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ON THE SECOND APPROXIMATE MATSUMOTO METRIC

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Abstract. In this paper, we study the second approximate Matsumoto metric $F = \alpha+\beta+\beta^2/\alpha+\beta^3/\alpha^2$ on a manifold M. We prove that F is of scalar flag curvature and isotropic S-curvature if and only if it is isotropic Berwald metric with almost isotropic flag curvature.

1. Introduction

The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry, which is first introduced by L. Berwald. For a Finsler manifold (M, F), the flag curvature is a function $\mathbf{K}(P, y)$ of tangent planes $P \subset T_xM$ and directions $y \in P$. F is said to be of scalar flag curvature if the flag curvature $\mathbf{K}(P, y) = \mathbf{K}(x, y)$ is independent of flags P associated with any fixed flagpole y. F is called of almost isotropic flag curvature if

$$\mathbf{K} = \frac{3c_{x^m}y^m}{F} + \sigma, \tag{1}$$

where $c = c(x)$ and $\sigma = \sigma(x)$ are scalar functions on M. One of the important problems in Finsler geometry is to characterize Finsler manifolds of almost isotropic flag curvature [10].

To study the geometric properties of a Finsler metric, one also considers non-Riemannian quantities. In Finsler geometry, there are several important non-Riemannian quantities: the Cartan torsion $\mathbf{C}$, the Berwald curvature $\mathbf{B}$, the mean Landsberg curvature $\mathbf{J}$ and S-curvature $\mathbf{S}$, etc. [3, 8, 10, 17]. These are geometric quantities which vanish for Riemnnian metrics.

Among the non-Riemannian quantities, the S-curvature $\mathbf{S} = \mathbf{S}(x, y)$ is closely related to the flag curvature which constructed by Shen for given comparison theorems on Finsler manifolds. A Finsler metric F is called of isotropic S-curvature if

$$\mathbf{S} = (n + 1)cF, \tag{2}$$

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for some scalar function $c = c(x)$ on M. In [10], it is proved that if a Finsler metric F of scalar flag curvature is of isotropic S-curvature (2), then it has almost isotropic flag curvature (1).

The geodesic curves of a Finsler metric $F = F(x, y)$ on a smooth manifold M, are determined by $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients. A Finsler metric F is called a Berwald metric, if G^i are quadratic in $y \in T_xM$ for any $x \in M$. A Finsler metric F is said to be isotropic Berwald metric if its Berwald curvature is in the following form

$$B^i{}_{jkl} = c\Big\{F_{y^jy^k}\delta^i{}_l + F_{y^ky^l}\delta^i{}_j + F_{y^ly^j}\delta^i{}_k + F_{y^jy^ky^l}y^i\Big\}, \tag{3}$$

where $c = c(x)$ is a scalar function on M [3].

As a generalization of Berwald curvature, Bácsó-Matsumoto proposed the notion of Douglas curvature [1]. A Finsler metric is called a Douglas metric if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k + P(x, y)y^i$.

In order to find explicit examples of Douglas metrics, we consider (α, β)-metrics. An (α, β)-metric is a Finsler metric of the form $F := \alpha\phi(\frac{\beta}{\alpha})$, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. This class of metrics is were first introduced by Matsumoto [9]. Among the (α, β)-metrics, the Matsumoto metric is special and significant metric which constitute a majority of actual research. The Matsumoto metric is expressed as

$$F = \alpha\Big[1 + \frac{\beta}{\alpha} + \big(\frac{\beta}{\alpha}\big)^2 + \big(\frac{\beta}{\alpha}\big)^3 + \cdots\Big].$$

This metric was introduced by Matsumoto as a realization of Finsler's idea "a slope measure of a mountain with respect to a time measure" [18]. In the Matsumoto metric, the 1-form $\beta = b_iy^i$ was originally to be induced by earth gravity. Hence, we could regard $b_i(x)$ as the infinitesimals and neglect the infinitesimals of degree of $b_i(x)$ more than two [11, 12, 13, 14, 15]. An approximate Matsumoto metric is a Finsler metric in the following form

$$F = \alpha\left[\sum_{k=0}^{r}\big(\frac{\beta}{\alpha}\big)^k\right], \tag{4}$$

where $|\beta| < |\alpha|$ (for more information, see [12]). This metric was introduced by Park-Choi in [12]. By definition, the Matsumoto metric is expressed as $\lim_{r\to\infty} L(\alpha, \beta) = \frac{\alpha^2}{\alpha-\beta}$.

In this paper, we consider second approximate Matsumoto metric $F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ with some non-Riemannian curvature properties and prove the following.

