Bull. Korean Math. Soc. **51** (2014), No. 1, pp. 99–113
http://dx.doi.org/10.4134/BKMS.2014.51.1.099

MULTIPLE SOLUTIONS FOR A p-LAPLACIAN SYSTEM WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. A nonlinear elliptic problem involving p-Laplacian and nonlinear boundary condition is considered in this paper. By using the method of Nehari manifold, it is proved that the system possesses two nontrivial nonnegative solutions if the parameter is small enough.

1. Introduction

This paper is devoted to the study of the following quasilinear elliptic problem with nonlinear boundary conditions:

$$\begin{cases} -\Delta_p u + m(x)|u|^{p-2}u = \lambda F_u(u,v), & x \in \Omega, \\ -\Delta_p v + n(x)|v|^{p-2}v = \lambda F_v(u,v), & x \in \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = G_u(u,v),\ |\nabla v|^{p-2}\frac{\partial v}{\partial \nu} = G_v(u,v), & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $p > 1$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and

- Δ_p denotes the p-Laplacian operator, defined by $\Delta_p z = \operatorname{div}\left(|\nabla z|^{p-2}\nabla z\right)$;
- $\lambda \in (0, +\infty)$, $m(x), n(x) \in C(\overline{\Omega})$, and there exist positive constants m_0 and n_0 such that $m(x) \geq m_0$ and $n(x) \geq n_0$ for all $x \in \overline{\Omega}$;
- ν is the unit outer normal to $\partial\Omega$;
- $F, G : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ satisfy:

(H1) $F, G \in C^1(\mathbb{R} \times \mathbb{R})$; $F(u,v) = F(|u|,|v|)$ and $G(u,v) = G(|u|,|v|)$; $F(u,v) \not\equiv 0$ and $G(u,v) \not\equiv 0$;

(H2) There exist constants $\alpha \in (p, p^*)$ and $\beta \in (1,p)$ such that

$$F(tu,tv) = t^\alpha F(u,v) \text{ and } G(tu,tv) = t^\beta G(u,v)$$

for all $t \geq 0$ and $(u,v) \in \mathbb{R} \times \mathbb{R}$.

Received November 14, 2012.
2010 *Mathematics Subject Classification.* 35J50, 35J57, 35J66.
Key words and phrases. p-Laplacian systems, nonlinear boundary condition, multiple solutions, Nehari manifold.

Here p^* denotes the Sobolev conjugate exponent of p, *i.e.*,

$$p^* = \begin{cases} \dfrac{Np}{N-p}, & p < N, \\ \infty, & p \geq N. \end{cases}$$

Problem involving the p-Laplacian operator appears in pure mathematics such as the theory of quasiregular and quasiconformal mapping [26, 39] as well as in applied mathematics. Indeed, it intervenes in numerous fields in experimental sciences: nonlinear reaction-diffusion problems, dynamics of populations, non-Newtonian fluids, flows through porous media, nonlinear elasticity, petroleum extraction, torsional creep problems, etc (see, e.g., [21, 22, 42]). In literature, there exist numerous papers dedicated to the study of such equations and systems. In fact, the study of scalar equations had really started in the middle of 80s by M. Ôtani [34] in one dimension and then in dimension N by F. de Thélin [19]. Later, the results are generalized to other kinds of equations or systems involving p-Laplacian in $\mathbb{R}^N$ or bounded open set $\Omega \subset \mathbb{R}^N$ (see, e.g., [1, 3, 4, 5, 6, 7, 8, 13, 18, 20, 23, 24, 27, 29, 30, 33, 35] and the references therein).

In recent years, the existence of solutions for the semilinear/quasilinear elliptic equations with nonlinear boundary conditions have been widely studied (see, e.g., [9, 12, 16, 17, 28, 36, 37, 41] and the references therein). In particular, in [37], the authors studied the multiple solutions of the following systems:

(1.2)

$$\begin{cases} -\Delta_p u + m(x)|u|^{p-2}u = \lambda_1 a(x)|u|^{\gamma-2}u, & x \in \Omega, \\ -\Delta_p v + m(x)|v|^{p-2}v = \lambda_2 b(x)|v|^{\gamma-2}v, & x \in \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \frac{\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta},\ |\nabla v|^{p-2}\frac{\partial v}{\partial \nu} = \frac{\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $p > 2$, is a bounded domain with smooth boundary $\partial\Omega$, $\lambda_1, \lambda_2 > 0$, and $2 < \alpha + \beta < p < \gamma < p^*$. Motivated by the results of the above works, we are interested in the existence of multiple nontrivial nonnegative solutions for problem (1.1). We remark that problem (1.2) is a special case of (1.1) with

$$F(u,v) = \frac{\lambda_1}{\lambda\gamma}a(x)|u|^{\gamma} + \frac{\lambda_2}{\lambda\gamma}b(x)|v|^{\gamma},\ G(u,v) = \frac{1}{\alpha+\beta}|u|^{\alpha}|v|^{\beta},\ \lambda = \lambda_1 + \lambda_2.$$

The main approach of this paper is the method of Nehari manifold, which was first introduced by Nehari in [31, 32], and the method turned out to be very useful in critical point theory (see, e.g., [1, 2, 10, 11, 14, 15, 25, 37, 38, 40, 41]) and eventually came to bear his name.

The rest of this work is organized as follows. In Section 2, we introduce some preliminaries including definitions and some lemmas for later use. In Section 3, the proof of the main result is given.

