complex conjugates of the corresponding entries in A and ${}^t\bar{A}$ is the transpose of $\bar{A}$.

2. The Bergman metric

Let $\Omega \subset \mathbb{C}^n$ be a domain and let $A^2(\Omega) := \{f \in \mathcal{O}(\Omega) : \int_\Omega |f|^2 dV < +\infty\}$. Define a map $\langle,\rangle_{A^2(\Omega)} : A^2(\Omega) \times A^2(\Omega) \longrightarrow \mathbb{C}$ by

$$\langle f, g\rangle_{A^2(\Omega)} := \int_\Omega f\bar{g}dV, \quad \forall f, g \in A^2(\Omega).$$

Then $(A^2(\Omega), \langle,\rangle_{A^2(\Omega)})$ forms a separable Hilbert space([4], [5], [10]). We call $(A^2(\Omega), \langle,\rangle_{A^2(\Omega)})$ the space of square integrable holomorphic functions on Ω. Note that $A^2(\mathbb{C}^n) = \{0\}$. Hence we are interested in the case when $A^2(\Omega)$ is nontrivial, in particular, when $\Omega \subset \mathbb{C}^n$ is a bounded domain.

Applying the Riesz representation theorem with a bounded domain $\Omega \subset \mathbb{C}^n$ and a fixed point $z \in \Omega$, there is a unique element in $A^2(\Omega)$, which we denote by $K(\cdot, z) \equiv K_\Omega(\cdot, z)$, such that

$$f(z) = \int_\Omega f(\zeta)K(z,\zeta)dV(\zeta) \tag{2.1}$$

for all $f \in A^2(\Omega)$. Here V denotes the usual Euclidean volume measure in $\mathbb{C}^n$. The function $K : \Omega\times\Omega \longrightarrow \mathbb{C}$ thus defined is called the Bergman kernel function for Ω. In particular, the property (2.1) is known as the reproducing property of the Bergman kernel function and it provides an example of an integral representation formula. The Bergman kernel function can almost never be calculated explicitly except for some special domains.

Proposition 2.1. ([5], [10]) *Let $K : \Omega \times \Omega \longrightarrow \mathbb{C}$ be the Bergman kernel function for a domain $\Omega \subset \mathbb{C}^n$. Then $K(z, w)$ is holomorphic as a function of $z \in \Omega$, but is conjugate-holomorphic as a function of $w \in \Omega$.*

Example 2.2. ([4], [5], [10]) *The Bergman kernel $K : \Omega \times \Omega \longrightarrow \mathbb{C}$ for the standard domains $\Omega = U^n$ and $\Omega = B_n$ is given, respectively, as follows*

- *If* $\Omega = U^n$*(the open unit polydisk in* $\mathbb{C}^n$*), then, for all* $z, w \in U^n$,

$$K(z,w) = \frac{1}{\pi^n} \prod_{j=1}^{n} \frac{1}{(1 - z_j \bar{w}_j)^2}.$$

- *If* $\Omega = B_n$*(the open unit ball in* $\mathbb{C}^n$*), then, for all* $z, w \in B_n$,

$$K(z,w) = \frac{n!}{\pi^n} \frac{1}{(1 - \langle z, w \rangle)^{n+1}}.$$

Let Ω_1 and Ω_2 be two domains in $\mathbb{C}^n$, and let $\psi : \Omega_1 \longrightarrow \Omega_2$ be a biholomorphic mapping of Ω_1 onto Ω_2. Put

$$A_\psi(z) := \det J_\psi(z), \ \ \forall z \in \Omega_1,$$

where J_ψ denotes the complex Jacobian matrix of the mapping $\psi : \Omega_1 \longrightarrow \Omega_2$. Then A_ψ is a holomorphic function on Ω_1 and we have the following:

Proposition 2.3. ([4], [5], [10]) *Let* Ω_1 *and* Ω_2 *be two domains in* $\mathbb{C}^n$ *and let* $\psi : \Omega \longrightarrow \Omega_2$ *be a biholomorphic mapping of* Ω_1 *onto* Ω_2*. Then*

$$K_{\Omega_1}(z,w) = K_{\Omega_2}(\psi(z), \psi(w)) A_\psi(z) \overline{A_\psi(w)}, \ \ \forall z, w \in \Omega_1,$$

where K_{Ω_j} *denotes the Bergman kernel of* Ω_j*. In particular, if* $\psi \in Aut(\Omega)$*, then*

$$K_\Omega(z,w) = K_\Omega(\psi(z), \psi(w)) A_\psi(z) \overline{A_\psi(w)}, \ \ \forall z, w \in \Omega.$$