Theorem 1.1. *Let $F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ be a non-Riemannian second approximate Matsumoto metric on a manifold M of dimension n. Then F is of scaler flag curvature with isotropic S-curvature* (2), *if and only if it has isotropic*

Berwald curvature (3) *with almost isotropic flag curvature* (1). *In this case, F must be locally Minkowskian.*

2. Preliminaries

Let M be a n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on M. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:

(i) F is C^∞ on TM_0;

(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;

(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y + su + tv)\right]|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u,v)\Big]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian [16]. For $y \in T_xM_0$, define mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$. By Diecke Theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$.

The horizontal covariant derivatives of $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y(u) := J_i(y)u^i$, where $J_i := I_{i|s}y^s$. A Finsler metric is said to be weakly Landsbergian if $\mathbf{J} = 0$.

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4}g^{il}\Big[\frac{\partial^2(F^2)}{\partial x^k \partial y^l}y^k - \frac{\partial(F^2)}{\partial x^l}\Big], \qquad y \in T_xM.$$

The $\mathbf{G}$ is called the spray associated to (M, F). In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For a tangent vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{B}_y(u,v,w) := B^i{}_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ and $\mathbf{E}_y(u,v) := E_{jk}(y)u^j v^k$ where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk} := \frac{1}{2}B^m{}_{jkm}.$$

The $\mathbf{B}$ and $\mathbf{E}$ are called the Berwald curvature and mean Berwald curvature, respectively. Then F is called a Berwald metric and weakly Berwald metric if $\mathbf{B} = \mathbf{0}$ and $\mathbf{E} = \mathbf{0}$, respectively.

A Finsler metric F is said to be isotropic mean Berwald metric if its mean Berwald curvature is in the following form

$$E_{ij} = \frac{n+1}{2F} c h_{ij}, \tag{5}$$

where $c = c(x)$ is a scalar function on M and h_{ij} is the angular metric [3].

Define $\mathbf{D}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $\mathbf{D}_y(u,v,w) := D^i{}_{jkl}(y) u^i v^j w^k \frac{\partial}{\partial x^i}|_x$ where

$$D^i{}_{jkl} := B^i{}_{jkl} - \frac{2}{n+1}\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk,l}y^i\}.$$

We call $\mathbf{D} := \{\mathbf{D}_y\}_{y \in TM_0}$ the Douglas curvature. A Finsler metric with $\mathbf{D} = 0$ is called a Douglas metric. The notion of Douglas metrics was proposed by Bácsó-Matsumoto as a generalization of Berwald metrics [1].

For a Finsler metric F on an n-dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x) dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\Big\{(y^i) \in R^n \ \Big|\ F\Big(y^i \frac{\partial}{\partial x^i}|_x\Big) < 1\Big\}}.$$

In general, the local scalar function $\sigma_F(x)$ can not be expressed in terms of elementary functions, even F is locally expressed by elementary functions. Let G^i denote the geodesic coefficients of F in the same local coordinate system. The S-curvature can be defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x,y) - y^i \frac{\partial}{\partial x^i}\Big[\ln \sigma_F(x)\Big],$$

where $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$. It is proved that $\mathbf{S} = 0$ if F is a Berwald metric. There are many non-Berwald metrics satisfying $\mathbf{S} = 0$. $\mathbf{S}$ said to be *isotropic* if there is a scalar functions $c(x)$ on M such that $\mathbf{S} = (n+1)c(x)F$.

The Riemann curvature $\mathbf{R}_y = R^i{}_k dx^k \otimes \frac{\partial}{\partial x^i}|_x : T_xM \to T_xM$ is a family of linear maps on tangent spaces, defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

For a flag $P = \mathrm{span}\{y, u\} \subset T_xM$ with flagpole y, the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(P,y) := \frac{\mathbf{g}_y(u, \mathbf{R}_y(u))}{\mathbf{g}_y(y,y)\mathbf{g}_y(u,u) - \mathbf{g}_y(y,u)^2}.$$

We say that a Finsler metric F is of scalar curvature if for any $y \in T_xM$, the flag curvature $\mathbf{K} = \mathbf{K}(x,y)$ is a scalar function on the slit tangent bundle TM_0. In this case, for some scalar function $\mathbf{K}$ on TM_0 the Riemann curvature is in the following form

$$R^i{}_k = \mathbf{K}F^2\{\delta^i_k - F^{-1}F_{y^k}y^i\}.$$

If $\mathbf{K} = constant$, then F is said to be of constant flag curvature. A Finsler metric F is called *isotropic flag curvature*, if $\mathbf{K} = \mathbf{K}(x)$.