2. Preliminaries

Let $W = W^{1,p}(\Omega) \times W^{1,p}(\Omega)$ be a Banach space with norm

$$\|(u,v)\| = \left(\int_\Omega (|\nabla u|^p + m(x)|u|^p)\, dx + \int_\Omega (|\nabla v|^p + n(x)|v|^p)\, dx \right)^{1/p}.$$

Definition 2.1. We say that $(u,v) \in W$ is a solution to (1.1) if for any $(\phi, \varphi) \in W$,

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u \cdot \nabla \phi + m(x)|u|^{p-2}u\phi\right) dx = \lambda \int_\Omega F_u(u,v)\phi dx + \int_{\partial\Omega} G_u(u,v)\phi ds,$$

$$\int_\Omega \left(|\nabla v|^{p-2}\nabla v \cdot \nabla \varphi + n(x)|v|^{p-2}v\varphi\right) dx = \lambda \int_\Omega F_v(u,v)\varphi dx + \int_{\partial\Omega} G_v(u,v)\varphi ds.$$

Let $\mathcal{J}_\lambda : W \to \mathbb{R}$ be the corresponding energy functional to problem (1.1) defined as

$$\mathcal{J}_\lambda(u,v) = \frac{1}{p}\|(u,v)\|^p - \lambda \int_\Omega F(u,v)dx - \int_{\partial\Omega} G(u,v)ds, \quad (u,v) \in W.$$

Furthermore the nonnegative solutions of problem (1.1) correspond to the critical points of $\mathcal{J}$. Define $I_\lambda : W \to \mathbb{R}$ as

$$I_\lambda(u,v) = \|(u,v)\|^p - \lambda\alpha \int_\Omega F(u,v)dx - \beta \int_{\partial\Omega} G(u,v)ds, \quad (u,v) \in W.$$

It follows from condition (H2) that $F_u(u,v)u + F_v(u,v)v = \alpha F(u,v)$ and $G_u(u,v)u + G_v(u,v)v = \beta G(u,v)$ for all $u, v \in \mathbb{R}$. Consequently,

$$I_\lambda(u,v) = \langle \mathcal{J}'_\lambda(u,v), (u,v)\rangle \quad \text{for all } (u,v) \in W.$$

Let us denote the Nehari manifold by $\mathcal{N}_\lambda$, i.e.,

$$\mathcal{N}_\lambda = \{(u,v) \in W \setminus \{(0,0)\} : I_\lambda(u,v) = 0\}.$$

It is easy to see that $(u,v) \in \mathcal{N}_\lambda$ if and only if

$$\|(u,v)\|^p = \lambda\alpha \int_\Omega F(u,v)dx + \beta \int_{\partial\Omega} G(u,v)ds. \tag{2.1}$$

Accordingly, for $(u,v) \in \mathcal{N}_\lambda$,

$$\begin{aligned} \langle I'_\lambda(u,v), (u,v)\rangle &= p\|(u,v)\|^p - \lambda\alpha^2 \int_\Omega F(u,v)dx - \beta^2 \int_{\partial\Omega} G(u,v)ds \\ &= -(\alpha - p)\|(u,v)\|^p + \beta(\alpha - \beta) \int_{\partial\Omega} G(u,v)ds \qquad (2.2) \\ &= (p - \beta)\|(u,v)\|^p - \lambda\alpha(\alpha - \beta) \int_\Omega F(u,v)dx. \end{aligned}$$

By (H2), F and G satisfy that, for all $u, v \in \mathbb{R}$,

$$F(u,v) \le M\left(|u|^p + |v|^p\right)^{\frac{\alpha}{p}}, \; G(u,v) \le M\left(|u|^p + |v|^p\right)^{\frac{\beta}{p}}, \tag{2.3}$$

where

$$M := \max\left\{\max_{|u|^p+|v|^p=1} F(u,v),\ \max_{|u|^p+|v|^p=1} G(u,v)\right\} > 0.$$

Since $\alpha \in (1, p^*)$ and $\beta \in (1, p)$, it follows from Sobolev and Sobolev trace inequalities that there exist positive constants C_1 and C_2 such that

$$\int_\Omega (|u|^p + |v|^p)^{\alpha/p}\, dx \le C_1\|(u,v)\|^\alpha,\ \int_{\partial\Omega} (|u|^p + |v|^p)^{\beta/p}\, ds \le C_2\|(u,v)\|^\beta \tag{2.4}$$

for all $(u,v) \in W$. By (2.3) and (2.4),

$$\int_\Omega F(u,v)dx \le MC_1\|(u,v)\|^\alpha \tag{2.5}$$

and

$$\int_{\partial\Omega} G(u,v)ds \le MC_2\|(u,v)\|^\beta \tag{2.6}$$

for all $(u,v) \in W$.

Define

$$\lambda^* := \left(\frac{p-\beta}{\alpha(\alpha-\beta)MC_1}\right)\left(\frac{\alpha-p}{\beta(\alpha-\beta)MC_2}\right)^{\frac{\alpha-p}{p-\beta}}$$

and

$$\lambda_* := \left(\frac{p-\beta}{\alpha(\alpha-\beta)MC_1}\right)\left(\frac{\alpha-p}{p(\alpha-\beta)MC_2}\right)^{\frac{\alpha-p}{p-\beta}},$$

where M is the constant in (2.3) and C_1, C_2 are the constants in (2.4). Note that $0 < \lambda_* < \lambda^*$.

Now we split $\mathcal{N}_\lambda$ into three parts:

$$\begin{aligned}
\mathcal{N}_\lambda^+ &= \{(u,v) \in \mathcal{N}_\lambda : \langle I'_\lambda(u,v),(u,v)\rangle > 0\},\\
\mathcal{N}_\lambda^0 &= \{(u,v) \in \mathcal{N}_\lambda : \langle I'_\lambda(u,v),(u,v)\rangle = 0\},\\
\mathcal{N}_\lambda^- &= \{(u,v) \in \mathcal{N}_\lambda : \langle I'_\lambda(u,v),(u,v)\rangle < 0\},
\end{aligned}$$

and present some properties of $\mathcal{N}_\lambda$.