Proposition 2.4. ([5], [10]) *Let* $\Omega \subset \mathbb{C}^n$ *be a bounded domain and let* $K : \Omega \times \Omega \longrightarrow \mathbb{C}$ *be the Bergman kernel of* Ω*. Then the following holds:*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \log K(z,z)}{\partial z_j \partial \bar{z}_k} \zeta_j \bar{\zeta}_k > 0, \quad \text{if } z \in \Omega \ \& \ \zeta \in \mathbb{C}^n - \{0\}.$$

Definition 2.5. *Let* $\Omega \subset \mathbb{C}^n$ *be a domain. A system* $g = (g_{\nu,\mu})_{1 \le \nu,\mu \le n}$ *of continuous functions* $g_{\nu,\mu} : \Omega \longrightarrow \mathbb{C}$ *is a Hermitian metric (resp. pseudo-metric) on* Ω *if the following conditions hold:*

$$\begin{cases} \circ \ \ g_{\nu,\mu} = \overline{g_{\mu,\nu}} \text{ for all } \nu, \mu; \\ \circ \ \ \sum_{\nu,\mu=1}^{n} g_{\nu,\mu}(z) \zeta_\nu \bar{\zeta}_\mu > 0 \, (\text{resp.} \ge 0) \text{ for all } z \in \Omega \ \& \ \zeta \in \mathbb{C}^n - \{0\}. \end{cases}$$

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain and define, for all $j, k \in \{1, 2, \cdots, n\}$,

$$g^\Omega_{j,k}(z) := \frac{\partial^2 \log K(z,z)}{\partial z_j \partial \bar{z}_k}, \ \ \forall z \in \Omega.$$

Then, Proposition 2.4 says that the Hermitian form

$$\sum_{j,k=1}^{n} g_{j,k}^{\Omega}(z) dz_j d\bar{z}_k, \quad \forall z \in \Omega,$$

is positive definite on $\mathbb{C}^n$. Hence a function $\beta_\Omega : \Omega \times \mathbb{C}^n \longrightarrow \mathbb{R}$ defined by

$$\beta_\Omega(z,\zeta) := \left[\sum_{j,k=1}^{n} g_{j,k}^{\Omega}(z)\zeta_j\bar{\zeta}_k \right]^{\frac{1}{2}}$$

is clearly the metric on Ω which is called the Bergman metric on Ω. In this case, $\beta_\Omega(z,\zeta)$ means the length of the vector $\zeta \in \mathbb{C}^n$ in the Bergman metric β_Ω on Ω.

For the standard domains $\Omega = U^n$ and $\Omega = B_n$, we can easily calculate from Example 2.2 that the entries of Hermitian matrix $(g_{jk}^{\Omega}(z))$ is given, respectively, by

$$g_{j,k}^{\Omega}(z) = \begin{cases} \frac{2}{(1-|z_j\bar{z}_k|)^2}\delta_{jk}, & \text{if } z \in U^n, \\ \frac{n+1}{(1-||z||^2)^2}[(1-||z||^2)\delta_{jk} + \bar{z}_j z_k], & \text{if } z \in B_n. \end{cases}$$

Hence we obtains the followings

Example 2.6. ([4], [5]) *We present effective formulas of Bergman metrics β_Ω for the standard domains $\Omega = U^n$ and $\Omega = B_n$:*

$$\beta_{U^n}(z,\zeta) = \sqrt{2}\sqrt{\sum_{j=1}^{n}\left(\frac{|\zeta_j|}{1-|z_j|^2}\right)^2}, \quad \textit{if } (z,\zeta) \in U^n \times \mathbb{C}^n,$$

$$\beta_{B_n}(z,\zeta) = \sqrt{n+1}\sqrt{\frac{||\zeta||^2}{1-||z||^2} + \frac{|\langle z,\zeta\rangle|^2}{(1-||z||^2)^2}}, \quad \textit{if } (z,\zeta) \in B_n \times \mathbb{C}^n.$$