3. Proof of Theorem 1.1

Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be an (α, β)-metric, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on a manifold M. Let

$$r_{ij} := \frac{1}{2}\Big[b_{i|j} + b_{j|i}\Big], \qquad s_{ij} := \frac{1}{2}\Big[b_{i|j} - b_{j|i}\Big].$$

$$r_j := b^i r_{ij}, \qquad s_j := b^i s_{ij},$$

where $b_{i|j}$ denote the coefficients of the covariant derivative of β with respect to α. Let

$$r_{i0} := r_{ij}y^j, \quad s_{i0} := s_{ij}y^j, \quad r_0 := r_j y^j, \quad s_0 := s_j y^j.$$

Put

$$Q = \frac{\phi'}{\phi - s\phi},$$
$$\Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'^2)}{2\phi\big[(\phi - s\phi') + (b^2 - s^2)\phi''\big]} \tag{6}$$
$$\Psi = \frac{\phi''}{2\big[(\phi - s\phi') + (b^2 - s^2)\phi''\big]}.$$

Then the S-curvature is given by

$$\mathbf{S} = \Big[Q' - 2\Psi Qs - 2(\Psi Q)'(b^2 - s^2) - 2(n+1)Q\Theta + 2\lambda\Big]s_0$$
$$+ 2(\Psi + \lambda)s_0 + \alpha^{-1}\Big[(b^2 - s^2)\Psi' + (n+1)\Theta\Big]r_{00}. \tag{7}$$

Let us put

$$\Delta := 1 + sQ + (b^2 - s^2)Q',$$
$$\Phi := -(n\Delta + 1 + sQ)(Q - sQ') - (b^2 - s^2)(1 + sQ)Q''.$$

In [5], Cheng-Shen characterize (α, β)-metrics with isotropic S-curvature.

Lemma 3.1 ([5])**.** *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric on an n-manifold. Then, F is of isotropic S-curvature $\mathbf{S} = (n+1)cF$, if and only if one of the following holds*

(i) *β satisfies*

$$r_{ij} = \varepsilon\Big\{b^2 a_{ij} - b_i b_j\Big\}, \qquad s_j = 0, \tag{8}$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2}, \tag{9}$$

where k is a constant. In this case, $c = k\epsilon$.

(ii) *β satisfies*

$$r_{ij} = 0, \qquad s_j = 0. \tag{10}$$

In this case, $c = 0$.

Let

$$\Psi_1 := \sqrt{b^2 - s^2}\Delta^{\frac{1}{2}}\Big[\frac{\sqrt{b^2 - s^2}\Phi}{\Delta^{\frac{3}{2}}}\Big]',$$
$$\Psi_2 := 2(n+1)(Q - sQ') + 3\frac{\Phi}{\Delta}, \tag{11}$$
$$\theta := \frac{Q - sQ'}{2\Delta}.$$

Then the formula for the mean Cartan torsion of an (α, β)-metric is given by following

$$I_i = \frac{1}{2}\frac{\partial}{\partial y^i}\Big[(n+1)\frac{\phi'}{\phi} - (n-2)\frac{s\phi''}{\phi - s\phi'} - \frac{3s\phi'' - (b^2 - s^2)\phi'''}{(\phi - s\phi') + (b^2 - s^2)\phi''}\Big]$$
$$= -\frac{\Phi(\phi - s\phi')}{2\Delta\phi\alpha^2}(\alpha b_i - s y_i). \tag{12}$$

In [6], it is proved that the condition $\Phi = 0$ characterizes the Riemannian metrics among (α, β)-metrics. Hence, in the continue, we suppose that $\Phi \neq 0$.

Let $G^i = G^i(x, y)$ and $\bar{G}^i_\alpha = \bar{G}^i_\alpha(x, y)$ denote the coefficients of F and α respectively in the same coordinate system. By definition, we have

$$G^i = \bar{G}^i_\alpha + P y^i + Q^i, \tag{13}$$

where

$$P := \alpha^{-1}\Theta\Big[-2Q\alpha s_0 + r_{00}\Big]$$
$$Q^i := \alpha Q {s^i}_0 + \Psi\Big[-2Q\alpha s_0 + r_{00}\Big]b^i.$$

Simplifying (13) yields the following

$$G^i = \bar{G}^i_\alpha + \alpha Q s^i_0 + \theta(-2\alpha Q s_0 + r_{00})\Big[\frac{y^i}{\alpha} + \frac{Q'}{Q - sQ'}b^i\Big]. \tag{14}$$

Clearly, if β is parallel with respect to α ($r_{ij} = 0$ and $s_{ij} = 0$), then $P = 0$ and $Q^i = 0$. In this case, $G^i = \bar{G}^i_\alpha$ are quadratic in y, and F is a Berwald metric.