Lemma 2.2. *Suppose that F and G satisfy* (H1) *and* (H2). *Then $\mathcal{N}_\lambda^0 = \emptyset$ for all $\lambda \in (0, \lambda^*)$.*

Proof. Let λ be a fixed number satisfying $\mathcal{N}_\lambda^0 \ne \emptyset$. Then for $(u,v) \in \mathcal{N}_\lambda^0$,

$$\begin{aligned}
0 = \langle I'_\lambda(u,v),(u,v)\rangle &= (p-\beta)\|(u,v)\|^p + \lambda\alpha(\beta-\alpha)\int_\Omega F(u,v)dx\\
&= (p-\alpha)\|(u,v)\|^p + \beta(\alpha-\beta)\int_{\partial\Omega} G(u,v)ds.
\end{aligned} \tag{2.7}$$

By (2.5), (2.6) and (2.7),

$$\left(\frac{p-\beta}{\lambda\alpha(\alpha-\beta)MC_1}\right)^{\frac{1}{\alpha-p}} \le \|(u,v)\| \le \left(\frac{\beta(\alpha-\beta)MC_2}{\alpha-p}\right)^{\frac{1}{p-\beta}},$$

and consequently

$$\lambda \geq \left(\frac{p-\beta}{\alpha(\alpha-\beta)MC_1}\right)\left(\frac{\alpha-p}{\beta(\alpha-\beta)MC_2}\right)^{\frac{\alpha-p}{p-\beta}} = \lambda^*. \qquad \square$$

Lemma 2.3. *Suppose that F and G satisfy* (H1) *and* (H2)*, and $\lambda \in (0, \lambda^*)$. Assume that (u_0, v_0) is a local minimizer for $\mathcal{J}_\lambda$ on $\mathcal{N}_\lambda$. Then (u_0, v_0) is a critical point of $\mathcal{J}_\lambda$, i.e., $\mathcal{J}_\lambda'(u_0, v_0) = 0$.*

Proof. Let (u_0, v_0) be a local minimizer for $\mathcal{J}_\lambda$ on $\mathcal{N}_\lambda$ and $\lambda \in (0, \lambda^*)$. Then (u_0, v_0) is a solution of the following optimization problem:

$$\text{minimize } \mathcal{J}_\lambda(u, v) = 0 \text{ subject to } I_\lambda(u, v) = 0.$$

Hence, by the theory of Lagrange multipliers, there exists a $\Lambda \in \mathbb{R}$ such that

$$\mathcal{J}_\lambda'(u_0, v_0) = \Lambda I_\lambda'(u_0, v_0).$$

Thus

$$\langle \mathcal{J}_\lambda'(u_0, v_0), (u_0, v_0)\rangle = \Lambda \langle I_\lambda'(u_0, v_0), (u_0, v_0)\rangle.$$

Since $(u_0, v_0) \in \mathcal{N}_\lambda$, $\langle \mathcal{J}_\lambda'(u_0, v_0), (u_0, v_0)\rangle = 0$. On the other hand, by Lemma 2.2,

$$\langle I_\lambda'(u_0, v_0), (u_0, v_0)\rangle \neq 0.$$

Hence $\Lambda = 0$, and this completes the proof. $\square$

Lemma 2.4. *$\mathcal{J}_\lambda$ is coercive and bounded below on $\mathcal{N}_\lambda$ for all $\lambda > 0$.*

Proof. For $u \in \mathcal{N}_\lambda$, it follows from (2.1) and (2.6) that

$$\begin{aligned} \mathcal{J}_\lambda(u, v) &= \frac{\alpha-p}{p\alpha}\|(u, v)\|^p - \frac{\alpha-\beta}{\alpha}\int_{\partial\Omega} G(u, v)ds \\ &\geq \frac{\alpha-p}{p\alpha}\|(u, v)\|^p - \frac{\alpha-\beta}{\alpha}MC_2\|(u, v)\|^\beta, \end{aligned} \tag{2.8}$$

which completes the proof since $\beta < p < \alpha$. $\square$

By Lemma 2.2, $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ for $0 < \lambda < \lambda^*$, and define

$$\gamma_\lambda^+ = \inf\{\mathcal{J}_\lambda(u, v) : (u, v) \in \mathcal{N}_\lambda^+\},$$
$$\gamma_\lambda^- = \inf\{\mathcal{J}_\lambda(u, v) : (u, v) \in \mathcal{N}_\lambda^-\}.$$

Lemma 2.5. *Suppose that F and G satisfy* (H1) *and* (H2)*. Then we have*

(i) *If $\lambda \in (0, +\infty)$, then $\gamma_\lambda^+ < 0$;*
(ii) *If $\lambda \in (0, \lambda_*)$, $\gamma_\lambda^- \geq d_0$ for some constant $d_0 = d_0(\lambda) > 0$.*

Proof. (i) Let $(u, v) \in \mathcal{N}_\lambda^+$ and $\lambda \in (0, \infty)$. By (2.2),

$$\frac{p-\beta}{\alpha(\alpha-\beta)}\|(u, v)\|^p > \lambda \int_\Omega F(u, v)dx,$$

and, by (2.1),

$$\begin{aligned}\mathcal{J}_\lambda(u,v) &= \left(\frac{1}{p}-\frac{1}{\beta}\right)\|(u,v)\|^p + \left(\frac{\alpha}{\beta}-1\right)\lambda\int_\Omega F(u,v)dx \\ &< \left(\frac{1}{p}-\frac{1}{\beta}+\frac{p-\beta}{\alpha\beta}\right)\|(u,v)\|^p \\ &= \frac{(p-\alpha)(p-\beta)}{p\alpha\beta}\|(u,v)\|^p.\end{aligned}$$

Since $\beta < p < \alpha$, $\mathcal{J}_\lambda(u,v) < 0$ for all $(u,v) \in \mathcal{N}_\lambda^+$, and thus $\gamma_\lambda^+ < 0$ for all $\lambda \in (0,\infty)$.

