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ON SOLVABILITY OF THE DISSIPATIVE KIRCHHOFF EQUATION WITH NONLINEAR BOUNDARY DAMPING

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Abstract. In this paper, we prove the global existence and uniqueness of the dissipative Kirchhoff equation

$$\begin{aligned}&u_{tt} - M(\|\nabla u\|^2) \triangle u + \alpha u_t + f(u) = 0 \text{ in } \Omega \times [0, \infty),\\ &u(x, t) = 0 \text{ on } \Gamma_1 \times [0, \infty),\\ &\tfrac{\partial u}{\partial \nu} + g(u_t) = 0 \text{ on } \Gamma_0 \times [0, \infty),\\ &u(x, 0) = u_0,\ u_t(x, 0) = u_1 \text{ in } \Omega\end{aligned}$$

with nonlinear boundary damping by Galerkin approximation benefited from the ideas of Zhang et al. [33]. Furthermore,we overcome some difficulties due to the presence of nonlinear terms $M(\|\nabla u\|^2)$ and $g(u_t)$ by introducing a new variables and we can transform the boundary value problem into an equivalent one with zero initial data by argument of compacity and monotonicity.

1. Introduction

Let Ω be a bounded domain of R^n and let Γ denote its C^2 boundary. Assume that Γ consists of two parts, Γ_0 and Γ_1, with positive measure and such that Γ_0 and Γ_1 are closed and disjoint. Let $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ denote the unit outward normal to Γ and let $\frac{\partial}{\partial \nu}$ denote the normal derivative.

This paper is concerned with the global existence and uniqueness of solutions of the following dissipative Kirchhoff equation with nonlinear boundary damping

$$(1.1) \qquad \begin{cases} u_{tt} - M(\|\nabla u\|^2) \triangle u + \alpha u_t + f(u) = 0 \text{ in } \Omega \times [0, \infty),\\ u(x, t) = 0 \text{ on } \Gamma_1 \times [0, \infty),\\ \frac{\partial u}{\partial \nu} + g(u_t) = 0 \text{ on } \Gamma_0 \times [0, \infty),\\ u(x, 0) = u^0(x), u_t(x, 0) = u^1(x) \text{ in } \Omega,\end{cases}$$

where $\|u\|^2 = \int_\Omega |u|^2 dx$, $\alpha > 0$ and $M(s)$, $f(u)$, $g(u_t)$ are functions enjoying some properties (see (A1)-(A3) below).

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For the case, $M(s) = 1$, we have the following wave equation with Dirichlet's boundary conditions

$$\begin{cases} u_{tt} - \triangle u + \alpha u_t + f(u) = 0 \text{ in } \Omega \times [0, \infty), \\ u(x,t) = 0 \text{ on } \partial\Omega \times [0, \infty), \\ u(x,0) = u^0(x), \ u_t(x,0) = u^1(x) \text{ in } \Omega. \end{cases} \tag{1.2}$$

The problem (1.1) arises in many fields such as classical mechanics, fluid dynamics, quantum field theory (see [27, 33]). Later, lots of researchers obtain global existence and uniqueness of solution of the equation (1.2) (see [8, 9, 10, 15, 24, 25, 26, 28]). It is worth mentioning that Z. Y. Zhang et al. [33] recently have investigated global existence and uniform decay for wave equation with dissipative term and boundary damping as follows

$$\begin{cases} u_{tt} - \triangle u + b(x)u_t + f(u) = 0 \text{ in } \Omega \times [0, \infty), \\ u = 0 \text{ on } \Gamma_1 \times [0, \infty), \\ \frac{\partial u}{\partial \nu} + g(u_t) = 0 \text{ on } \Gamma_0 \times [0, \infty), \\ u(x,0) = u^0(x), \ u_t(x,0) = u^1(x) \text{ in } \Omega, \end{cases}$$

under some assumptions on nonlinear feedback function $g(u_t)$. They have obtain the results by means of Galerkin method and the multiplier technique. More precisely, they introduced a new variables and transformed the boundary value problem into an equivalent one with zero initial data by argument of compacity and monotonicity. If $M(s) = 1$ and $\alpha = b(x)$, then the above problem reduce to problem (1.2). More details are present in [33]. Later on, Zhang et al. [32] studied the wellposedness and uniform stability of strong and weak solutions of the nonlinear generalized dissipative Klein-Gordon equation with nonlinear damped boundary conditions given by

$$\begin{cases} u_{tt} - \triangle u + b(x)u_t + f(u) + h(\nabla u) = 0 \text{ in } \Omega \times (0, \infty), \\ u = 0 \text{ on } \Gamma_0 \times (0, \infty), \\ \frac{\partial u}{\partial \nu} + g(u_t) = 0 \text{ on } \Gamma_1 \times (0, \infty), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) \text{ in } \Omega. \end{cases}$$

Also, the authors proved the wellposedness by means of nonlinear semigroup method and obtain the uniform stabilization by using the perturbed energy functional method.

