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A COUPLED $\mathcal{N}$-STRUCTURE WITH AN APPLICATION IN A SUBTRACTION ALGEBRA

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Abstract. In this paper, we introduce a coupled $\mathcal{N}$-structure which is the generalization of $\mathcal{N}$-structure. Using this coupled $\mathcal{N}$-structure, we have applied in a subtraction algebra and have introduced the notion of a coupled $\mathcal{N}$-subalgebra, a coupled $\mathcal{N}$-ideal. Also the characterization of coupled $\mathcal{N}$-ideal is presented.

1. Introduction

The system of the form Φ, where $(\Phi; \circ, \backslash)$, considered by B. M. Schein [13], is a set of functions closed under the composition " $\circ$ " of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction " $\backslash$ " (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [15] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [5] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [6], Y. B. Jun et al. established the ideal generated by a set, and discussed related results.

To solve complicated problems in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics

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which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. The fundamental concept of fuzzy set is a mapping from non-empty set S to unit closed interval was published by Zadeh in his paper [14] in 1965, was applied to generalize some of the basic concepts of algebra. The fuzzy algebraic structures play a prominent role in mathematics with wide applications in many other branches such as theoretical physics, computer science, control engineering, information sciences, coding theory, topological spaces, logic, set theory, group theory, groupoids, real analysis, measure theory etc.The fuzzification of ideal theory in subtraction algebras were discussed detail in [12].

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to 0, 1$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point 1 into the interval $[0, 1]$. Because no negative meaning of information is suggested. To attain such object, Jun et al. [7] introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. After that, in [9] Khan et al. have introduced the notion of of $\mathcal{N}$-fuzzy left (right) ideals in ordered semigroups and characterize ordered semigroups in terms of $\mathcal{N}$-fuzzy left (right) ideals and have characterized left regular (right regular) and left simple (right simple) ordered semigroups in terms of $\mathcal{N}$-fuzzy left ($\mathcal{N}$-fuzzy right) ideals. Also, in [11] Khan et al. have introduced the concepts $\mathcal{N}$-fuzzy filters, $\mathcal{N}$-fuzzy bi-ideal subsets and $\mathcal{N}$-fuzzy bi-filters of ordered semigroups and have characterized ordered semigroups in terms of $\mathcal{N}$-fuzzy filters, $\mathcal{N}$-fuzzy bi-ideal subsets and $\mathcal{N}$-fuzzy bi-filters. They have discussed the relationship of $\mathcal{N}$-fuzzy bi-filters and prime $\mathcal{N}$-fuzzy bi-ideal subsets of ordered semigroups. Recently, using $\mathcal{N}$-structures, Jun et al. [8] have introduced the notion of an $\mathcal{N}$-ideals in a subtraction algebras. The characterizations of an $\mathcal{N}$-ideal were discussed in detail. Conditions for an $\mathcal{N}$-structure to be an $\mathcal{N}$-ideal are provided. The description of a created $\mathcal{N}$-ideal was established.

Antanassov [2, 3] introduces the concept of intuitionistic fuzzy sets, which constitute a extension of fuzzy sets theory: intuitionistic fuzzy sets give both a membership degree and a non-membership degree. The only constraint on these two degrees is that the sum must be smaller than or equal to 1. In this intuitionistic fuzzy sets, the negative meanings of information are not suggested. This motivate us to develop a coupled $\mathcal{N}$-structures. In [10] Jun et al., have introduced the concept of coupled $\mathcal{N}$-structures and applied it in the subtraction algebras. In this paper, we have introduced a notion of coupled $\mathcal{N}$-ideals of subtraction algebras and have studied their related properties.

2. Preliminary

In this section, we cite the fundamental definitions that will be used in the sequel:

Definition 2.1. [5] A non-empty set X together with a binary operation "–" is said to be a *subtraction algebra* if it satisfies the following:
(S1) $x - (y - x) = x$.
(S2) $x - (x - y) = y - (y - x)$.
(S3) $(x - y) - z = (x - z) - y$, for all $x, y, z \in X$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X : $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$; then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c)). \end{aligned}$$

In a subtraction algebra, the following are true (see [5]):

(a1) $(x - y) - y = x - y$,
(a2) $x - 0 = x$ and $0 - x = 0$,
(a3) $(x - y) - x = 0$,
(a4) $x - (x - y) \leq y$,
(a5) $(x - y) - (y - x) = x - y$,
(a6) $x - (x - (x - y)) = x - y$,

(a7) $(x - y) - (z - y) \leq x - z$,
(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$,
(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$,
(a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$,
(a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$,
(a12) $(x - y) - z = (x - z) - (y - z)$.

Definition 2.2. [5] A nonempty subset A of a subtraction algebra X is called an *ideal* of X; denoted by $A \lhd X$; if it satisfies:

(b1) $a - x \in A$ for all $a \in A$ and $x \in X$,
(b2) for all $a, b \in A$, whenever $a \vee b$ exists in X then $a \vee b \in A$.

Proposition 2.3. [5] *A nonempty subset A of a subtraction algebra X is an ideal of X if and only if it satisfies:*

(b3) $0 \in A$,
(b4) $(\forall x \in X)\ (\forall y \in A)\ (x - y \in A \Rightarrow x \in A)$.

Proposition 2.4. [5] *Let X be a subtraction algebra and let $x, y \in X$. If $w \in X$ is an upper bound for x and y, then the element*

$$x \vee y := w - ((w - y) - x)$$

is a least upper bound for x and y.

Definition 2.5. [4] Let X be a subtraction algebra and Y be a nonempty subset of X. Then Y is called a *subalgebra* of X if $x - y \in Y$, whenever $x, y \in Y$.

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a negative-valued function from X to $[-1, 0]$ (briefly, $\mathcal{N}$-function on X). By an $\mathcal{N}$-structure we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X.

For any $\mathcal{N}$-function f on X and $t \in [-1, 0)$, the set

$$C(f, t) := \{x \in X \mid f(x) \leq t\}$$

is called a *closed*$(f, t) - $*cut* of (X, f).

Definition 2.6. [8] By *an ideal (resp. subalgebra)* of X based on $\mathcal{N}$-function f (briefly, *$\mathcal{N}$-ideal (resp. $\mathcal{N}$-subalgebra)* of X), we mean an $\mathcal{N}$-structure (X, f) in which every nonempty closed $(f, t) - $*cut* of (X, f) is *an ideal* (*resp. subalgebra*) of X, for all $t \in [-1, 0)$.