(ii) Let $(u,v) \in \mathcal{N}_\lambda^-$ and $\lambda \in (0,\lambda_*)$. By (2.2) and (2.5),

$$(p-\beta)\|(u,v)\|^p < \lambda\alpha(\alpha-\beta)\int_\Omega F(u,v)dx \le \lambda\alpha(\alpha-\beta)MC_1\|(u,v)\|^\alpha,$$

and

$$\|(u,v)\| > \left(\frac{p-\beta}{\lambda\alpha(\alpha-\beta)MC_1}\right)^{\frac{1}{\alpha-p}}. \tag{2.9}$$

By (2.8) and (2.9),

$$\begin{aligned}&\mathcal{J}_\lambda(u,v) \\ &\ge \|(u,v)\|^\beta\left[\frac{\alpha-p}{p\alpha}\|(u,v)\|^{p-\beta} - \frac{\alpha-\beta}{\alpha}MC_2\right] \\ &> \left(\frac{p-\beta}{\lambda\alpha(\alpha-\beta)MC_1}\right)^{\frac{\beta}{\alpha-p}}\left[\left(\frac{\alpha-p}{p\alpha}\right)\left(\frac{p-\beta}{\lambda\alpha(\alpha-\beta)MC_1}\right)^{\frac{p-\beta}{\alpha-p}} - \left(\frac{\alpha-\beta}{\alpha}\right)MC_2\right].\end{aligned}$$

Thus, for each $\lambda \in (0,\lambda_*)$, there exists a positive constant $d_0 = d_0(\lambda)$ such that

$$\gamma_\lambda^- \ge d_0.$$

□

Lemma 2.6. *Suppose that F and G satisfy* (H1) *and* (H2). *Let $\lambda \in (0,\lambda_*)$ and $(u,v) \in W$. Then we have*

(i) *If $\int_\Omega F(u,v)dx > 0$, there exists a unique $t_2 = t_2(u,v)$ with*

$$t_2 > t_1^* = t_1^*(u,v) := \left[\frac{(p-\beta)\|(u,v)\|^p}{\lambda\alpha(\alpha-\beta)\int_\Omega F(u,v)dx}\right]^{\frac{1}{\alpha-p}} > 0$$

such that $(t_2u,t_2v) \in \mathcal{N}_\lambda^-$ and $\mathcal{J}_\lambda(t_2u,t_2v) = \sup_{t\ge 0}\mathcal{J}_\lambda(tu,tv)$.

(ii) *If $\int_{\partial\Omega} G(u,v)ds > 0$, there exists a unique $t_3 = t_3(u,v)$ with*

$$0 < t_3 < t_2^* = t_2^*(u,v) := \left[\frac{\beta(\alpha-\beta)\int_{\partial\Omega} G(u,v)ds}{(\alpha-p)\|(u,v)\|^p}\right]^{\frac{1}{p-\beta}}$$

such that $(t_3u, t_3v) \in \mathcal{N}_\lambda^+$ *and* $\mathcal{J}_\lambda(t_3u, t_3v) = \inf_{0\le t\le t_2^*} \mathcal{J}_\lambda(tu, tv)$.

Proof. (i) Fix $(u, v) \in W$ with $\int_\Omega F(u, v)dx > 0$. Then $(u, v) \neq (0, 0)$. Let

$$a_{(u,v)}(t) = t^{p-\beta}\|(u, v)\|^p - \lambda\alpha t^{\alpha-\beta} \int_\Omega F(u, v)dx \quad \text{for } t \geq 0.$$

Then $a_{(u,v)}(0) = 0$ and $a_{(u,v)}(t) \to -\infty$ as $t \to +\infty$. Since

$$a'_{(u,v)}(t) = (p - \beta)t^{p-\beta-1}\|(u, v)\|^p - \lambda\alpha(\alpha - \beta)t^{\alpha-\beta-1} \int_\Omega F(u, v)dx,$$

we see that $a'_{(u,v)}(t) = 0$ for $t = t_1^*$, $a'_{(u,v)}(t) > 0$ for $t \in (0, t_1^*)$ and $a'_{(u,v)}(t) < 0$ for $t \in (t_1^*, +\infty)$. Moreover, by (2.5),

$$\begin{aligned} a_{(u,v)}(t_1^*) &= (t_1^*)^{p-\beta}\|(u, v)\|^p \left(\frac{\alpha - p}{\alpha - \beta}\right) \\ &= \left(\frac{(p - \beta)\|(u, v)\|^\alpha}{\lambda\alpha(\alpha - \beta) \int_\Omega F(u, v)dx}\right)^{\frac{p-\beta}{\alpha-p}} \|(u, v)\|^\beta \left(\frac{\alpha - p}{\alpha - \beta}\right) \\ &> \left(\frac{p - \beta}{\lambda_*\alpha(\alpha - \beta)MC_1}\right)^{\frac{p-\beta}{\alpha-p}} \left(\frac{\alpha - p}{\alpha - \beta}\right) \|(u, v)\|^\beta. \end{aligned} \tag{2.10}$$