Now, we pay attention to our problem. In [11], G. Kirchhoff firstly proposed the so called Kirchhoff string model in the study nonlinear vibration of an elastic string

$$\rho h u_{tt} + \delta u_t = p_0 + \frac{Eh}{2L} \left(\int_0^L |u_x|^2 dx \right) u_{xx} + f, \quad 0 < x < L, \ t > 0, \tag{1.3}$$

where $u = u(x,t)$ is the lateral displacement at the space coordinate x and the time t, E is the Young modules, h is the cross-section area, ρ is the mass density, L is the length, p_0 is the initial axial tension, δ is the resistance modules and f is the external force. More details are present in [30, 31, 34, 35].

Lots of papers have devoted to the Kirchhoff equation with other boundary conditions. For instance, we can see [18, 19, 21, 22, 23, 30, 31, 34, 35]. It is interesting to observe that problems with condition $M(s) = 1$ and a feedback occurs on the boundary were studied by many authors (see [3, 4, 5, 6, 7, 12, 13, 14, 20]). It is worth mentioning that M. Aassila [1] has employed Yosida approximation to obtain the global existence for the Kirchhoff equation with nonlinear damping as follows

$$\begin{cases} u_{tt} - M(\|\nabla u\|^2) \triangle u + g(u_t) = h(x,t) \quad \text{in } \Omega \times [0,\infty), \\ u(x,t) = 0 \text{ on } \Gamma_1 \times [0,\infty), \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_0 \times [0,\infty), \\ u(x,0) = u^0(x), \ u_t(x,0) = u^1(x) \quad \text{in } \Omega. \end{cases}$$

It is worth mentioning that Z. Y. Zhang [34] recently have considered the global existence and asymptotic behavior of solutions to initial boundary problem

$$\begin{cases} u_{tt} + \gamma \triangle^2 u - M(\|\nabla u\|^2)\triangle u + \alpha u_t + f(u) = 0, \ (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = 0, \ \ (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), \end{cases}$$

by Banach contraction mapping principle and modified energy functional method.

However, according to our best knowledge, in the present paper, we have to treat Eq.(1.1) with a nonlinear boundary damping condition $g(u_t)$ and nonlinear term $M(\|\nabla u\|^2)$ and it is not considered in the literature. The proof of the existence is based on the Galerkin approximation. For strong solutions to (1.1), this approximation requires a change of variables to transform (1.1) into an equivalent problem with initial value equals zero. Especially, we overcome some difficulties, that is, the presence of nonlinear terms $M(\|\nabla u\|^2)$ and nonlinear boundary damping $g(u_t)$ bring up serious difficulties when passing to the limit, which overcome combining arguments of compacity and monotonicity.

We organize the paper as follows. In Section 2, we give the notations and state our main results. In Section 3, we prove the existence and uniqueness of strong and weak solutions to the problem (1.1) by Galerkin method.

2. Notations and main results

In this section, we give some notations which will be used throughout this paper and will state our main results.

Let

$$V = \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma_1\}$$

and we define

$$(u,v) = \int_\Omega u(x)v(x)dx, (u,v)_{\Gamma_0} = \int_{\Gamma_0} u(x)v(x)dx,$$
$$\|u\|^2 = \int_\Omega u^2 dx, |u|^2_{\Gamma_0} = \int_{\Gamma_0} u^2 dx.$$

Now, we state the general assumptions:

(A1) Assumptions on f:

Let $f : R \to R$ be a $W^{1,\infty}_{loc}(R)$, piecewise $C^1(R)$ function,

$$f(s)s \geq 0 \quad \text{for } s \in R. \tag{2.1}$$

Assume that there exists $C_1 > 0$ such that

$$|f'(s)| \leq C_1(1 + |s|^{p-1}), \ 1 < p \leq \tfrac{n}{n-2} \quad \text{for } s \in R. \tag{2.2}$$