−	**0**	**a**	**b**
0	0	0	0
a	a	0	a
b	b	b	0

TABLE 1

Theorem 2.7. *Let (X, f) be an $\mathcal{N}$-structure of X. Then (X, f) is called a $\mathcal{N}$-subalgebra of X if and only if it satisfies*
(NS) $f(x - y) \leq max\{f(x), f(y)\}$, *whenever* $x, y \in X$.

Proof. The proof is straight forward from the definition 2.6. □

Theorem 2.8. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-ideal of X if and only if it satisfies the following assertions:*
(NI1) $(x, y \in X)$ $(f(x - y) \leq f(x))$,
(NI2) $(x, y \in X)$ $(\exists(x \vee y) \Rightarrow f(x \vee y) \leq max\{f(x), f(y)\})$.

Theorem 2.9. *Let (X, f) be an $\mathcal{N}$-structure of X. Then the following conditions are holds:*

1. $(\forall x, a, b \in X)$ $(a - x \leq b \Rightarrow f(x) \leq\ max\{f(a), f(b)\})$ *then* (X, f) *is an $\mathcal{N}$-subalgebra of X,*
2. *Any $\mathcal{N}$-ideal (X, f) of X is an $\mathcal{N}$-subalgebra of X.*

Proof. (1) Let $x, y \in X$ be such that by (a4), we have $x - (x - y) \leq y$. Thus

$$f(x - y) \leq max\{f(x), f(y)\}.$$

Hence f is an $\mathcal{N}$-subalgebra of X.

(2) Assume that (X, f) is an $\mathcal{N}$-ideal of X. Then, there exists $x \vee y \in X$ such that

$$x - y \leq x \leq w - ((w - y) - x) := x \vee y$$

for all $w, x, y \in X$. Thus, by (NI2)

$$f(x - y) \leq f(x \vee y) \leq \max\{f(x), f(y)\}.$$

Hence (X, f) is an $\mathcal{N}$-subalgebra of X. □

Example 2.10. Let $X = \{0, a, b\}$ be a subtraction algebra with the Cayley table which is given in Table 1. Let (X, f) be a $\mathcal{N}$-structure defined by:

$$f(0) = -0.7, \quad f(a) = -0.3 \quad \text{and} \quad f(b) = -0.6.$$

−	**0**	**a**	**b**	**c**
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Table 2

It is easy to check that (X, f) is both an $\mathcal{N}$-subalgebra and an $\mathcal{N}$-ideal of X.

However, the following example shows that the converse of Theorem 2.9 of (2) is not true.

Example 2.11. Let $X = \{0, a, b, c\}$ be a subtraction algebra with the Cayley table which is given in Table 2. Let (X, f) be an $\mathcal{N}$-structure defined by:

$$f(0) = -0.7, \quad f(a) = -0.4, \quad f(b) = -0.5 \quad \text{and} \quad f(c) = -0.6.$$

It is easy to check that (X, f) is an $\mathcal{N}$-subalgebra, but not an $\mathcal{N}$-ideal of X. Since

$$f(b \vee b) = f(a - ((a - b) - b)) = f(a) = -0.4 \nleq -0.5 = \max\{f(b), f(b)\}.$$

3. Coupled $\mathcal{N}$-ideals of subtraction algebras

Let X be a nonempty fixed set. A coupled $\mathcal{N}$-structure (BNS for short) is an object of the form

$$N = \{(x, f_N, g_N) : x \in X\}$$

where f_N and g_N are negative-valued functions from X to $[-1, 0]$ with $0 \geq f_N(x) + g_N(x) \geq -1$, for all $x \in X$. For the sake of simplicity, we use the symbol (X, f_N, g_N) for the BNS $N = \{(x, f_N, g_N) : x \in X\}$.

In what follows, let X denote a subtraction algebra and f_N, g_N are $\mathcal{N}$-functions on X unless otherwise specified.

For any $\mathcal{N}$-functions f_N and g_N on X and $t \in [-1, 0)$, the set

$$\mathcal{C}(f_N, g_N; t) := \{x \in X | f_N(x) \leq t \text{ and } g_N(x) \geq t\}$$

is called *level set* of (X, f_N, g_N).

Definition 3.1. A coupled $\mathcal{N}$-structure (X, f_N, g_N) of X is called a *coupled $\mathcal{N}$-ideal* of X, if it satisfies:

(BNI1) $(\forall x, y \in X), (f_A(x - y) \leq f_A(x),$
(BNI2) $(\forall x, y \in X), (\exists\, x \vee y \Rightarrow f_A(x \vee y) \leq \max\{f_A(x), f_A(y)\},$
(BNI3) $(\forall x, y \in X), (g_A(x - y) \geq g_A(x),$
(BNI4) $(\forall x, y \in X), (\exists\, x \vee y \Rightarrow g_A(x \vee y) \geq \min\{g_A(x), g_A(y)\},$

Example 3.2. Let $X = \{0, a, b, c\}$ be a subtraction algebra with the Cayley table which is given in Table 2. Let (X, f_N, g_N) be a coupled $\mathcal{N}$-structure defined by:
$f_N(0) = -0.7$, $f_N(a) = -0.6$, $f_N(b) = -0.3$, $f_N(c) = -0.1$.
and
$g_N(0) = -0.1$, $g_N(a) = -0.3$, $g_N(b) = -0.5$, $g_N(c) = -0.6$.

By routine calculations, we can prove that (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal .

The following theorem provide the relation between the coupled $\mathcal{N}$-ideal and an ideal on $\mathcal{C}(f_N, g_N; t)$.

Theorem 3.3. *Let (X, f_N, g_N) be a coupled $\mathcal{N}$-structure of X and $\mathcal{C}(f_N, g_N; t)$ be a level set of X. If (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X if and only if $\mathcal{C}(f_N, g_N; t)$ is an ideal of X , for all $t \in [-1, 0)$.*

Proof. Let (X, f_N, g_N) be a coupled $\mathcal{N}$-ideal of X and $\mathcal{C}(f_N, g_N; t)$ be a level set in X.

(i) For all $x, y \in \mathcal{C}(f_N, g_N; t)$, $\forall t \in [-1, 0)$ then $f_N(x - y) \leq f_N(x) \leq t$ and $g_N(x - y) \geq g_N(x) \geq t$. This implies $x - y \in \mathcal{C}(f_N, g_N; t)$. This satisfies the condition (b1).

(ii) For all $x, y \in \mathcal{C}(f_N, g_N; t)$, $\forall t \in [-1, 0)$. Then

$$f_N(x \vee y) \leq \max\{f_N(x), f_N(y)\} \leq t.$$

and

$$g_N(x \vee y) \geq \min\{g_N(x), g_N(y)\} \geq t.$$

This implies there exist $x \vee y \in \mathcal{C}(f_N, g_N; t)$. Thus $\mathcal{C}(f_N, g_N; t)$ is an ideal of (X, f_N, g_N).