On the other hand, by (2.6),

$$\begin{aligned} 0 &\leq \beta \int_{\partial\Omega} G(u, v)ds \\ &\leq \beta MC_2\|(u, v)\|^\beta \\ &\leq \left(\frac{\alpha - p}{\alpha - \beta}\right) \left(\frac{p - \beta}{\lambda_*\alpha(\alpha - \beta)MC_1}\right)^{\frac{p-\beta}{\alpha-p}} \|(u, v)\|^\beta \end{aligned}$$

and, by (2.10),

$$\beta \int_{\partial\Omega} G(u, v)ds < a_{(u,v)}(t_1^*).$$

Hence, there are unique $t_1 = t_1(u, v)$ and $t_2 = t_2(u, v)$ such that

$$0 \leq t_1 < t_1^* < t_2,\ a_{(u,v)}(t_1) = a_{(u,v)}(t_2) = \beta \int_{\partial\Omega} G(u, v)ds \quad \text{and } a'_{(u,v)}(t_2) < 0.$$

Clearly $(t_2u, t_2v) \neq (0, 0)$, and $(t_2u, t_2v) \in \mathcal{N}_\lambda$ since

$$\begin{aligned} I_\lambda(t_2u, t_2v) &= t_2^p\|(u, v)\|^p - \lambda\alpha t_2^\alpha \int_\Omega F(u, v)dx - \beta t_2^\beta \int_{\partial\Omega} G(u, v)ds \\ &= t_2^\beta \left(a_{(u,v)}(t_2) - \beta \int_{\partial\Omega} G(u, v)ds\right) = 0. \end{aligned}$$

It follows from (2.2) that

$$\begin{aligned} \langle I'_\lambda(t_2u, t_2v), (t_2u, t_2v)\rangle &= (p - \beta)t_2^p\|(u, v)\|^p - \lambda\alpha(\alpha - \beta)t_2^\alpha \int_\Omega F(u, v)dx \\ &= t_2^{\beta+1} a'_{(u,v)}(t_2) < 0. \end{aligned}$$

Thus $(t_2u, t_2v) \in \mathcal{N}_\lambda^-$. Moreover

$$\begin{aligned}\frac{d}{dt}\mathcal{J}_\lambda(tu,tv) &= t^{p-1}\|(u,v)\|^p - \lambda\alpha t^{\alpha-1}\int_\Omega F(u,v)dx - \beta t^{\beta-1}\int_{\partial\Omega} G(u,v)ds \\ &= t^{\beta-1}\left(a_{(u,v)}(t) - \beta\int_{\partial\Omega} G(u,v)ds\right),\end{aligned}$$

which implies that $\frac{d}{dt}\mathcal{J}_\lambda(tu,tv) = 0$ for $t = t_1$ and $t = t_2$; $\frac{d}{dt}\mathcal{J}_\lambda(tu,tv) < 0$ for $t \in (0,t_1)\cup(t_2,+\infty)$; $\frac{d}{dt}\mathcal{J}_\lambda(tu,tv) > 0$ for $t \in (t_1,t_2)$. From Lemma 2.5(ii) and $\mathcal{J}_\lambda(0,0) = 0$, it follows that

$$\mathcal{J}_\lambda(t_2u,t_2v) = \sup_{t\geq 0}\mathcal{J}_\lambda(tu,tv).$$

(ii) Fix $(u,v) \in W$ with $\int_{\partial\Omega} G(u,v)ds > 0$. Then $(u,v) \neq (0,0)$. Let

$$b_{(u,v)}(t) = t^{p-\alpha}\|(u,v)\|^p - \beta t^{\beta-\alpha}\int_{\partial\Omega} G(u,v)ds \quad \text{for } t > 0.$$

Then $b_{(u,v)}(t) \to -\infty$ as $t \to 0^+$ and $b_{(u,v)}(t) \to 0$ as $t \to +\infty$. Since

$$b'_{(u,v)}(t) = (p-\alpha)t^{p-\alpha-1}\|(u,v)\|^p - \beta(\beta-\alpha)t^{\beta-\alpha-1}\int_{\partial\Omega} G(u,v)ds,$$

we see that $b'_{(u,v)}(t) = 0$ for $t = t_2^*$, $b'_{(u,v)}(t) > 0$ for $t \in (0,t_2^*)$ and $b'_{(u,v)}(t) < 0$ for $t \in (t_2^*,+\infty)$. Moreover, by (2.6),

$$\begin{aligned}b_{(u,v)}(t_2^*) &= (t_2^*)^{p-\alpha}\|(u,v)\|^p\left(\frac{p-\beta}{\alpha-\beta}\right) \\ &= \left(\frac{(\alpha-p)\|(u,v)\|^\beta}{\beta(\alpha-\beta)\int_{\partial\Omega} G(u,v)ds}\right)^{\frac{\alpha-p}{p-\beta}}\|(u,v)\|^\alpha\left(\frac{p-\beta}{\alpha-\beta}\right) \\ &\geq \left(\frac{\alpha-p}{\beta(\alpha-\beta)MC_2}\right)^{\frac{\alpha-p}{p-\beta}}\|(u,v)\|^\alpha\left(\frac{p-\beta}{\alpha-\beta}\right).\end{aligned} \tag{2.11}$$

On the other hand, by (2.5),

$$\begin{aligned}0 &\leq \lambda\alpha\int_\Omega F(u,v)dx \\ &< \lambda_*\alpha MC_1\|(u,v)\|^\alpha \\ &\leq \|(u,v)\|^\alpha\left(\frac{p-\beta}{\alpha-\beta}\right)\left[\frac{\alpha-p}{\beta MC_2(\alpha-\beta)}\right]^{\frac{\alpha-p}{p-\beta}},\end{aligned}$$

and, by (2.11),

$$0 \leq \lambda\alpha\int_\Omega F(u,v)dx < b_{(u,v)}(t_2^*) \quad \text{for } 0 < \lambda < \lambda_*.$$