Setting $F(s) = \int_0^s f(\lambda)d\lambda$, there exist C_2, $C_3 > 0$ satisfying

$$C_2|s|^{p+1} \leq F(s) \leq C_3 s f(s) \quad \text{for } s \in R. \tag{2.3}$$

We notice that from (2.2), we derive that there exists $C_4 > 0$ such that

$$|f(s)| \leq C_4(1 + |s|^p) \quad \text{for } s \in R. \tag{2.4}$$

Also, we assume that there exists $C_4 > 0$ such that

$$|f(x) - f(y)| \leq C_5(|x|^{p-1} + |y|^{p-1})|x - y| \quad \text{for } x, y \in R. \tag{2.5}$$

(A2) Assumptions on g:

Let $g : R \to R$ be non-decrease $C^1(R)$ function,

$$g(s)s \geq 0 \quad \text{for } s \neq 0.$$

There exist $C_i (i = 6, 7, 8, 9)$, such that

$$C_6|s| \leq |g(s)| \leq C_7|s| \quad \text{if } |s| \leq 1, \tag{2.6}$$

$$C_8|s|^q \leq |g(s)| \leq C_9|s|^q, \ 1 < q \leq \tfrac{n-1}{n-2} \quad \text{if } |s| > 1. \tag{2.7}$$

(A3) Assumptions on $M(s)$:

Let $M : R \to R$ be a $C^1(R)$ function. Assume that there exist C_{10}, $C_{11} > 0$ such that

$$0 < C_{10} \leq M(s) \leq C_{11} \quad \text{for } s > 0. \tag{2.8}$$

Also, we define

$$\overline{M(s)} = \int_0^s M(s)ds.$$

Furthermore, $\overline{M(s)}$ satisfies the following condition

$$|\overline{M(s)}| \leq sM(s).$$

(A4) Assume that

$$\{u^0, u^1\} \in D(A) \times D(A), \tag{2.9}$$

satisfying the compatibility condition

$$\frac{\partial u}{\partial \nu} + g(u_t) = 0 \quad \text{on } \Gamma_0, \tag{2.10}$$

where $D(A) = \{u \in V, \ \triangle u \in H\}$.

Notations. In this paper C, and C_i will denote various positive constants which may be different from line to line and we denote $L^2(\Omega) = H$.

Now, we are position to state our main results.

Theorem 2.1. *Under assumption* (A1)-(A4)*, the problem* (1.1) *admits a unique strong solution, that is, a function* $u(x,t) : [0,\infty) \times \Omega \to R$*, such that*

$$u \in L^\infty[0,\infty; V),\ \ u_t \in L^\infty[0,\infty; V),\ \ u_{tt} \in L^\infty[0,\infty; L^2(\Omega)).$$

Theorem 2.2. *Assume that* $\{u^0, u^1\} \in V \times L^2(\Omega)$ *and assumptions* (A1)-(A3) *hold. Then the problem* (1.1) *has a unique weak solutions,* $u(x,t) : [0,\infty)\times\Omega \to R$*, in the class*

$$u \in C[0,\infty; V) \cap C^1[0,\infty; L^2(\Omega)).$$

3. Existence and uniqueness of strong and weak solutions

In this section, we prove the existence and uniqueness of strong solutions of the problem (1.1), when u^0, u^1 are smooth. Firstly, we consider strong solutions by Galerkin approximation and we extend the same result to weak solutions using a density argument.

Now, we consider the variational function of (1.1) as follows

$$(u_{tt}, w) + M(\|\nabla u\|^2)(\nabla u, \nabla w) + \alpha(u_t, w) + (g(u_t), w)_{\Gamma_0} + (f(u), w) = 0, \forall w \in V. \tag{3.1}$$

Strong solutions to (1.1) with boundary condition $(g(u_t), w)_{\Gamma_0}$ can not be obtained by the method of "special basis", hence, basis formed by eigenfunctions of the operator $-\triangle$ can not be used for it. This leads us to differentiate the variational formulation related with (1.1) with respect to time t. But, this brings up serious difficulties when we shall estimate $u_{tt}(0)$.

In order to overcome this difficulties, we can transform the boundary value problem (1.1) into an equivalent one with zero initial data. In fact, we introduce the new variables

$$v(x,t) = u(x,t) - \phi(x,t), \tag{3.2}$$

where

$$\phi(x,t) = u^0(x) + tu^1(x),\ \ t \in [0,T]. \tag{3.3}$$

Due to (2.11), (2.12), (3.1)-(3.3), we get the equivalent problem for variables v :

$$\begin{cases} v_{tt} - M(\|\nabla v + \nabla\phi\|^2)\triangle v + \alpha v_t + f(v+\phi) = \mathcal{F} \text{ in } \Omega \times [0,\infty), \\ v = 0 \text{ on } \Gamma_1 \times [0,\infty), \\ \frac{\partial v}{\partial \nu} + g(v_t + \phi_t) = \mathcal{G} \text{ on } \Gamma_0 \times [0,\infty), \\ v(0) = v_t(0) = 0 \text{ in } \Omega, \end{cases} \tag{3.4}$$

where

$$\mathcal{F} = \triangle\phi + \alpha\phi_t,\ \mathcal{G} = -\frac{\partial\phi}{\partial\nu}. \tag{3.5}$$

We note that if v is a solution of (3.4) in $[0,T]$, then $u = v + \phi$ is a solution of (1.1) in the same interval.