For an (α, β)-metric $F = \alpha\phi(s)$, the mean Landsberg curvature is given by

$$J_i = -\frac{1}{2\Delta\alpha^4}\Bigg[\frac{2\alpha^2}{b^2 - s^2}\big[\frac{\Phi}{\Delta} + (n+1)(Q - sQ')\big](r_0 + s_0)h_i$$
$$+ \frac{\alpha}{b^2 - s^2}(\Psi_1 + s\frac{\Phi}{\Delta})(r_{00} - 2\alpha Q s_0)h_i + \alpha\Big[-\alpha Q' s_0 h_i + \alpha Q(\alpha^2 s_i - y_i s_0)$$
$$+ \alpha^2\Delta s_{i0} + \alpha^2(r_{i0} - 2\alpha Q s_i) - (r_{00} - 2\alpha Q s_0)y_i\Big]\frac{\Phi}{\Delta}\Bigg]. \tag{15}$$

Contracting (15) with $b^i = a^{im} b_m$ yields

$$\bar{J} := J_i b^i = -\frac{1}{2\Delta\alpha^2}\Big[\Psi_1(r_{00} - 2\alpha Q s_0) + \alpha\Psi_2(r_0 + s_0)\Big]. \tag{16}$$

The horizontal covariant derivatives $J_{i;m}$ and $J_{i|m}$ of J_i with respect to F and α, respectively, are given by

$$J_{i;m} = \frac{\partial J_i}{\partial x^m} - J_l \Gamma^l_{im} - \frac{\partial J_i}{\partial y^l} N^l_m,$$

$$J_{i|m} = \frac{\partial J_i}{\partial x^m} - J_l \bar{\Gamma}^l_{im} - \frac{\partial J_i}{\partial y^l} \bar{N}^l_m.$$

Then we have

$$J_{i;m} y^m = J_{i|m} y^m - J_l (N^l_i - \bar{N}^l_i) - 2\frac{\partial J_i}{\partial y^l}(G^l - \bar{G}^l). \tag{17}$$

Let F be a Finsler metric of scalar flag curvature $\mathbf{K}$. By Akbar-Zadeh's theorem it satisfies following

$$A_{ijk;s;m} y^s y^m + \mathbf{K} F^2 A_{ijk} + \frac{F^2}{3}\Big[h_{ij}\mathbf{K}_k + h_{jk}\mathbf{K}_j + h_{ki}\mathbf{K}_j\Big] = 0, \tag{18}$$

where $A_{ijk} = F C_{ijk}$ is the Cartan torsion and $\mathbf{K}_i = \frac{\partial \mathbf{K}}{\partial y^i}$ [2]. Contracting (18) with g^{ij} yields

$$J_{i;m} y^m + \mathbf{K} F^2 I_i + \frac{n+1}{3} F^2 \mathbf{K}_i = 0. \tag{19}$$

By (17) and (19), for an (α,β)-metric $F = \alpha\phi(s)$ of constant flag curvature $\mathbf{K}$, the following holds

$$J_{i|m} - J_l \frac{\partial (G^l - \bar{G}^l)}{\partial y^i} b^i - 2\frac{\partial \bar{J}}{\partial y^l}(G^l - \bar{G}^l)\mathbf{K}\alpha^2\phi^2 I_i = 0. \tag{20}$$

Contracting (20) with b^i implies that

$$\bar{J}_{|m} y^m - J_i a^{ik} b_{k|m} y^m - J_l \frac{\partial (G^l - \bar{G}^l)}{\partial y^i} b^i - 2\frac{\partial \bar{J}}{\partial y^l}(G^l - \bar{G}^l) + \boldsymbol{K}\alpha^2\phi^2 I_i b^i = 0. \tag{21}$$

There exists a relation between mean Berwald curvature $\mathbf{E}$ and the S-curvature $\mathbf{S}$. Indeed, taking twice vertical covariant derivatives of the S-curvature gives rise the E-curvature. It is easy to see that, every Finsler metric of isotropic S-curvature (2) is of isotropic mean Berwald curvature (5). Now, is the equation $\mathbf{S} = (n+1)cF$ equivalent to the equation $\mathbf{E} = \frac{n+1}{2} c F^{-1}\mathbf{h}$?

Recently, Cheng-Shen prove that a Randers metric $F = \alpha + \beta$ is of isotropic S-curvature if and only if it is of isotropic E-curvature [4]. Then, Chun-Huan-Cheng extend this equivalency to the Finsler metric $F = \alpha^{-m}(\alpha+\beta)^{m+1}$ for every real constant m, including Randers metric [20]. In [7], Cui extend their result and show that for the Matsumoto metric $F = \frac{\alpha^2}{\alpha-\beta}$ and the special (α,β)-metric $F = \alpha + \epsilon\beta + \kappa(\beta^2/\alpha)$ $(\kappa \neq 0)$, these notions are equivalent.

To prove Theorem 1.1, we need the following.

Proposition 3.2. *Let $F=\alpha+\beta+\frac{\beta^2}{\alpha}+\frac{\beta^3}{\alpha^2}$ be a second approximate Matsumoto metric on a manifold M of dimension n. Then the following are equivalent*

(i) *F has isotropic S-curvature, $\mathbf{S}=(n+1)c(x)F$;*
(ii) *F has isotropic mean Berwald curvature, $\mathbf{E}=\frac{n+1}{2}c(x)F^{-1}\mathbf{h}$;*

where $c=c(x)$ is a scalar function on the manifold M. In this case, $\mathbf{S}=0$. Then β is a Killing 1-form with constant length with respect to α, that is, $r_{00}=0$.