Theorem 2.7. ([4], [5], [10]) *Let Ω_1, Ω_2 be domains in $\mathbb{C}^n$ and let $\psi : \Omega_1 \longrightarrow \Omega_2$ be a biholomorphic mapping. Then we have the following formula*

$$\beta_{\Omega_1}(z,\zeta) = \beta_{\Omega_2}(\psi(z), J_\psi(z)\zeta), \quad \textit{if } (z,\zeta) \in \Omega_1 \times \mathbb{C}^n.$$

In other words,

$$\sum_{j,k=1}^{n} g_{j,k}^{\Omega_1}(z)\zeta_j\bar{\zeta}_k = \sum_{l,m=1}^{n} g_{l,m}^{\Omega_2}(\psi(z))(J_\psi(z)\zeta)_l(\overline{J_\psi(z)\zeta})_m,$$

Here $(J_\psi(z))\zeta)_j$ denotes the j-th component of the vector $J_\psi(z)\zeta$.

3. The metric induced by the Hermitian matrix $(g^{j,k})$

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain and define, for all $j,k \in \{1,2,\cdots,n\}$,

$$g^{\Omega}_{j,k}(z) := \frac{\partial^2 \log K(z,z)}{\partial z_j \partial \bar{z}_k}, \quad \forall z \in \Omega.$$

Then the Hermitian $n \times n$ matrix $(g^{\Omega}_{j,k}(z))$ for $z \in \Omega$ is positive definite. Hence it is invertible. Let $(g^{j,k}_{\Omega}(z))$ denote the inverse matrix of the Hermitian $n \times n$ matrix $(g^{\Omega}_{j,k}(z))$ for $z \in \Omega$.

In case of the unit open ball $B_n \subset \mathbb{C}^n$, we have, for all $z \in B_n$,

$$g^{B_n}_{j,k}(z) = \frac{n+1}{(1-||z||^2)^2}[(1-||z||^2)\delta_{jk} + \bar{z}_j z_k].$$

Let us take $n = 2$ to simplify the calculation. Then we have the Hermitian 2×2 matrix

$$\left(g^{B_2}_{i,j}(z)\right) = \frac{3}{(1-||z||^2)^2}\begin{pmatrix} 1-|z_2|^2 & \bar{z}_1 z_2 \\ \bar{z}_2 z_1 & 1-|z_1|^2 \end{pmatrix}.$$

An elementary computation now shows that

$$\left(g^{i,j}_{B_2}(z)\right) = \frac{1-||z||^2}{3}\begin{pmatrix} 1-|z_1|^2 & -\bar{z}_1 z_2 \\ -\bar{z}_2 z_1 & 1-|z_2|^2 \end{pmatrix} = \frac{1-||z||^2}{3}\left(\delta_{ij} - \bar{z}_i z_j\right).$$

Continuing in this fashion, we obtains the following:

Proposition 3.1. *The entries of Hermitian $n \times n$ matrix $(g^{j,k}_{\Omega}(z))$ for the standard domains $\Omega = U^n$ and $\Omega = B_n$ is given, respectively, by*

$$g^{j,k}_{\Omega}(z) = \begin{cases} \frac{1}{2}(1-|z_j\bar{z}_k|)^2\delta_{jk}, & \text{if } z \in \Omega = U^n, \\ \frac{(1-||z||^2)}{n+1}[\delta_{jk} - \bar{z}_j z_k], & \text{if } z \in \Omega = B_n. \end{cases}$$

The Hermitian form

$$\sum_{j,k=1}^{n} g^{j,k}_{\Omega}(z) dz_j d\bar{z}_k, \quad \forall z \in \Omega,$$

is positive definite on $\mathbb{C}^n$. Hence it defines a metric $\gamma_\Omega : \Omega \times \mathbb{C}^n \longrightarrow \mathbb{R}$ by

$$\gamma_\Omega(z,\zeta) =: \left[\sum_{j,k=1}^{n} g^{j,k}_{\Omega}(z)\zeta_j\bar{\zeta}_k\right]^{\frac{1}{2}}, \quad \text{if } (z,\zeta) \in \Omega \times \mathbb{C}^n.$$

We call the mapping $\gamma_\Omega : \Omega \times \mathbb{C}^n \longrightarrow \mathbb{R}$ the metric induced by Hermitian matrix $(g_\Omega^{j,k}(z))$ on Ω. Then we obtain the followings from Proposition 3.1:

Proposition 3.2. *The effective formulas of the metrics γ_Ω for the standard domains U^n and B_n is given, respectively, by*

$$\gamma_{U^n}(z,\zeta) = \left[\frac{1}{2} \sum_{j=1}^{n} (1-|z_j|^2)^2 |\zeta_j|^2 \right]^{\frac{1}{2}}, \quad \textit{if } (z,\zeta) \in U_n \times \mathbb{C}^n,$$

and

$$\gamma_{B_n}(z,\zeta) = \left[\frac{(1-||z||^2)}{n+1} \left(||\zeta||^2 - |\langle z,\zeta\rangle|^2 \right) \right]^{\frac{1}{2}}, \quad \textit{if } (z,\zeta) \in B_n \times \mathbb{C}^n.$$

Theorem 3.3. *Let Ω_1, Ω_2 be domains in $\mathbb{C}^n$ and let $\psi : \Omega_1 \longrightarrow \Omega_2$ be a biholomorphic mapping. Then we have the following formula*

$$\sum_{j,k=1}^{n} g_{\Omega_1}^{j,k}(z)\zeta_j\bar{\zeta}_k = \sum_{l,m=1}^{n} g_{\Omega_2}^{l,m}(\psi(z))({}^tJ_{\psi^{-1}}(\psi(z))\bar{\zeta})_l^*({}^tJ_{\psi^{-1}}(\psi(z))\bar{\zeta})_m,$$

where $(J_{\psi^{-1}}(\psi(z))\zeta)_j$ denotes the j-th component of the vector $J_{\psi^{-1}}(\psi(z))\zeta$.

Proof. By Theorem 2.7, we have the equality

$$g_{j,k}^{\Omega_1}(z) = \sum_{l,m=1}^{n} g_{l,m}^{\Omega_2}(\psi(z)) \frac{\partial \psi_l(z)}{\partial z_j} \overline{\frac{\partial \psi_m(z)}{\partial z_k}}, \quad \forall j,k \in \{1,2,\cdots,n\}. \tag{3.2}$$

Hence if we use the following $n \times n$ matrices representations

$$A = (g_{j,k}^{\Omega_1}(z)), \quad B = (g_{j,k}^{\Omega_2}(\psi(z))) \;\;\&\;\; C = {}^tJ_\psi(z),$$

we can rewrite the equality (3.2) as $A = CBC^*$. Then since

$$A^{-1} = (C^*)^{-1}B^{-1}C^{-1} = (C^{-1})^*B^{-1}C^{-1},$$

we get the equality, with $\psi^{-1} =: (\eta_1, \cdots, \eta_n)$,

$$g_{\Omega_1}^{j,k}(z) = \sum_{l,m=1}^{n} g_{\Omega_2}^{l,m}(\psi(z)) \overline{\frac{\partial \eta_l(\psi(z))}{\partial z_j}} \frac{\partial \eta_m(\psi(z))}{\partial z_k}, \quad \forall j,k \in \{1,2,\cdots,n\}.$$

Hence we have

$$\sum_{j,k=1}^{n} g_{\Omega_1}^{j,k}(z)\zeta_j\bar{\zeta}_k = \sum_{l,m=1}^{n} g_{\Omega_2}^{l,m}(\psi(z))({}^tJ_{\psi^{-1}}(\psi(z))\bar{\zeta})_l^*({}^tJ_{\psi^{-1}}(\psi(z))\bar{\zeta})_m,$$

for all $(z,\zeta) \in \Omega_1 \times \mathbb{C}^n$. $\square$

Corollary 3.4. *Let Ω_1, Ω_2 be domains in $\mathbb{C}^n$ and let $\psi : \Omega_1 \longrightarrow \Omega_2$ be a biholomorphic mapping. Then we have the following formula, for all $(z, \zeta) \in \Omega_1 \times \mathbb{C}^n$,*

$$\gamma_{\Omega_1}(z, \zeta) = \gamma_{\Omega_2}(\psi(z), \overline{J_{\psi^{-1}}(\psi(z))\bar{\zeta}}).$$

In order to simplify the calculation, let us take a simple biholomorphic mapping and check our assertion in Theorem 3.3 for the case of the standard domains.