Hence, there is a unique $t_3 = t_3(u,v) \in (0,t_2^*)$ such that

$$b_{(u,v)}(t_3) = \lambda\alpha\int_\Omega F(u,v)dx \quad \text{and} \quad b'_{(u,v)}(t_3) > 0.$$

Clearly $(t_3u, t_3v) \neq (0,0)$, and $(t_3u, t_3v) \in \mathcal{N}_\lambda$ since

$$\begin{aligned} I_\lambda(t_3u, t_3v) &= t_3^p\|(u,v)\|^p - \lambda\alpha t_3^\alpha \int_\Omega F(u,v)dx - \beta t_3^\beta \int_{\partial\Omega} G(u,v)ds \\ &= t_3^\alpha \left(b_{(u,v)}(t) - \lambda\alpha \int_\Omega F(u,v)dx \right) = 0. \end{aligned}$$

It follows from (2.2) that

$$\begin{aligned} \langle I'_\lambda(t_3u, t_3v), (t_3u, t_3v)\rangle &= (p-\alpha)t_3^p\|(u,v)\|^p - \beta(\beta-\alpha)t_3^\beta \int_{\partial\Omega} G(u,v)ds \\ &= t_3^{\alpha+1} b'_{(u,v)}(t_3) > 0. \end{aligned}$$

Thus $(t_3u, t_3v) \in \mathcal{N}_\lambda^+$. Moreover,

$$\begin{aligned} \frac{d}{dt}\mathcal{J}_\lambda(tu, tv) &= t^{p-1}\|(u,v)\|^p - \lambda\alpha t^{\alpha-1} \int_\Omega F(u,v)dx - \beta t^{\beta-1} \int_{\partial\Omega} G(u,v)ds \\ &= t^{\alpha-1} \left(b_{(u,v)}(t) - \lambda\alpha \int_\Omega F(u,v)dx \right). \end{aligned}$$

So, $\frac{d}{dt}\mathcal{J}_\lambda(tu,tv) = 0$ for $t = t_3$; $\frac{d}{dt}\mathcal{J}_\lambda(tu,tv) < 0$ for $t \in (0, t_3)$; $\frac{d}{dt}\mathcal{J}_\lambda(tu,tv) > 0$ for $t \in (t_3, t_2^*)$. Hence, $\mathcal{J}_\lambda(t_3u, t_3v) = \inf_{0\le t\le t_2^*} \mathcal{J}_\lambda(tu, tv)$. □

3. Main result

Now we state our main result.

Theorem 3.1. *Suppose* (H1) *and* (H2) *hold. Then problem* (1.1) *has at least two nontrivial nonnegative solutions for* $\lambda \in (0, \lambda_*)$.

The proof of this theorem will be a consequence of the next two propositions.

Proposition 3.2. *Suppose* (H1) *and* (H2) *hold and* $\lambda \in (0, \lambda_*)$. *Then the functional* $\mathcal{J}_\lambda$ *has a minimizer* (u_0^+, v_0^+) *in* $\mathcal{N}_\lambda^+$, *and it satisfies*

(i) $\mathcal{J}_\lambda(u_0^+, v_0^+) = \gamma_\lambda^+$;
(ii) (u_0^+, v_0^+) *is a nontrivial nonnegative solution of problem* (1.1).

Proof. By Lemma 2.4, $\mathcal{J}_\lambda$ is coercive and bounded below on $\mathcal{N}_\lambda$. By assumption (H1) and Lemma 2.6(ii), $\mathcal{N}_\lambda^+ \neq \emptyset$. Let $\{(u_n, v_n)\}$ be a minimizing sequence for $\mathcal{J}_\lambda$ on $\mathcal{N}_\lambda^+$, *i.e.*, $\lim_{n\to+\infty} \mathcal{J}_\lambda(u_n, v_n) = \inf_{(u,v)\in\mathcal{N}_\lambda^+} \mathcal{J}_\lambda(u,v) = \gamma_\lambda^+ < 0$. Then, by Lemma 2.4 and the Rellich-Kondrachov theorem, there exist a subsequence of $\{(u_n, v_n)\}$, denoted by itself, and $(u_0^+, v_0^+) \in W$ such that

$$\begin{aligned} &(u_n, v_n) \rightharpoonup (u_0^+, v_0^+) \text{ weakly in } W, \\ &u_n \to u_0^+,\ v_n \to v_0^+ \text{ strongly in } L^\alpha(\Omega) \text{ and } L^\beta(\partial\Omega),\ \text{respectively}. \end{aligned}$$

Thus, by (2.3),

$$\begin{aligned}\int_\Omega F(u_n,v_n)dx &\to \int_\Omega F(u_0^+,v_0^+)dx \text{ as } n\to+\infty,\\ \int_{\partial\Omega} G(u_n,v_n)ds &\to \int_{\partial\Omega} G(u_0^+,v_0^+)ds \text{ as } n\to+\infty.\end{aligned} \tag{3.1}$$

From the facts that

$$\mathcal{J}_\lambda(u_n,v_n)=\frac{\alpha-p}{\alpha p}\|(u_n,v_n)\|^p-\frac{\alpha-\beta}{\alpha}\int_{\partial\Omega}G(u_n,v_n)ds,$$

and

$$\mathcal{J}_\lambda(u_n,v_n)\to\gamma_\lambda^+<0 \text{ as } n\to+\infty,$$

it follows that

$$\int_{\partial\Omega}G(u_0^+,v_0^+)ds>0.$$

In particular, $(u_0^+,v_0^+)\neq(0,0)$. Now, we prove that $(u_n,v_n)\to(u_0^+,v_0^+)$ strongly in W. Suppose otherwise, then

$$\|(u_0^+,v_0^+)\|<\lim_{n\to+\infty}\|(u_n,v_n)\|, \tag{3.2}$$

and

$$\mathcal{J}_\lambda(u_0,v_0)<\lim_{n\to\infty}\mathcal{J}_\lambda(u_n,v_n)=\gamma_\lambda^+. \tag{3.3}$$