Proof. (i)⇒(ii) is obvious. Conversely, suppose that F has isotropic mean Berwald curvature, $\mathbf{E}=\frac{(n+1)}{2}c(x)F^{-1}\mathbf{h}$. Then we have

$$\mathbf{S}=(n+1)[cF+\eta], \tag{22}$$

where $\eta=\eta_i(x)y^i$ is a 1-form on M. For the second approximate Matsumoto metric, (6) reduces to following

$$\begin{aligned}
Q&=-\frac{1+2s+3s^2}{-1+s^2+2s^3},\\
\Theta&=\frac{1}{2}\frac{1-6s^2-12s^3-15s^4-12s^5}{(1+s+s^2+s^3)(1-3s^2-8s^3+2b^2+6b^2s)},\\
\Psi&=\frac{1+3s}{(1-3s^2-8s^3+2b^2+6b^2s)}.
\end{aligned}\tag{23}$$

By substituting (22) and (23) in (7), we have

(24)

$$\begin{aligned}
\mathbf{S}=\Bigg[&\frac{2(1+3s)(1+s+s^2+s^3)}{(-1+s^2+2s^3)^2}+\frac{2(1+3s)(1+2s+3s^2)s}{(1-3s^2-8s^3+2b^2+6b^2s)(-1+s^2+2s^3)}\\
&-\frac{2(5+26s+77s^2+88s^3-61s^4-430s^5-805s^6+4b^2+40b^2s+148b^2s^2)(b^2-s^2)}{(1-3s^2-8s^3+2b^2+6b^2s)^2(-1+s^2+2s^3)^2}\\
&-\frac{2(256s^3b^2+252s^4b^2+216s^5b^2+108s^6b^2-828s^7-432s^8)(b^2-s^2)}{(1-3s^2-8s^3+2b^2+6b^2s)^2(-1+s^2+2s^3)^2}\\
&+\frac{(n+1)(1+2s+3s^2)(1-6s^2-12s^3-15s^4-12s^5)}{(-1+s^2+2s^3)(1+s+s^2+s^3)(1-3s^2-8s^3+2b^2+6b^2s)}+2\lambda\Bigg]s_0\\
&+2\left[\frac{(1+3s)}{1-3s^2-8s^3+2b^2+6b^2s}+\lambda\right]+\left[\frac{3(b^2-s^2)(1+11s^2+16s^3+2s)}{\alpha(1-3s^2-8s^3+2b^2+6b^2s)^2}\right]r_{00}\\
&+\left[\frac{(n+1)(1-6s^2-12s^3-15s^4-12s^5)}{2\alpha(1+s+s^2+s^3)(1-3s^2-8s^3+2b^2+6b^2s)}\right]r_{00}\\
=&\,(n+1)\big[c\alpha(1+s+s^2+s^3)+\eta\big].
\end{aligned}$$

Multiplying (24) with $(-1+s^2+2s^3)(1+s+s^2+s^3)(1-3s^2-8s^3+2b^2+6b^2s)^2\alpha^{14}$ implies that

$$M_1+M_2\alpha^2+M_3\alpha^4+M_4\alpha^6+M_5\alpha^8+M_6\alpha^{10}+M_7\alpha^{12}+M_8\alpha^{14}$$

$$+\alpha\Big[M_9+M_{10}\alpha^2+M_{11}\alpha^4+M_{12}\alpha^6+M_{13}\alpha^8+M_{14}\alpha^{10}$$