Example 3.5. *For a point $(a_1, a_2, \cdots, a_n) \in U^n$ fixed, the mapping $\psi : U^n \longrightarrow U^n$ defined by*

$$\psi(z) = \left(\frac{a_1 - z_1}{1 - \bar{a}_1 z_1}, \cdots, \frac{a_n - z_n}{1 - \bar{a}_n z_n} \right)$$

is in $Aut(U^n)$ and $\psi(\psi(z)) = z$ for all $z \in U_n$. By the simple calculation we have the diagonal matrix

$$J_{\psi^{-1}}(\psi(z)) = \begin{pmatrix} -\frac{(1-\bar{a}_1 z_1)^2}{1-|a_1|^2} & & & \\ & -\frac{(1-\bar{a}_2 z_2)^2}{1-|a_2|^2} & & \\ & & \ddots & \\ & & & -\frac{(1-\bar{a}_n z_n)^2}{1-|a_n|^2} \end{pmatrix}$$

and also, by Proposition 3.1, we have

$$g_{U^n}^{l,l}(\psi(z)) = \frac{1}{2} \frac{(1 - |a_l|^2)^2 (1 - |z_l|^2)^2}{|1 - \bar{a}_l z_l|^4}, \quad \forall l \in \{1, 2, \cdots, n\}.$$

From these facts, we obtains the following equality

$$\overline{J_{\psi^{-1}}(\psi(z))}(g_{U^n}^{i,j}(\psi(z)))^t J_{\psi^{-1}}(\psi(z)) = (g_{U^n}^{i,j}(z)).$$

In other words,

$$\sum_{l,m=1}^{n} g_{U^n}^{l,m}(\psi(z))({}^t J_{\psi^{-1}}(\psi(z))\bar{\zeta})_l^* ({}^t J_{\psi^{-1}}(\psi(z))\bar{\zeta})_m = \sum_{j,k=1}^{n} g_{U^n}^{j,k}(z)\zeta_j \bar{\zeta}_k$$

for all $(z, \zeta) \in U^n \times \mathbb{C}^n$.

Example 3.6. *In order to consider the case of the unit open n-ball $B_n \subset \mathbb{C}^n$, let us take a complex number $a \in \mathbb{C}$ with $|a| < 1$. Then the mapping $\psi : B_n \longrightarrow B_n$ defined by*

$$\psi(z) = \left(\frac{a - z_1}{1 - \bar{a} z_1}, -\frac{\sqrt{1 - |a|^2}}{1 - \bar{a} z_1} z_2, \cdots, -\frac{\sqrt{1 - |a|^2}}{1 - \bar{a} z_1} z_n \right)$$

is in $Aut(B_n)$ *and* $\psi(\psi(z)) = z$ *for all* $z \in B_n$*. By the simple calculation we have*

$$J_{\psi^{-1}}(\psi(z)) = \begin{pmatrix} -\frac{(1-\bar{a}z_1)^2}{1-|a|^2} & 0 & 0 & \cdots & 0 \\ \frac{\bar{a}(1-\bar{a}z_1)}{1-|a|^2} z_2 & -\frac{1-\bar{a}z_1}{\sqrt{1-|a|^2}} & 0 & \cdots & 0 \\ \frac{\bar{a}(1-\bar{a}z_1)}{1-|a|^2} z_3 & 0 & -\frac{1-\bar{a}z_1}{\sqrt{1-|a|^2}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\bar{a}(1-\bar{a}z_1)}{1-|a|^2} z_n & 0 & 0 & \cdots & -\frac{1-\bar{a}z_1}{\sqrt{1-|a|^2}} \end{pmatrix}$$