Conversely, (i) Suppose that (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X, that is, every non-empty $\mathcal{C}(f_N, g_N; t)$ of (X, f_N, g_N) is an ideal of X for all $t \in [-1, 0)$. If there are $a, b \in X$ such that $f_N(a-b) > f_N(a)$ and $g_N(a-b) > g_N(a)$, then $f_N(a - b) > t_0 \geq f_N(a)$ and $g_N(a - b) < t_0 \leq g_N(a)$ for some $t_0 \in [-1, 0)$. Thus $a \in \mathcal{C}(f_N, g_N; t_0)$, but $a - b \notin \mathcal{C}(f_N, g_N; t_0)$.

This is a contradiction, and so $f_N(x-y) \leq f_N(x)$ and $g_N(x-y) \geq g_N(x)$ for all $x, y \in X$. Thus $\mathcal{C}(f_N, g_N; t)$ satisfies (BNI1) and (BNI3).

(ii) Assume that there exist $a, b \in X$ such that $a \vee b$ exists such that $f_N(a \vee b) > \max\{f_N(a), f_N(b)\}$ and $g_N(a \vee b) < \max\{g_N(a), g_N(b)\}$. Then $f_N(a \vee b) > t_1 \geq \max\{f(a), f(b)\}$ and $g_N(a \vee b) < t_1 \leq \max\{g_N(a), g_N(b)\}$ for some $t_1 \in [-1, 0)$. It follows that $a, b \in \mathcal{C}(f_N, g_N; t_1)$ and $a \vee b \notin \mathcal{C}(f_N, g_N; t_1)$ which is a contradiction. Thus $\mathcal{C}(f_N, g_N; t)$ satisfies (BNI2) and (BNI4). Hence $\mathcal{C}(f_N, g_N; t)$ is a coupled $\mathcal{N}$-ideal of $\mathcal{C}(f_N, g_N ; t)$. □

Theorem 3.4. *Let A be an ideal of X and let (X, f_N, g_N) be a coupled $\mathcal{N}$-ideal in X defined by*

$$f_N(x) = \begin{cases} \alpha_0, & \text{if } x \in A, \\ \alpha_1, & \text{otherwise.} \end{cases} \qquad g_N(x) = \begin{cases} \beta_0, & \text{if } x \in A, \\ \beta_1, & \text{otherwise.} \end{cases}$$

for all $x \in X$ and $\alpha_i, \beta_i \in [-1, 0]$ such that $\alpha_0 < \alpha_1$, $\beta_0 > \beta_1$ and $\alpha_i + \beta_i \geq -1$ for $i = 0, 1$. Then (X, f_N, g_N) coupled $\mathcal{N}$-ideal of X and $(X, f_N, g_N) = A$.

Proof. Let $x, y \in X$. If $x, y \notin A$, then

$$f_N(x-y) \leq \alpha_1 = f_N(x),$$

and

$$g_N(x-y) \geq \beta_1 = g_N(x).$$

Also, we have

$$f_N(x \vee y) \leq \alpha_1 = \max\{f_N(x), f_N(y)\}$$

and

$$g_N(x \vee y) \geq \beta_1 = \min\{g_N(x), g_N(y)\}$$

Other cases are trivial, and we omit the proof.
Hence (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X. □

Theorem 3.5. *If a coupled $\mathcal{N}$-structure (X, f_N, g_N) in X is a coupled $\mathcal{N}$-ideal of X, then so is $\Box N := (X, f_N, \bar{f}_N)$, $\bar{f}_N = 1 - f_N$.*

Proof. It is sufficient to show that $\bar{f}_N$ satisfies conditions (BNI2) and (BNI4). For all $x, y \in X$, we have

$$\bar{f}_N(x-y) = 1 - f_N(x-y) \geq 1 - f_N(x) = \bar{f}_N(x)$$

and there exists $x \vee y \in X$, such that

$$\begin{aligned}\bar{f}_N(x \vee y) &= 1 - f_N(x \vee y) \\ &\geq 1 - \min\{f_N(x), f_N(y)\} \\ &= \min\{1 - f_N(x), 1 - f_N(y)\} \\ &= \min\{\bar{f}_N(x), \bar{f}_N(y)\}.\end{aligned}$$

□

Theorem 3.6. *A coupled $\mathcal{N}$-structure (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X if and only if the $\mathcal{N}$-functions f_N and $\bar{g}_N$ are $\mathcal{N}$-ideals of X.*

Proof. Let (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X. Clearly f_N is a $\mathcal{N}$-ideal of X. For all $x, y \in X$, we have

$$\bar{g}_N(x - y) = 1 - g_N(x - y) \leq 1 - g_N(x) = \bar{g}_N(x)$$

and there exists $x \vee y \in X$, such that

$$\begin{aligned}\bar{g}_N(x \vee y) &= 1 - g_N(x \vee y) \\ &\leq 1 - \max\{g_N(x), g_N(y)\} \\ &= \max\{1 - g_N(x), 1 - g_N(y)\} \\ &= \max\{\bar{g}_N(x), \bar{g}_N(y)\}.\end{aligned}$$

Hence $\bar{g}_N$ is a $\mathcal{N}$-ideal of X.

Conversely, suppose that f_N and $\bar{g}_N$ are $\mathcal{N}$-ideals of X. Let $x, y \in X$. Then

$$1 - g_N(x - y) = \bar{g}_N(x - y) \leq \bar{g}_N(x) = 1 - g_N(x)$$

and there exists $x \vee y \in X$, such that

$$\begin{aligned}1 - g_N(x \vee y) &= \bar{g}_N(x \vee y) \\ &\leq \max\{\bar{g}_N(x), \bar{g}_N(y)\} \\ &= \max\{1 - g_N(x), 1 - g_N(y)\} \\ &= 1 - \max\{g_N(x), g_N(y)\}.\end{aligned}$$

which implies that

$$g_N(x - y) \geq g_N(x)$$

and

$$g_N(x \vee y) \geq \min\{g_N(x), g_N(y)\}.$$

This completes the proof. □

Corollary 3.7. *A coupled $\mathcal{N}$-structure (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X if and only if $\Box N = (X, f_N, \bar{f}_N)$ and $\Diamond N = (X, \bar{g}_N, g_N)$ are coupled $\mathcal{N}$-ideals of X.*