Hence, using standard methods, we can extend the solution u to the interval $[0,\infty)$. It is sufficient to prove that (3.4) has a local solution by using Galerkin method.

Let $(\omega_j)_{j\in N}$ be basis in $D(A) = V \cap H^2(\Omega)$ which is orthonormal in H and setting $V_m = span\{\omega_1, \omega_2, \ldots, \omega_m\}$.

Now, we define $v_m(t) = \sum_{j=1}^{n} g_{jm}(t)\omega_j$, where $v_m(t)$ is the solution the Cauchy problem as follows:

$$\begin{aligned}&(v''_m, w) + M(\|\nabla v + \nabla\phi\|^2)(\nabla v_m, \nabla w) + (\alpha v'_m, w) + (g(v'_m + \phi'), w)_{\Gamma_0}\\ &+ (f(v_m + \phi), w) = (\mathcal{F}, w) + (\mathcal{G}, w)_{\Gamma_0},\ \forall w \in V_m,\\ &v_m(0) = v'_m(0) = 0.\end{aligned} \tag{3.6}$$

By standard methods of differential equations, we prove the existence of a solution (3.6) on some interval $[0,t_m]$, then, this solution can be extended to the whole interval $[0,T]$ by use the first estimate as follows.

First Estimate Taking $w = v'_m$ in (3.6), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{\|v'_m\|^2 + \overline{M(\|\nabla v + \nabla\phi\|^2)} + 2\int_\Omega F(v_m + \phi)dx\right\} + \frac{1}{2}\alpha\|v'_m\|^2\\ &+ (g(v'_m + \phi'), v'_m + \phi')_{\Gamma_0} = (\mathcal{F}, v'_m) + \frac{d}{dt}(\mathcal{G}, v_m)_{\Gamma_0} - (\mathcal{G}', v_m)_{\Gamma_0}\\ &+ (f(v_m + \phi), v'_m) + (g(v'_m + \phi'), \phi')_{\Gamma_0} - M(\|\nabla v + \nabla\phi\|^2)(\nabla v_m, \nabla u^1)\\ &- M(\|\nabla v + \nabla\phi\|^2)(\nabla v'_m, \nabla\phi) - M(\|\nabla v + \nabla\phi\|^2)(\nabla\phi, \nabla\phi'),\end{aligned}$$

i.e.,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{\|v'_m\|^2 + \overline{M(\|\nabla v + \nabla\phi\|^2)} + 2\int_\Omega F(v_m + \phi)dx\right\}\\ &+ (g(v'_m + \phi'), v'_m + \phi')_{\Gamma_0}\\ &\le (\mathcal{F}, v'_m) + \frac{d}{dt}I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + I_6(t) + I_7(t).\end{aligned} \tag{3.7}$$

Next, we shall estimate for $I_i(t)(i = 1, 2, \ldots, 7)$ in (3.7), respectively.

Due to (2.4) and applying Young's inequality, we get

$$\begin{aligned}|I_3| &= |(f(v_m + \phi), v'_m)|\\ &\le C\int_\Omega (1 + |v_m + \phi|^p)|\phi'|dx\\ &\le C\left\{\int_\Omega |\phi'|dx + \int_\Omega |v_m + \phi|^{p+1}dx + \int_\Omega |\phi'|^{p+1}dx\right\}\\ &\le C + C\int_\Omega |v_m + \phi|^{p+1}dx.\end{aligned} \tag{3.8}$$

At the same time, due to $\frac{q}{q+1}+\frac{1}{q+1}=1$, we obtain

$$\tag{3.9} \begin{aligned} |I_4| &= |(g(v_m'+\phi'),\phi')| \\ &\le \varepsilon \int_{\Gamma_0} g(v_m'+\phi')^{\frac{q+1}{q}} d\Gamma + C(\varepsilon)\int_{\Gamma_0} |\phi'|^{q+1} d\Gamma, \end{aligned}$$

where ε is an arbitrary positive constant.