Since $\int_{\partial\Omega}G(u_0^+,v_0^+)ds>0$, by Lemma 2.6(ii), there exists a unique $t_3=t_3(u_0^+,v_0^+)\in(0,t_2^*(u_0^+,v_0^+))$ such that $(t_3u_0^+,t_3v_0^+)\in\mathcal{N}_\lambda^+$ and $\mathcal{J}_\lambda(t_3u_0^+,t_3v_0^+)=\inf_{0\le t\le t_2^*(u_0^+,v_0^+)}\mathcal{J}_\lambda(tu_0^+,tv_0^+)$. Furthermore,

$$\frac{d}{dt}\mathcal{J}_\lambda(tu_0^+,tv_0^+)<0 \quad \text{for } t\in(0,t_3). \tag{3.4}$$

Recall that $b_{(u,v)}(t)=t^{p-\alpha}\|(u,v)\|^p-\beta t^{\beta-\alpha}\int_{\partial\Omega}G(u,v)ds$ for $t>0$. Then

$$b_{(u_0^+,v_0^+)}(t_3)=\lambda\alpha\int_\Omega F(u_0^+,v_0^+)dx. \tag{3.5}$$

By (3.1), (3.2) and (3.5),

$$\begin{aligned}&\lim_{n\to+\infty}\left(b_{(u_n,v_n)}(t_3)-\lambda\alpha\int_\Omega F(u_n,v_n)dx\right)\\ &=\lim_{n\to+\infty}\left(t_3^{p-\alpha}\|(u_n,v_n)\|^p-\beta t_3^{\beta-\alpha}\int_{\partial\Omega}G(u_n,v_n)ds-\lambda\alpha\int_\Omega F(u_n,v_n)dx\right)\\ &=t_3^{p-\alpha}\lim_{n\to+\infty}\|(u_n,v_n)\|^p-\beta t_3^{\beta-\alpha}\int_{\partial\Omega}G(u_0^+,v_0^+)ds-\lambda\alpha\int_\Omega F(u_0^+,v_0^+)dx\\ &>t_3^{p-\alpha}\|(u_0^+,v_0^+)\|^p-\beta t_3^{\beta-\alpha}\int_{\partial\Omega}G(u_0^+,v_0^+)ds-\lambda\alpha\int_\Omega F(u_0^+,v_0^+)dx\\ &=b_{(u_0^+,v_0^+)}(t_3)-\lambda\alpha\int_\Omega F(u_0^+,v_0^+)dx=0,\end{aligned}$$

which implies that, for n large enough,

$$b_{(u_n,v_n)}(t_3) > \lambda\alpha \int_\Omega F(u_n,v_n)dx. \tag{3.6}$$

On the other hand, since $(u_n,v_n) \in \mathcal{N}_\lambda^+$, by (2.2),

$$\int_{\partial\Omega} G(u_n,v_n)ds > \frac{(\alpha-p)\|(u_n,v_n)\|^p}{\beta(\alpha-\beta)},$$

which implies that $t_2^*(u_n,v_n) > 1$ by Lemma 2.6(ii). Moreover, we obtain

$$b_{(u_n,v_n)}(1) = \|(u_n,v_n)\|^p - \beta\int_{\partial\Omega} G(u_n,v_n)ds = \lambda\alpha\int_\Omega F(u_n,v_n)dx,$$

and $b_{(u_n,v_n)}(t)$ is increasing for $t \in (0, t_2^*(u_n,v_n))$. Thus

$$b_{(u_n,v_n)}(t) \le b_{(u_n,v_n)}(1) = \lambda\alpha\int_\Omega F(u_n,v_n)dx \text{ for all } t \in (0,1]. \tag{3.7}$$

For n sufficiently large, by (3.6) and (3.7),

$$1 < t_3 < t_2^*(u_0^+, v_0^+). \tag{3.8}$$

By (3.4) and (3.8),

$$\mathcal{J}_\lambda(t_3u_0^+, t_3v_0^+) < \mathcal{J}_\lambda(u_0^+, v_0^+),$$

which contradicts $(t_3u_0^+, t_3v_0^+) \in \mathcal{N}_\lambda^+$ by (3.3). Hence

$$(u_n,v_n) \to (u_0^+, v_0^+) \text{ strongly in } W,$$

and

$$\mathcal{J}_\lambda(u_n,v_n) \to \mathcal{J}_\lambda(u_0^+,v_0^+) = \gamma_\lambda^+ \text{ as } n \to +\infty.$$

By Lemma 2.2, $(u_0^+,v_0^+) \in \mathcal{N}_\lambda^+$ and (u_0^+,v_0^+) is a local minimizer for $\mathcal{J}_\lambda$ on $\mathcal{N}_\lambda$. Since $\mathcal{J}_\lambda(u_0^+,v_0^+) = \mathcal{J}_\lambda(|u_0^+|,|v_0^+|)$ and $(|u_0^+|,|v_0^+|) \in \mathcal{N}_\lambda^+$, by Lemma 2.3, we may assume (u_0^+,v_0^+) is a nontrivial nonnegative solution of (1.1), and thus the proof is complete. □

Proposition 3.3. *Suppose* (H1) *and* (H2) *hold and* $\lambda \in (0,\lambda_*)$. *Then the functional* $\mathcal{J}_\lambda$ *has a minimizer* (u_0^-,v_0^-) *in* $\mathcal{N}_\lambda^-$ *and it satisfies*

(i) $\mathcal{J}_\lambda(u_0^-,v_0^-) = \gamma_\lambda^-$;

(ii) (u_0^-,v_0^-) *is a nontrivial nonnegative solution of problem* (1.1).