Proof. The proof is straightforward. □

Definition 3.8. A $\mathcal{N}-$function χ on B is called a $\mathcal{N}$-*charateristic function* if

$$\chi_B(x) = \begin{cases} -1, & \text{if } x \in B, \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 3.9. *Let* χ_A *be the* $\mathcal{N}$*-characteristic function of an ideal* A *of* X*. Then the* $\bar{A} = (X, \chi_A, \bar{\chi}_A)$ *is a coupled* $\mathcal{N}$*-ideal of* X*.*

Proof. The proof is straightforward. □

Theorem 3.10. *Every coupled* $\mathcal{N}$*-structure* (X, f_N, g_N) *of* X *is a coupled* $\mathcal{N}$*-ideal if and only if it satisfies*

(c1) $(\forall a, b \in X)$ $(f_N(x - ((x - a) - b)) \leq max\{f_N(a), f_N(b)\})$,
(c2) $(\forall a, b \in X)$ $(g_N(x - ((x - a) - b)) \geq min\{g_N(a), g_N(b)\})$.

Proof. Let (X, f_N, g_N) be a coupled $\mathcal{N}$-structure of X satisfying (c1) and (c2). Using (a2) and (S3), we have

$$x - y = (x - y) - (((x - y) - x) - x),$$

for all $x, y \in X$. From (c1) and (c2), we have

$$\begin{aligned} f_N(x - y) &= f_N((x - y) - (((x - y) - x) - x)) \\ &\leq \max\{f_N(x), f_N(x)\} \\ &= f_N(x). \end{aligned}$$

and

$$\begin{aligned} g_N(x - y) &= g_N((x - y) - (((x - y) - x) - x)) \\ &\geq \min\{g_N(x), g_N(x)\} \\ &= g_N(x). \end{aligned}$$

Thus (BNI1) and (BNI3) are valid.

Suppose $x \in X$ is an upper bound for a and b, for all $a, b \in X$ that is $x := a \vee b$, then by proposition 2.4 and (S3), we have

$$a \vee b = x - ((x - a) - b).$$

Thus we have

$$f_N(a \vee b) = f_N(x - ((x - a) - b)) \leq \max\{f_N(a), f_N(b)\}$$

and

$$g_N(a \vee b) = g_N(x - ((x - a) - b)) \geq \min\{g_N(a), g_N(b)\}.$$

Thus (BNI2) and (BNI4) are valid. Hence coupled $\mathcal{N}$-structure is a coupled $\mathcal{N}$-ideal of X.

On the other hand, (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X. If $x \in X$ is an upper bound for a and b, for all $a, b \in X$ that is $x := a \vee b$, then we have

$$a \vee b = x - ((x - a) - b).$$

Thus

$$f_N(a \vee b) = f_N(x - ((x - a) - b)) \leq \max\{f_N(a), f_N(b)\},$$

and

$$g_N(a \vee b) = g_N(x - ((x - a) - b)) \geq \min\{g_N(a), g_N(b)\}.$$

Hence (c1) and (c2) are valid. This completes the proof. □

Proposition 3.11. *Every coupled $\mathcal{N}$-ideal of X satisfies the following inequality*

(c3) $(\forall x \in X)\ (f_N(0) \leq f_N(x))$,
(c4) $(\forall x \in X)\ (g_N(0) \geq g_N(x))$.

Proof. By taking $y := x$ in (BNI1) and (BNI2), we have

$$f_N(0) = f_N(x - x) \leq f_N(x)$$

and

$$g_N(0) = g_N(x - x) \geq g_N(x).$$ □

Corollary 3.12. *Every coupled $\mathcal{N}$-ideal (X, f_N, g_N) satisfies:*

(c5) $(\forall x, y \in X)\ (x \leq y \Rightarrow f_N(x) \geq f_N(y)$ *and* $g_N(x) \leq g_N(y))$.

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x - y = 0$ and so

$$f_N(x) = f_N(x-0) = f_N(x-((x-y)-y)) \leq \max\{f_N(y), f_N(y)\} = f_N(y),$$

and

$$g_N(x) = g_N(x-0) = g_N(x-((x-y)-y) \geq \min\{g_N(y), g_N(y)\} = g_N(y)$$

by using (a2), (c1) and (c2). This completes the proof. □

Theorem 3.13. *A coupled $\mathcal{N}$-structure (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X if and only if it satisfies the (c3), (c4) and*

(c6) $(\forall x, y, z \in X)\ (f_N(x - z) \leq$*max*$\{f_N((x - y) - z), f_N(y)\})$,
(c7) $(\forall x, y, z \in X)\ (g_N(x - z) \geq$*min*$\{g_N((x - y) - z), g_N(y)\})$.

Proof. Assume that (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X and satisfies (c5). Then the conditions (c3) and (c4) are holds by Proposition 3.11. Then, by (c1) and (c2), we have

$$f_N(x - ((x - a) - y)) \leq \max\{f_N(a), f_N(y)\}, \tag{1}$$

and

$$g_N(x-((x-a)-y)) \geq \min\{g_N(a), g_N(y)\}, \tag{2}$$

for all $a, x, y \in X$. By setting,

$$a = (x-y)-z, \tag{3}$$

for all $z \in X$. Now

$$\begin{aligned}
((x-a)-y) &= (x-((x-y)-z))-y, \\
&= (x-y)-(((x-y)-z)-y), \text{ by (a12)} \\
&= (x-y)-(((x-y)-y)-(z-y)), \text{ by (a12)} \\
&= (x-y)-((x-y)-(z-y)), \text{ by (a1)} \\
&= (x-y)-((x-z)-y), \text{ by (a12)} \\
&= (x-y)-((x-y)-z), \text{ by (S3)} \\
&\leq z, \text{ by (a4)}.
\end{aligned}$$

Thus, by using (a9), for all $x \in X$

$$x-z \leq x-((x-a)-y).$$

This implies, by using (c5)

$$f_N((x-z)) \leq f_N(x-((x-a)-y)), \tag{4}$$

and

$$g_N((x-z)) \geq g_N(x-((x-a)-y)). \tag{5}$$

Using (3), (4) in (1) and (3), (5) in (2), we have

$$f_N(x-z) \leq f_N(x-((x-((x-y)-z)-y))) \leq \max\{f_N((x-y)-z), f_N(y)\},$$

and

$$g_N(x-z) \geq g_N(((x-y)-z)_y) \geq \min\{g_N((x-y)-z), g_N(y)\},$$

for all $x, y, z \in X$.