$$+M_{15}\alpha^{12}+M_{16}\alpha^{14}\Big]=0, \tag{25}$$

where

$$
\begin{aligned}
M_1 :=\ & -128(n+1)c\beta^{15},\\
M_2 :=\ & 2\Big[(n+1)\big[(-385+96b^2)c\beta^2-64\eta\beta\big]+128\lambda(r_0+s_0)\beta+48nr_{00}\Big]\beta^{11},\\
M_3 :=\ & -\Big[(n+1)[4(-283b^2+243+18b^4)c\beta^2-6(32b^2-59)\eta\beta]\\
& +[12(-59+32b^2)\lambda(r_0+s_0)-96((3n-1)s_0-r_0]\beta\\
& +3(-24b^2+36-219n+72nb^2)r_{00}\Big]\beta^9,\\
M_4 :=\ & \Big[(n+1)\big[-2(29-738b^2+208b^4)c\beta^2-3(-172b^2+21+24b^4)\eta\beta\big]\\
& +(-159nb^2-132-39n+96b^2)r_{00}\\
& -3(-48b^2+123+72nb^2-277n)s_0\beta\\
& +3(24b^2+86)r_0\beta+6\lambda(-172b^2+21+24b^4)(s_0+r_0)\beta\Big]\beta^7,\\
M_5 :=\ & \Big[(n+1)[-8(-27b^2+70b^4-41)c\beta^2-4(47b^4-40-32b^2)\eta\beta]\\
& +(15nb^2+20-55n+108b^2)r_{00}+8\lambda(47b^4-40-32b^2)(r_0+s_0)\beta\\
& -2\big[(-262b^2+278+303nb^2-147n)s_0-(94b^2+32)r_0\big]\beta\Big]\beta^5\\
M_6 :=\ & 2\Big[(n+1)\big[-2(4b+1)(4b-1)(4b^2+13)c\beta^2-10(1+20b^2+6b^4)\eta\beta\big]\\
& +(11n-40b^2+35nb^2+20)r_{00}+20\lambda(1+20b^2+6b^4)(r_0+s_0)\beta\\
& -2(32n+51+129nb^2-294b^2)s_0\beta-(30b^2-50)r_0\beta\Big]\beta^3,\\
M_7 :=\ & -\Big[(n+1)\big[-4(-17b^2-7+30b^4)c\beta^2-6(-1+10b^4)\beta\big]\\
& +(3n+12)b^2r_{00}\\
& -12\lambda(1-10b^4)(s_0+r_0)-6\big[(nb^2-n-2+14b^2)s_0-10b^2r_0\big]\beta\Big]\beta,\\
M_8 :=\ & 4\Big[(n+1)\Big[2(1+2b^2)(8b^2+1)c\beta+(1+2b^2)^2\eta\Big]\\
& -2\lambda(1+2b^2)^2(s_0+r_0)\\
& +\big[-57nb^2-64(n+1)b^4-8n-55b^2+6\big]r_{00}-(2-4b^2)r_0\\
& +\big[n+(4+2n)b^2-1\big]s_0\Big],\\
M_9 :=\ & -416(n+1)c\beta^{14},
\end{aligned}
$$

$$
\begin{aligned}
M_{10} :=& \Big[(n+1)[(-1037+616b^2)c\beta^2 - 288\eta\beta]\\
&+ 576\lambda(r_0+s_0)\beta + (204n-6)r_{00}\Big]\beta^{12},\\
M_{11} :=& -\frac{1}{2}\Big[(n+1)[8(57b^4-385b^2+143)c\beta^2 - 2(424b^2-267)\eta\beta]\\
&+ \big[4\lambda(424b^2-267)(r_0+s_0) - 4(-115+330n)s_0 + 1320r_0\big]\beta\\
&+ (300nb^2+249-189n-120b^2)r_{00}\Big]\beta^8,\\
M_{12} :=& 4\Big[(n+1)[-(572b^4-932b^2-275)c\beta^2 - 4(39b^4-28-102b^2)\eta\beta]\\
&- 75nb^2 - 62 - 89n + 144b^2r_{00} + 8\lambda(39b^4-28-102b^2)(r_0+s_0)\beta\\
&- 6[(-58b^2+100+81nb^2-109n)s_0 - (26b^2+34)r_0]\beta\Big]\beta^6,\\
M_{13} :=& \Big[(n+1)[-8(-24+35b^2+47b^4)c\beta^2 - 6(26b^4-11+20b^2)\eta\beta]\\
&+ (51nb^2+33-6n+24b^2)r_{00} + 12\lambda(26b^4-11+20b^2)(s_0+r_0)\\
&- 6(-118b^2+54+83nb^2-3n)s_0\beta + 6(26b^2-10)r_0\beta\Big]\beta^4,\\
M_{14} :=& -\Big[(n+1)[-(56b^4-272b^2-39)c\beta^2 - 4(7(n+1)b^4\\
&- 6(1+n) - 22nb^2)\eta\beta]\\
&+ (-7nb^2-8-5n+32b^2)r_{00} + 8\lambda(7b^4-6-22b^2)(r_0+s_0)\beta\\
&+ 2(-154b^2+12+33nb^2+19n)s_0\beta + 2(14b^2-22)s_0\beta\Big]\beta^2\\
M_{15} :=& \Big[(n+1)[4(23b^4+5b^2-1)c\beta^2 - (14b^2+1)(2b^2+1)\eta\beta]\\
&- \frac{n+1}{2}(8b^2+1)r_{00} - 2\lambda(14b^2+1)(2b^2+1)(s_0+r_0)\beta\\
&- 2(-6b^2+3-5nb^2)s_0\beta - 2(4+14b^2)r_0\beta\Big],\\
M_{16} :=& (n+1)(1+2b^2)^2c.
\end{aligned}
$$