Proof. By assumption (H1) and Lemma 2.6(i), $\mathcal{N}_\lambda^- \neq \emptyset$. Let $\{(u_n,v_n)\}$ be a minimizing sequence for $\mathcal{J}_\lambda$ on $\mathcal{N}_\lambda^-$, *i.e.*,

$$\lim_{n\to+\infty} \mathcal{J}_\lambda(u_n,v_n) = \inf_{(u,v)\in\mathcal{N}_\lambda^-} \mathcal{J}_\lambda(u,v).$$

Then by Lemma 2.4 and the Rellich-Kondrachov theorem, there exists a subsequence of $\{(u_n,v_n)\}$, denoted by itself, and $(u_0^-,v_0^-) \in W$ such that

$$(u_n,v_n) \rightharpoonup (u_0^-,v_0^-) \text{ weakly in } W,$$

$$u_n \to u_0^-,\ v_n \to v_0^- \text{ strongly in } L^\alpha(\Omega) \text{ and } L^\beta(\partial\Omega), \text{ respectively.}$$

Thus, by (2.3),

$$\int_\Omega F(u_n, v_n)dx \to \int_\Omega F(u_0^-, v_0^-)dx \text{ as } n \to +\infty,$$
$$\int_{\partial\Omega} G(u_n, v_n)ds \to \int_{\partial\Omega} G(u_0^-, v_0^-)ds \text{ as } n \to +\infty.$$

Moreover, by (2.2),

$$\int_\Omega F(u_n, v_n)dx > \frac{p-\beta}{\lambda_* \alpha(\alpha-\beta)} \|(u_n, v_n)\|^p. \tag{3.9}$$

By (2.9) and (3.9), there exists a positive number D such that

$$\int_\Omega F(u_n, v_n)dx > D,$$

which implies

$$\int_\Omega F(u_0^-, v_0^-)dx \geq D. \tag{3.10}$$

Now we prove that $(u_n, v_n) \to (u_0^-, v_0^-)$ strongly in W. Suppose otherwise, then

$$\|(u_0^-, v_0^-)\| < \lim_{n\to+\infty} \|(u_n, v_n)\|. \tag{3.11}$$

By Lemma 2.6(i) and (3.10), there exists a unique $t_2 = t_2(u_0, v_0)$ such that $t_2 > t_1^*(u_0, v_0)$, $(t_2 u_0^-, t_2 v_0^-) \in \mathcal{N}_\lambda^-$ and

$$\mathcal{J}_\lambda(t_2 u_0^-, t_2 v_0^-) = \sup_{t\geq 0} \mathcal{J}_\lambda(t u_0^-, t v_0^-).$$

Since $(u_n, v_n) \in \mathcal{N}_\lambda^-$, $t_1^*(u_n, v_n) < 1$ and $a_{(u_n,v_n)}(1) = \beta \int_{\partial\Omega} G(u_n, v_n)ds$ for all $n \in \mathbb{N}$. Thus $t_2(u_n, v_n) = 1$ and $\mathcal{J}_\lambda(u_n, v_n) \geq \mathcal{J}_\lambda(t_2 u_n, t_2 v_n)$ by Lemma 2.6(i). On the other hand, by (3.11),

$$\mathcal{J}_\lambda(t_2 u_0^-, t_2 v_0^-) < \lim_{n\to+\infty} \mathcal{J}_\lambda(t_2 u_n, t_2 v_n)$$

and thus

$$\mathcal{J}_\lambda(t_2 u_0^-, t_2 v_0^-) < \lim_{n\to+\infty} \mathcal{J}_\lambda(u_n, v_n) = \gamma_\lambda^-.$$

This is a contradiction to the fact that $(t_2 u_0^-, t_2 v_0^-) \in \mathcal{N}_\lambda^-$. Hence

$$(u_n, v_n) \to (u_0^-, v_0^-) \text{ strongly in } W \text{ as } n \to +\infty.$$

This implies

$$\mathcal{J}_\lambda(u_n, v_n) \to \mathcal{J}_\lambda(u_0^-, v_0^-) = \gamma_\lambda^- \text{ as } n \to +\infty.$$

By Lemma 2.2, $(u_0^-, v_0^-) \in \mathcal{N}_\lambda^-$ and (u_0^-, v_0^-) is a local minimizer for $\mathcal{J}_\lambda$ on $\mathcal{N}_\lambda$. Since $\mathcal{J}_\lambda(u_0^-, v_0^-) = \mathcal{J}_\lambda(|u_0^-|, |v_0^-|)$ and $(|u_0^-|, |v_0^-|) \in \mathcal{N}_\lambda^-$, by Lemma 2.3, we may assume that (u_0^-, v_0^-) is a nontrivial nonnegative solution of (1.1), and thus the proof is complete. □

Proof of Theorem 3.1. By Propositions 3.2 and 3.3, we obtain problem (1.1) has two nontrivial nonnegative solutions (u_0^+, v_0^+) and (u_0^-, v_0^-) such that $(u_0^+, v_0^+) \in \mathcal{N}_\lambda^+$ and $(u_0^-, v_0^-) \in \mathcal{N}_\lambda^-$. Since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, (u_0^+, v_0^+) and (u_0^-, v_0^-) are distinct, and thus the proof is complete. □

Acknowledgments. This work was done when the authors visited College of William and Mary in 2012. They would like to thank Department of Mathematics, College of William and Mary for warm hospitality, and thank Professor Junping Shi for constant encouragement and helpful discussions.

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