Conversely, assume that coupled $\mathcal{N}$-structure satisfied (c3), (c4), (c6) and (c7). By using (a2), (a3), (c6), we have

$$\begin{aligned}
f_N(x-y) &= f_N((x-y)-(((x-y)-x)-x)) \\
&\leq \max\{f_N(x), f_N(x)\} \\
&= f_N(x).
\end{aligned}$$

By using (a2), (a3) and (c7), we have

$$\begin{aligned}
g_N(x-y) &= g_N((x-y)-(((x-y)-x)-x)) \\
&\geq \min\{g_N(x), g_N(x)\} \\
&= g_N(x).
\end{aligned}$$

Now, suppose $x \vee y$ exists for $x, y \in X$. Also, we have $f_N(x \vee y) \leq f_N(x)$ and $g_N(x \vee y) \geq g_N(x)$, such that

$$f_N(x \vee y) \leq \max\{f_N(x), f_N(y)\},$$

and

$$g_N(x \vee y) \geq \min\{f_N(x), f_N(y)\}.$$

Thus coupled $\mathcal{N}$-structure (X, f_N, g_N) satisfies BNI1, BNI2, BNI3 and BNI4. Hence (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal . □

Theorem 3.14. *A coupled $\mathcal{N}$-structure (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X if and only if it satisfies the condition (c3), (c4) and*

(c8) $(\forall x, y \in X)$ $(f_N(x) \leq \max\{f_N(x-y), f_N(y)\})$ *and*
(c9) $(\forall x, y \in X)$ $(g_N(x) \geq \min\{g_N(x-y), g_N(y)\})$.

Proof. Assume that (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X. Then the condition (c3) and (c4) are valid by Proposition 3.11, and the conditions (c8) and (c9) are valid by taking $z = 0$ in (c6) and (c7) respectively.

Conversely, (X, f_N, g_N) satisfying (c3), (c4), (c8) and (c9). Then

$$\begin{aligned}(x-((x-a)-b))-b &= (x-b)-(((x-a)-b)-b), \text{ by (a12)} \\ &= (x-b)-((x-a)-b), \text{ by (a1)} \\ &\leq x-(x-a), \text{ by (a7)} \\ &\leq a, \text{ by (a4)}.\end{aligned}$$

That is $((x-((x-a)-b))-b)-a = 0$, for all $x, a, b \in X$, it follows from (c3) and (c8) that

$$\begin{aligned}f_N(x-((x-a)-b)) &\leq \max\{f_N((x-((x-a)-b))-b), f_N(b)\} \\ &\leq \max\{\max\{f_N(((x-((x-a)-b))-b)-a), \\ &\qquad f_N(a)\}, f_N(b)\} \\ &= \max\{\max\{f_N(0), f_N(a)\}, f_N(b)\}\} \\ &= \max\{f_N(a), f_N(b)\}\end{aligned}$$

and from (c4) and (c9) that

$$\begin{aligned}g_N(x-((x-a)-b)) &\geq \min\{g_N((x-((x-a)-b))-b), g_N(b)\} \\ &\geq \min\{\min\{g_N(((x-((x-a)-b))-b)-a), \\ &\qquad g_N(a)\}, g_N(b)\} \\ &= \min\{\min\{g_N(0), g_N(a)\}, g_N(b)\}\} \\ &= \min\{g_N(a), g_N(b)\}.\end{aligned}$$

By Theorem 3.10, (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X. This completes the proof. □

Proposition 3.15. *Every coupled $\mathcal{N}$-ideal (X, f_N, g_N) of X satisfies the following assertions:*

(c10) $(f_N(x-y) \leq f_N((x-y)-y)) \Leftrightarrow (f_N((x-z)-(y-z)) \leq f_N((x-y)-z))$, *for all* $x, y, z \in X$ *and*

(c11) $(g_N(x-y) \geq g_N((x-y)-y)) \Leftrightarrow (g_N((x-z)-(y-z)) \geq g_N((x-y)-z))$, *for all* $x, y, z \in X$.

Proof. Assume that $f_N(x-y) \leq f_N((x-y)-y)$ and $g_N(x-y) \geq g_N((x-y)-y)$, for all $x, y \in X$. Also

$$((x-(y-z))-z)-z = ((x-z)-(y-z))-z \leq (x-y)-z,$$

then, by Corollary 3.12

$$f_N(((x-(y-z))-z)-z) \leq f_N((x-y)-z)$$

and

$$g_N(((x-(y-z))-z)-z) \geq g_N((x-y)-z).$$

Thus

$$\begin{aligned} f_N((x-z)-(y-z)) &= f_N((x-(y-z))-z)-z), \text{ by (a1 and a12)} \\ &\leq f_N(((x-(y-z))-z)-z), \text{ by (c10)} \\ &\leq f_N((x-y)-z), \end{aligned}$$

and

$$\begin{aligned} g_N((x-z)-(y-z)) &= g_N((x-(y-z))-z)-z), \text{ by (a1 and a12)} \\ &\geq g_N(((x-(y-z))-z)-z), \text{ by (c11)} \\ &\geq g_N((x-y)-z). \end{aligned}$$

Conversely, suppose $f_N((x-z)-(y-z) \leq f_N((x-y)-z)$ and $g_N((x-z)-(y-z)) \geq g_N((x-y)-z)$, for all $x, y, z \in X$

$$f_N(x-y) = f_N((x-y)-0) = f_N((x-y)-(y-y)) \leq f_N((x-y)-y),$$

and

$$g_N(x-y) = g_N((x-y)-0) = g_N((x-y)-(y-y)) \geq g_N((x-y)-y).$$

This completes the proof. □

Theorem 3.16. *A coupled $\mathcal{N}$-structure (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X if and only if it satisfies:*

(c12) $(\forall a, b, x \in X)$ $(x-a \leq b \Rightarrow f(x) \leq \max\{f_N(a), f_N(b)\})$, *and*

(c13) $(\forall a, b, x \in X)$ $(x-a \leq b \Rightarrow g_N(x) \geq \min\{g_N(a), g_N(b)\})$.