The term of (25) which is seemingly does not contain α^2 is M_1. Since β^{15} is not divisible by α^2, then $c=0$ which implies that

$$M_1 = M_9 = 0.$$

Therefore (25) reduces to following

$$M_2 + M_3\alpha^2 + M_4\alpha^4 + M_5\alpha^6 + M_6\alpha^8 + M_7\alpha^{10} + M_8\alpha^{12} = 0, \tag{26}$$

$$M_{10} + M_{11}\alpha^2 + M_{12}\alpha^4 + M_{13}\alpha^6 + M_{14}\alpha^8 + M_{15}\alpha^{10} + M_{16}\alpha^{12} = 0. \tag{27}$$

By plugging $c = 0$ in M_2 and M_{10}, the only equations that don't contain α^2 are the following

$$8\Big[8(2\lambda(r_0+s_0)-(n+1)\eta)+6nr_{00}\Big]=\tau_1\alpha^2, \tag{28}$$

$$6\Big[48(2\lambda(r_0+s_0)-(n+1)\eta)+(34n-1)r_{00}\Big]=\tau_2\alpha^2, \tag{29}$$

where $\tau_1 = \tau_1(x)$ and $\tau_2 = \tau_2(x)$ are scalar functions on M. By eliminating $[2\lambda(r_0 + s_0) - (n + 1)\eta]$ from (28) and (29), we get

$$r_{00} = \tau\alpha^2, \tag{30}$$

where $\tau = \frac{\tau_2-\tau_1}{-(18n+1)}$. By (28) or (29), it follows that

$$2\lambda(r_0+s_0)-(n+1)\eta=0. \tag{31}$$

By (30), we have $r_0 = \tau\beta$. Putting (30) and (31) in M_{10} and M_{11} yield

(32)

$$M_{10} = (204n-6)\tau\alpha^2\beta^{12},$$

(33)

$$M_{11} = \Big[[(660n-230)s_0-660r_0]\beta-\frac{(300n-120)b^2+249-189n}{2}r_{00}\tau\alpha^2\Big]\beta^9.$$

By putting (32) and (33) into (27), we have

$$\begin{aligned}(34)\quad &[(660n-230)s_0-660r_0]\beta^{10}-\frac{300nb^2+249-189n-120b^2}{2}r_{00}\tau\alpha^2\beta^9\\ &+(204n-6)\tau\beta^{12}-M_{12}\alpha^2+M_{13}\alpha^4+M_{14}\alpha^6+M_{15}\alpha^8+M_{16}\alpha^{10}=0.\end{aligned}$$

The only equations of (34) that do not contain α^2 is $[(204n-6)\tau\beta^2 + (660n - 230)s_0 - 660r_0]\beta^{10}$. Since β^{10} is not divisible by α^2, then we have

$$[(204n-6)\tau\beta^2+(660n-230)s_0-660r_0]=0. \tag{35}$$

By Lemma 3.1, we always have $s_j = 0$. Then (35), reduces to following

$$(204n-6)\tau\beta^2-660r_0=0. \tag{36}$$

Thus

$$2(204n-6)\tau b_i\beta-660\tau b_i=0. \tag{37}$$

By multiplying (37) with b^i, we have

$$\tau = 0.$$

Thus by (31), we get $\eta = 0$ and then $\mathbf{S} = (n + 1)cF$. By (30), we get $r_{ij} = 0$. Therefore Lemma 3.1, implies that $\mathbf{S} = 0$. This completes the proof. □

Proof of Theorem 1.1. Let F be an isotropic Berwald metric (3) with almost isotropic flag curvature (1). In [19], it is proved that every isotropic Berwald metric (3) has isotropic S-curvature (2).

Conversely, suppose that F is of isotropic S-curvature (2) with scalar flag curvature $\mathbf{K}$. In [10], it is showed that every Finsler metric of isotropic S-curvature (2) has almost isotropic flag curvature (1). Now, we are going to prove that F is a isotropic Berwald metric. In [3], it is proved that F is an isotropic Berwald metric (3) if and only if it is a Douglas metric with isotropic mean Berwald curvature (5). On the other hand, every Finsler metric of isotropic S-curvature (2) has isotropic mean Berwald curvature (5). Thus for completing the proof, we must show that F is a Douglas metric. By Proposition 3.2, we have $\mathbf{S} = 0$. Therefore by Theorem 1.1 in [10], F must be of isotropic flag curvature $\boldsymbol{K} = \sigma(x)$. By Proposition 3.2, β is a Killing 1-form with constant length with respect to α, that is, $r_{ij} = s_j = 0$. Then (14), (15) and (16) reduce to

$$G^i - \bar{G}^i = \alpha Q s^i{}_0, \quad J_i = -\frac{\Phi s_{i0}}{2\alpha\Delta}, \quad \bar{J} = 0. \tag{38}$$

By (12), we get

$$I_i b^i = \frac{-\Phi}{2\Delta F}(\phi - s\phi')(b^2 - s^2). \tag{39}$$

We consider two case:

Case 1. Let $\dim M \geq 3$. In this case, by Schur Lemma F has constant flag curvature and (21) holds. Thus by (38) and (39), the equation (21) reduces to following

$$\frac{\Phi s_{i0}}{2\alpha\Delta} a^{ik} s_{k0} + \frac{\Phi s_{l0}}{2\alpha\Delta}\Big(\alpha Q s^l{}_0\Big)_{.i} b^i - \boldsymbol{K} F \frac{\Phi}{2\Delta}(\phi - s\phi')(b^2 - s^2) = 0. \tag{40}$$

By assumption $\Phi \neq 0$. Thus by (40), we get

$$s_{i0} s^i{}_0 + s_{l0}\Big(\alpha Q s^l{}_0\Big)_{.i} b^i - \boldsymbol{K} F \alpha(\phi - s\phi')(b^2 - s^2) = 0. \tag{41}$$

The following holds

$$\Big(\alpha Q s^l{}_0\Big)_{.i} b^i = sQ s^i{}_0 + Q' s^i{}_0 (b^2 - s^2).$$

Then (41) can be rewritten as follows

$$s_{i0} s^i{}_0 \Delta - \boldsymbol{K}\alpha^2 \phi(\phi - s\phi')(b^2 - s^2) = 0. \tag{42}$$

By (11), (23) and (42), we obtain

(43)
$$\Big[1 - \frac{s(1+2s+3s^2)}{(-1+s^2+2s^3)} + \frac{2(b^2-s^2)(1+3s)(1+s+s^2+s^3)}{(-1+s^2+2s^3)^2}\Big] s_{i0} s^i{}_0$$
$$- \boldsymbol{K}(1+s+s^2+s^3)\alpha^2\big[1+s+s^2+s^3 - s(1+2s+3s^2)\big](b^2 - s^2) = 0.$$

Multiplying (43) with $(-1+s^2+2s^3)^2\alpha^{12}$ yields

$$A+\alpha B=0,$$

where

$$\begin{aligned}
A = &-\boldsymbol{K}b^2\alpha^{14}+(2b^2+1)(\boldsymbol{K}\beta^2+s_{i0}s^i{}_0)\alpha^{12}\\
&+2(3\boldsymbol{K}b^2\beta^2+4s_{i0}s^i{}_0b^2-\boldsymbol{K}\beta^2-s_{i0}s^i{}_0)\beta^2\alpha^{10}\\
&-(6\boldsymbol{K}\beta^2+11s_{i0}s^i{}_0+20\boldsymbol{K}\beta^2b^2-6s_{i0}s^i{}_0)\beta^4\alpha^8\\
&-(-20\boldsymbol{K}\beta^2+5\boldsymbol{K}\beta^2b^2+8s_{i0}s^i{}_0)\beta^6\alpha^6+(\boldsymbol{K}\beta^{10})(26b^2+5)\alpha^4\\
&-2\boldsymbol{K}\beta^{12}(13-4b^2)\alpha^2-8\boldsymbol{K}\beta^{14},\\
B = &-(\boldsymbol{K}b^2\beta)\alpha^{12}+(1+8b^2)(\boldsymbol{K}b^2\beta^2+s_{i0}s^i{}_0)\beta\alpha^{10}\\
&-2(3\boldsymbol{K}b^2\beta^2-4s_{i0}s^i{}_0b^2+4\boldsymbol{K}\beta^2+5s_{i0}s^i{}_0)\beta^3\alpha^8\\
&+(6\boldsymbol{K}\beta^2-11s_{i0}s^i{}_0-20\boldsymbol{K}\beta^2b^2)\beta^5\alpha^6\\
&+(5\boldsymbol{K}\beta^9)(3b^2+4)\alpha^4+(5\boldsymbol{K}\beta^{11})(-3+4b^2)\alpha^2-20\alpha\boldsymbol{K}\beta^{13}.
\end{aligned} \tag{44}$$

Obviously, we have $A=0$ and $B=0$.

By $A=0$ and the fact that β^{14} is not divisible by α^2, we get $\mathbf{K}=0$. Therefore (43) reduces to following

$$s_{i0}s^i{}_0=a_{ij}s^j{}_0s^i{}_0=0.$$

Because of positive-definiteness of the Riemannian metric α, we have $s^i{}_0=0$, i.e., β is closed. By $r_{00}=0$ and $s_0=0$, it follows that β is parallel with respect to α. Then $F=\alpha+\beta+\frac{\beta^2}{\alpha}+\frac{\beta^3}{\alpha^2}$ is a Berwald metric. Hence F must be locally Minkowskian.

Case 2. Let dim $M=2$. Suppose that F has isotropic Berwald curvature (3). In [19], it is proved that every isotropic Berwald metric (3) has isotropic S-curvature, $\mathbf{S}=(n+1)cF$. By Proposition 3.2, $c=0$. Then by (3), F reduces to a Berwald metric. Since F is non-Riemannian, then by Szabó's rigidity Theorem for Berwald surface (see [2] page 278), F must be locally Minkowskian. □

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