and

$$\left(g_{B_n}^{i,j}(\psi(z))\right)_{n\times n} = \alpha \begin{pmatrix} 1-\frac{|a-z_1|^2}{|1-\bar{a}z_1|^2} & \frac{\sqrt{1-|a|^2}(\bar{a}-\bar{z}_1)}{|1-\bar{a}z_1|^2} z_2 & \cdots & \frac{\sqrt{1-|a|^2}(\bar{a}-\bar{z}_1)}{|1-\bar{a}z_1|^2} z_n \\ \frac{\sqrt{1-|a|^2}(a-z_1)}{|1-\bar{a}z_1|^2} \bar{z}_2 & 1-\frac{(1-|a|^2)}{|1-\bar{a}z_1|^2} \bar{z}_2 z_2 & \cdots & -\frac{1-|a|^2}{|1-\bar{a}z_1|^2} \bar{z}_2 z_n \\ \frac{\sqrt{1-|a|^2}(a-z_1)}{|1-\bar{a}z_1|^2} \bar{z}_3 & -\frac{(1-|a|^2)}{|1-\bar{a}z_1|^2} \bar{z}_3 z_2 & \cdots & -\frac{1-|a|^2}{|1-\bar{a}z_1|^2} \bar{z}_3 z_n \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{1-|a|^2}(a-z_1)}{|1-\bar{a}z_1|^2} \bar{z}_n & -\frac{1-|a|^2}{|1-\bar{a}z_1|^2} \bar{z}_n z_2 & \cdots & 1-\frac{(1-|a|^2)}{|1-\bar{a}z_1|^2} \bar{z}_n z_n \end{pmatrix}.$$

Here $\alpha = \frac{1-||z||^2}{n+1} \frac{1-|a|^2}{|1-a\bar{z}_1|^2}$*. Therefore, the matrix multiplication*

$$\overline{J_{\psi^{-1}}(\psi(z))} \left(g_{B_n}^{i,j}(\psi(z))\right) {}^t J_{\psi^{-1}}(\psi(z))$$

results in a matrix expression

$$\alpha \begin{pmatrix} \frac{|1-a\bar{z}_1|^2}{1-|a|^2}(1-|z_1|^2) & -\frac{|1-a\bar{z}_1|^2}{1-|a|^2} \bar{z}_1 z_2 & \cdots & -\frac{|1-a\bar{z}_1|^2}{1-|a|^2} \bar{z}_1 z_n \\ -\frac{|1-a\bar{z}_1|^2}{1-|a|^2} \bar{z}_2 z_1 & \frac{|1-a\bar{z}_1|^2}{1-|a|^2}(1-|z_2|^2) & \cdots & -\frac{|1-a\bar{z}_1|^2}{1-|a|^2} \bar{z}_2 z_n \\ -\frac{|1-a\bar{z}_1|^2}{1-|a|^2} \bar{z}_3 z_1 & -\frac{|1-a\bar{z}_1|^2}{1-|a|^2} \bar{z}_3 z_2 & \cdots & -\frac{|1-a\bar{z}_1|^2}{1-|a|^2} \bar{z}_3 z_n \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{|1-a\bar{z}_1|^2}{1-|a|^2} \bar{z}_n z_1 & -\frac{|1-a\bar{z}_1|^2}{1-|a|^2} \bar{z}_n z_2 & \cdots & \frac{|1-\bar{a}z_1|^2}{1-|a|^2}(1-|z_n|^2) \end{pmatrix}.$$

Equivalently,

$$\overline{J_{\psi^{-1}}(\psi(z))} \left(g_{B_n}^{i,j}(\psi(z))\right) {}^t J_{\psi^{-1}}(\psi(z)) = \left(g_{B_n}^{i,j}(z)\right).$$

Thus we have, for all $(z,\zeta) \in B_n \times \mathbb{C}^n$,

$$\begin{aligned}
&\sum_{l,m=1}^{n} g_{B_n}^{l,m}(\psi(z))({}^tJ_{\psi^{-1}}(\psi(z))\bar{\zeta})_l^*({}^tJ_{\psi^{-1}}(\psi(z))\bar{\zeta})_m \\
&\qquad = \frac{1-||z||^2}{n+1}[(1-|z_1|^2)\bar{\zeta}_1\zeta_1 - z_1\bar{z}_2\bar{\zeta}_1\zeta_2 - \cdots - z_1\bar{z}_n\bar{\zeta}_1\zeta_n \\
&\qquad\qquad - z_2\bar{z}_1\zeta_1\bar{\zeta}_2 + (1-|z_2|^2)\zeta_2\bar{\zeta}_2 - \cdots - z_2\bar{z}_n\bar{\zeta}_2\zeta_n \\
&\qquad\qquad - z_n\bar{z}_1\zeta_1\bar{\zeta}_n - z_n\bar{z}_2\zeta_2\bar{\zeta}_n - \cdots + (1-|z_n|^2)\zeta_n\bar{\zeta}_n] \\
&\qquad = \sum_{j,k=1}^{n} g_{B_n}^{j,k}(z)\zeta_j\bar{\zeta}_k.
\end{aligned}$$

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