Proof. Assume that (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X. Let $a, b, x \in X$ be such that $x - a \leq b$. Then $(x - a) - b = 0$, and so

$$\begin{aligned} f_N(x) &\leq \max\{f_N(x-a), f_N(a)\}, \text{ by (c8)} \\ &\leq \max\{\max\{f_N((x-a)-b), f_N(b)\}, f_N(a)\}, \text{ by (c8)} \\ &= \max\{\max\{f_N(0), f_N(b)\}, f_N(a)\} \\ &= \max\{f_N(a), f_N(b)\}, \end{aligned}$$

and

$$\begin{aligned} g_N(x) &\geq \min\{g_N(x-a), g_N(a)\}, \text{ by (c9)} \\ &\geq \min\{\min\{g_N((x-a)-b), g_N(b)\}, g_N(a)\}, \text{ by (c9)} \\ &= \min\{\min\{g_N(0), g_N(b)\}, g_N(a)\} \\ &= \min\{g_N(a), g_N(b)\}. \end{aligned}$$

Conversely, let (X, f_N, g_N) be a $\mathcal{N}$-structure satisfying the conditions (c12) and (c13). Since $0 - x \leq x$ for all $x \in X$, it follows from (c12) and (c13) that $f_N(0) \leq \max\{f_N(x), f_N(x)\} = f_N(x)$ and $g_N(0) \geq \min\{g_N(x), g_N(x)\} = g_N(x)$ for all $x \in X$. Note that $x-(x-y) \leq y$ for all $x, y \in X$. Using (c12) and (c13), we have $f_N(x) \leq \max\{f_N(x-y), f_N(y)\}$ and $g_N(x) \geq \min\{f_N(x-y), f_N(y)\}$ for all $x, y \in X$. Hence (X, f_N, g_N) is an $\mathcal{N}$-ideal of X by Theorem 3.14. This completes the proof. □

Theorem 3.17. *A coupled $\mathcal{N}$-structure (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X if and only if it satisfies:*

(c14) $f_N(x) \leq max\{f_N(a_i) \,|\, i = 1, 2, ..., n\}$ *and* $g_N(x) \geq min\{g_N(a_i) \,|\, i = 1, 2, ..., n\}$ *for all* $x, a_1, a_2, ..., a_n \in X$ *with* $(\cdots((x-a_1)-a_2)-\cdots)-a_n = 0$.

Proof. Assume that (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X. If $x-a = 0$ for any $x, a \in X$, then $f_N(x) \leq f_N(a)$ and $g_N(x) \geq g_N(a)$ by Corollary 3.12. Let $a, b, x \in X$ be such that $(x - a) - b = 0$. Then

$$f_N(x) \leq \max\{f_N(a), f_N(b)\}$$

and

$$g_N(x) \geq \max\{g_N(a), g_N(b)\}$$

by Theorem 3.16. Now let $x, a_1, a_2, ..., a_n \in X$ be such that $(\cdots((x - a_1) - a_2) - \cdots) - a_n = 0$. By induction on n, we conclude that $f_N(x) \leq \max\{f_N(a_i) \,|\, i = 1, 2, ..., n\}$ and $g_N(x) \geq \min\{g_N(a_i) \,|\, i = 1, 2, ..., n\}$.

Conversely, let (X, f_N, g_N) be a coupled $\mathcal{N}$-structure in X which (c14) is valid for all $x, a_1, a_2, ..., a_n \in X$ with $(\cdots((x - a_1) - a_2) - \cdots) - a_n = 0$. Using (c12) and (c13), we have $f_N(x) \leq \max\{f_N(y), f_N(z)\}$ and

$g_N(x) \leq \max\{g_N(y), g_N(z)\}$ for all $x, y, z \in X$ with $(x-y)-z = 0$. Since $(0-x)-x = 0$ for all $x \in X$, it follows from (c12) and (c13) that $f_N(0) \leq \min\{f_N(x), f_N(x)\} = f_N(x)$ and $g_N(0) \geq \min\{g_N(x), g_N(x)\} = g_N(x)$. Hence (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X by Theorem 3.16. □

For any element $w \in X$, we consider the set

$$X_w := \{y \in X \mid f_N(y) \leq f_N(w) \text{ and } g_N(y) \geq g_N(w)\}.$$

Clearly, $w \in X_w$, and so X_w is a non-empty subset of X.

Theorem 3.18. *Let w be an element of X. If (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X. Then X_w is an ideal of X.*

Proof. Obviously, $0 \in X_w$, by (c3) and (c4). Let $x \in X$ and $y \in X_w$ be such that $x - y \in X_w$. Then

$$f_N(x-y) \leq f_N(w) \text{ and } f_N(x) \leq f_N(w),$$

and

$$g_N(x-y) \geq g_N(w) \text{ and } g_N(x) \geq g_N(w).$$

Since (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X, it follows from (c8) and (c9) that

$$f_N(x) \leq \max\{f_N(x-y), f_N(y)\} \leq f_N(w),$$

and

$$g_N(x) \geq \min\{f_N(x-y), f_N(y)\} \geq f_N(w)$$

so that $x \in X_w$. Hence X_w is an ideal of X. □

Theorem 3.19. *Let w be an element of X. If (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X. Then*

(i) If X_w is an ideal of X, then (X, f_N, g_N) satisfies the following condition:

(c15) $(\forall x, y, z \in X)$ $(f_N(x) \geq \max\{f_N(y-z), f_N(z)\} \Rightarrow f_N(x) \geq f_N(y)$, *and*

(c16) $(\forall x, y, z \in X)$ $(g_N(x) \leq \min\{g_N(y-z), g_N(z)\} \Rightarrow g_N(x) \leq g_N(y)$.

(ii) If (X, f_N, g_N) satisfies (c3), (c4), (c15) and (c16), then X_w is an ideal of X.

Proof. (i) Assume that X_w is an ideal of X for each $w \in X$. Let $x, y, z \in X$ be such that

$$f_N(x) \geq \max\{f_N(y-z), f_N(z)\} \text{ and } g_N(x) \geq \max\{g_N(y-z), g_N(z)\}.$$

Then $y - z \in X_x$ and $z \in X_x$. Since X_x is an ideal of X, it follows that $y \in X_x$, that is, $f_N(y) \leq f_N(x)$ and $g_N(y) \geq g_N(x)$.

(ii) Suppose that (X, f_N, g_N) satisfies (c3), (c4), (c15) and (c16). For each $w \in X$, let $x, y \in X$ be such that $x - y \in X_w$ and $y \in X_w$. Then $f_N(x-y) \leq f(w)$ and $f(y) \leq f(w)$, Also $g_N(x-y) \geq g_N(w)$ and $g_N(y) \geq g_N(w)$, which implies that $f_N(w) \geq \max\{f_N(x-y), f_N(y)\}$ and $g_N(w) \geq \max\{g_N(x-y), g_N(y)\}$. Using (c11) and (c12), we have $f_N(w) \geq f_N(x)$ and $g_N(w) \leq g_N(x)$. Hence $x \in X_w$. Obviously $0 \in X_w$. Therefore X_w is an ideal of X. □

For any $\mathcal{N}$-function f on X,we denote

$$\perp := \inf\{f(x) | x \in X\} - 1.$$

For any $\alpha \in [-1, 0]$, we denote $f^\alpha(x) = f(x) + \alpha$, for all $x \in X$. Obviously, $f^\alpha(x)$ is a mapping from X to $[-1, 0]$, that is, $f^\alpha(x)$ is an $\mathcal{N}$-function on X. We say that $(X, f_N^\alpha, g_N^\alpha)$ is a coupled α-translation of (X, f_N, g_N).

Theorem 3.20. *For every $\alpha \in [\perp, 0]$, the coupled α-translation $(X, f_N^\alpha, g_N^\alpha)$ of a coupled $\mathcal{N}$-ideal (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X.*

Proof. For any $x, y \in X$, we have

$$f_N^\alpha(x-y) = f_N(x-y) + \alpha \leq f_N(x) + \alpha = f_N^\alpha(x),$$

and

$$g_N^\alpha(x-y) = g_N(x-y) + \alpha \leq g_N(x) + \alpha = g_N^\alpha(x).$$

Also, there exist $x \vee y \in X$ such that

$$\begin{aligned} f_N^\alpha(x \vee y) &= f_N(x \vee y) + \alpha \\ &\leq \max\{f_N(x), f_N(y)\} + \alpha \\ &\leq \max\{f_N(x) + \alpha, f_N(y) + \alpha\} \\ &\leq \max\{f_N^\alpha(x), f_N^\alpha(y)\} \end{aligned}$$

and

$$\begin{aligned} g_N^\alpha(x \vee y) &= g_N(x \vee y) + \alpha \\ &\geq \min\{g_N(x), g_N(y)\} + \alpha \\ &\geq \min\{g_N(x) + \alpha, g_N(y) + \alpha\} \\ &\geq \min\{g_N^\alpha(x), g_N^\alpha(y)\}. \end{aligned}$$

Hence $(X, f_N^\alpha, g_N^\alpha)$ is a coupled $\mathcal{N}$-ideal of X. □

Theorem 3.21. *For every $\alpha \in [\perp, 0]$, the coupled α-translation $(X, f_N^\alpha, g_N^\alpha)$ is a coupled $\mathcal{N}$-ideal (X, f_N, g_N), then (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X.*

Proof. For any $x, y \in X$, we have

$$f_N(x-y)+\alpha = f_N^{\alpha}(x-y) \leq f_N(x)+\alpha \Rightarrow f_N(x-y) \leq f_N(x),$$

and

$$g_N(x-y)+\alpha = g_N^{\alpha}(x-y) \geq g_N(x)+\alpha \Rightarrow g_N(x-y) \geq g_N(x).$$

Also, there exist $x \vee y \in X$ such that

$$\begin{aligned} f_N(x \vee y)+\alpha &= f_N^{\alpha}(x \vee y) \\ &\leq \max\{f_N^{\alpha}(x), f_N^{\alpha}(y)\} \\ &\leq \max\{f_N(x)+\alpha, f_N(y)+\alpha\} \\ &\leq \max\{f_N(x), f_N(y)\}+\alpha. \end{aligned}$$

Thus

$$f_N(x \vee y) \leq \max\{f_N(x), f_N(y)\},$$

and

$$\begin{aligned} g_N(x \vee y)+\alpha &= g_N^{\alpha}(x \vee y) \\ &\geq \min\{g_N^{\alpha}(x), g_N^{\alpha}(y)\} \\ &\geq \min\{g_N(x)+\alpha, g_N(y)+\alpha\} \\ &\geq \min\{g_N(x), g_N(y)\}+\alpha. \end{aligned}$$

Thus

$$g_N(x \vee y) \geq \min\{g_N(x), g_N(y)\}.$$

Hence (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X. □

Let (X, f_N, g_N) and (X, p_N, q_N) be two $\mathcal{N}$-structures. Then (X, f_N, g_N) is said to be a *coupled-retrenchment* of (X, f_N, g_N) if $f_N(x) \leq p_N(x)$ and $g_N(x) \geq q_N(x)$.

Definition 3.22. Let (X, f_N, g_N) be a coupled $\mathcal{N}$-structure. A coupled $\mathcal{N}$-structure (X, p_N, q_N) is called a *coupled-retrenchment coupled $\mathcal{N}$-ideal* of (X, f_N, g_N) if it satisfies

(i) (X, p_N, q_N) is a coupled-retrenchment of (X, f_N, g_N).

(ii) If (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X, then (X, p_N, q_N) is a coupled $\mathcal{N}$-ideal of X.

Theorem 3.23. *Let (X, f_N, g_N) be a coupled $\mathcal{N}$-ideal of X. For every $\alpha \in [\perp, 0]$ the coupled α-translation $(X, f_N^{\alpha}, g_N^{\alpha})$ of (X, f_N, g_N) is a coupled-retrenchment coupled $\mathcal{N}$-ideal of (X, f_N, g_N).*

Proof. Obviously, $(X, f_N^{\alpha}, g_N^{\alpha})$ is a coupled-retrenchment of (X, f_N, g_N). Using Theorem 3.21, we conclude that $(X, f_N^{\alpha}, g_N^{\alpha})$ is a coupled-retrenchment coupled $\mathcal{N}$-ideal of (X, f_N, g_N). □

−	**0**	**a**	**b**	**c**	**d**
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

Table 3

The converse of Theorem 3.23 is not true as seen in the following example.

Example 3.24. Let $X = \{0, a, b, c\}$ be a subtraction algebra with the Cayley table which is given in Table 3. Let (X, f_N, g_N) be a coupled $\mathcal{N}$-structure defined by:

$$f_N(0) = -0.7, \quad f_N(a) = -0.5, \quad f_N(b) = -0.6,$$
$$f_N(c) = -0.4, \quad \text{and} \quad f_N(d) = -0.2,$$

and

$$g_N(0) = -0.1, \quad g_N(a) = -0.3, \quad g_N(b) = -0.2,$$
$$g_N(c) = -0.5, \quad \text{and} \quad g_N(d) = -0.2.$$

It is easy to check that (X, f_N, g_N) is a coupled $\mathcal{N}$-ideal of X.
Also, define (X, p_N, q_N) as follows:

$$p_N(0) = -0.72, \quad p_N(a) = -0.53, \quad p_N(b) = -0.65,$$
$$p_N(c) = -0.45, \quad \text{and} \quad p_N(d) = -0.26,$$

and

$$q_N(0) = -0.16, \quad q_N(a) = -0.34, \quad q_N(b) = -0.24,$$
$$q_N(c) = -0.52, \quad \text{and} \quad q_N(d) = -0.21.$$

Then (X, p_N, q_N) is a coupled-retrenched coupled $\mathcal{N}$-ideal of (X, f_N, g_N), which is not a coupled α-translation of (X, f_N, g_N) for $\alpha \in [\perp, 0]$.

Definition 3.25. Let (X, f_N, g_N) be a coupled $\mathcal{N}$-structure. An coupled $\mathcal{N}$-structure (X, p_N, q_N) is called a *created coupled $\mathcal{N}$-ideal* of (X, f_N, g_N) if it satisfies

(i) (X, p_N, q_N) is a coupled $\mathcal{N}$-ideal of X.
(ii) (X, p_N, q_N) is a coupled-retrenchment of (X, f_N, g_N).

(iii) For any coupled $\mathcal{N}$-ideal (X, r_N, s_N) of X, if (X, r_N, s_N) is a coupled-retrenchment of (X, f_N, g_N), then (X, r_N, s_N) is a coupled-retrenchment of (X, p_N, q_N).

The created coupled $\mathcal{N}$-ideal (X, f_N, g_N) of X will be denoted by $(X, [f_N], [g_N])$. Note that the created coupled $\mathcal{N}$-ideal of (X, f_N, g_N) is the greatest coupled $\mathcal{N}$-ideal in X which is a retrenchment of (X, f_N, g_N). We discuss how to make a created coupled $\mathcal{N}$-ideal of a coupled $\mathcal{N}$-structure (X, f_N, g_N).

Theorem 3.26. *For any coupled $\mathcal{N}$-structure (X, f_N, g_N), the created coupled $\mathcal{N}$-ideal $(X, [f_N], [g_N])$ of (X, f_N, g_N) is described as follows:*

$[f_N](x) = \inf\{max\{f(a_i) \mid i = 1, 2, ..., n\} \mid (\cdots((x - a_1) - a_2) - \cdots) - a_n = 0\}$,

and

$[g_N](x) = \sup\{min\{f(a_i) \mid i = 1, 2, ..., n\} \mid (\cdots((x - a_1) - a_2) - \cdots) - a_n = 0\}$.

Proof. Let (X, r_N, s_N), be a coupled $\mathcal{N}$-structure in which r_N and s_N are defined by

$p_N(x) = \inf\{\max\{f(a_i) \mid i = 1, 2, ..., n\} \mid (\cdots((x - a_1) - a_2) - \cdots) - a_n = 0\}$,

and

$q_N(x) = \sup\{\min\{f(a_i) \mid i = 1, 2, ..., n\} \mid (\cdots((x - a_1) - a_2) - \cdots) - a_n = 0\}$.

Let $x, a, b \in X$ be such that

$$(x - a) - b = 0. \tag{6}$$

For any $\epsilon > 0$, there exist $a_1, a_2, ..., a_n, b_1, b_2, ..., b_m \in X$ such that

$$(\cdots((a - a_1) - a_2) - \cdots) - a_n = 0, \quad (\cdots((b - b_1) - b_2) - \cdots) - b_m = 0. \tag{7}$$

Also, we have

$$p_N(x) > \max\{f(a_i)|i = 1, 2, ..., n\} - \epsilon \text{ and } r_N(x) > \max\{f(b_i)|i = 1, 2, ..., m\} - \epsilon, \tag{8}$$

and

$$q_N(x) < \min\{f(a_i)|i = 1, 2, ..., n\} - \epsilon \text{ and } s_N(x) < \min\{f(b_i)|i = 1, 2, ..., m\} - \epsilon. \tag{9}$$

Using (6) in (7), we have

$$(\cdots((((\cdots((x - a_1) - a_2) - \cdots) - a_n) - b_1) - b_2) - \cdots) - b_m = 0.$$

Using (8) and (9) in the definition of p_N and q_N respectively, we have

$$\begin{aligned} p_N(x) &\leq \max\{f(a_1), f(a_2), ..., f(a_n), f(b_1), f(b_2), ..., f(b_m)\} \\ &< \max\{p_N(a) + \epsilon, p_N(b) + \epsilon\} \\ &= \max\{p_N(a), p_N(b)\} + \epsilon, \end{aligned}$$

and

$$\begin{aligned} q_N(x) &\geq \min\{f(a_1), f(a_2), ..., f(a_n), f(b_1), f(b_2), ..., f(b_m)\} \\ &> \min\{q_N(a)+\epsilon, q_N(b)+\epsilon\} \\ &= \min\{q_N(a), q_N(b)\}+\epsilon. \end{aligned}$$

Since ϵ is arbitrary, it follows that

$$p_N(x) \leq \max\{p_N(a), p_N(b)\} \quad \text{and} \quad q_N(x) \geq \min\{q_N(a), q_N(b)\},$$

so from Theorem 3.16 that (X, p_N, q_N) is a coupled $\mathcal{N}$-ideal in X. Now since $x - x = 0$ for all $x \in X$, we obtain $p_N(x) \leq f_N(x)$ and $q_N(x) \geq f_N(x)$ for all $x \in X$, and so (X, p_N, q_N) is a coupled-retrenchment of (X, f_N, g_N). Let (X, r_N, s_N) be an coupled $\mathcal{N}$-ideal in X which is a coupled-retrenchment of (X, f_N, g_N). For any $x \in X$, we have

$$\begin{aligned} p_N(x) &= \inf\{\max\{f_N(a_i)|i=1,2,...,n\}|(\cdots((x-a_1)-a_2)-\cdots)-a_n=0\} \\ &\geq \inf\{\max\{r_N(a_i)|i=1,2,...,n\}|(\cdots((x-a_1)-a_2)-\cdots)-a_n=0\} \\ &= \inf\{r_N(x)\} = r_N(x), \end{aligned}$$

and

$$\begin{aligned} q_N(x) &= \sup\{\min\{g_N(a_i)|i=1,2,...,n\}|(\cdots((x-a_1)-a_2)-\cdots)-a_n=0\} \\ &\geq \sup\{\min\{s_N(a_i)|i=1,2,...,n\}|(\cdots((x-a_1)-a_2)-\cdots)-a_n=0\} \\ &= \sup\{s_N(x)\} = s_N(x). \end{aligned}$$

Thus (X, r_N, s_N) is a coupled-retrenchment of (X, p_N, q_N). Therefore (X, p_N, q_N) is a created coupled $\mathcal{N}$-ideal of (X, f_N, g_N). Since $(X, [f_N])$ is greatest and $(X, [g_N])$ is lowest, we have $p_N = [f_N]$ and $q_N = [g_N]$. This completes the proof. □

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