Also, from (2.7), we deduce

$$\tag{3.10} |g(s)|^{\frac{q+1}{q}} = |g(s)||g(s)|^{\frac{1}{q}} \le C_{11}|g(s)||s|,\ |s|>1.$$

Owing to Poincaré inequality, we conclude

$$\tag{3.11} \|v\|_{\Gamma_0} \le C\|\nabla v\|,\ \forall v\in V.$$

From Cauchy-Schwarz's inequality, we have

$$\tag{3.12} |I_2| = |(\mathcal{G}',v_m)_{\Gamma_0}| \le C\{\|\mathcal{G}'\|_{\Gamma_0}^2+\|\nabla v_m\|^2\}.$$

Combining (3.7)-(3.12), it follows that

$$\tag{3.13} \begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{\|v_m'\|^2+\overline{M(\|\nabla v+\nabla\phi\|^2)}+2\int_\Omega F(v_m+\phi)dx\right\} \\ &\quad+(1-\varepsilon)\int_{|v_m'+\phi'|>1} |g(v_m'+\phi')|^{\frac{q+1}{q}} d\Gamma \\ &\le C(\varepsilon)+\|\mathcal{F}\|^2+\|\mathcal{G}'\|_{\Gamma_0}^2+\frac{d}{dt}I_1(t) \\ &\quad+C\left\{\int_\Omega |v_m+\phi|^{p+1}dx+\|v_m'\|^2+\|\nabla v_m\|^2\right\}+I_5(t)+I_6(t)+I_7(t). \end{aligned}$$

Now we pay attention to estimating the terms $I_1(t),\ I_5(t),\ I_6(t),\ I_7(t)$ in (3.13) as follows.

Due to (3.11), we have

$$\tag{3.14} |I_1| = |(\mathcal{G},v_m)_{\Gamma_0}| \le \frac{C^2}{4\varepsilon}\|\mathcal{G}\|_{\Gamma_0}^2+\varepsilon\|\nabla v_m\|^2.$$

From (A3) and Young's inequality,we get

$$\tag{3.15} |I_5| \le C\|\nabla v_m\|^2+C,$$

$$\tag{3.16} |I_6| \le C\|\nabla v_m'\|^2+C,$$

$$\tag{3.17} |I_7| \le C.$$

From (3.13)-(3.17), choosing $\varepsilon>0$ small enough and using Gronwall's inequality and integrating (3.13) over $(0,t)$, observing that $v_m(0)=v_m'(0)=0$, and taking (2.3) into account, it follows that

$$\tag{3.18} \|v_m'\|^2+\overline{M(\|\nabla v+\nabla\phi\|^2)}+\int_\Omega F(v_m+\phi)dx+\int_0^t\int_{\Gamma_0} |g(v_m'+\phi')|^{\frac{q+1}{q}} d\Gamma ds \le C.$$

Second Estimate Taking $w = v''_m(0)$ in (3.6) and noticing that $v_m(0) = v'_m(0) = 0$, we obtain

$$\begin{aligned}&\|v''_m(0)\|^2 + (g(u^1), v''_m(0))_{\Gamma_0} + (f(u^0), v''_m(0))\\&= (\overline{M(\|u^0\|^2)}\triangle u^0 - \alpha\phi_t, v''_m(0)) + (-\frac{\partial u^1}{\partial \nu}, v''_m(0))_{\Gamma_0}.\end{aligned} \tag{3.19}$$

From (A3), (2.12) and (3.19), we obtain

$$\|v''_m(0)\|^2 \leq (\|f(u^0)\| + \|\triangle u^0(0)\|)\|v''_m(0)\|.$$

Owing to (A4), we deduce that

$$\|v''_m(0)\| \leq C. \tag{3.20}$$

Also, taking the derivative of (3.6) with respect to time t and taking $w = v''_m(t)$, we get

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\|v''_m\|^2 + \overline{M(\|\nabla v + \nabla\phi\|^2)}\right\} + \int_{\Gamma_0} g(v'_m + \phi')(v''_m)^2 d\Gamma\\&+ \int_\Omega f'(v_m + \phi)(v'_m + \phi')v''_m d\Gamma \leq (\mathcal{F}', v''_m) + \frac{d}{dt}(\mathcal{G}', v'_m)_{\Gamma_0}.\end{aligned} \tag{3.21}$$

Next, we shall estimate some terms of (3.21).

Firstly, we estimate $I_8 = \int_\Omega f'(v_m + \phi)(v'_m + \phi')v''_m d\Gamma$.

Owing to (2.2), we get

$$|I_8| \leq C\int_\Omega (1 + |v_m + \phi|^{p-1})|v'_m + \phi'||v''_m|dx. \tag{3.22}$$

Noticing that $\frac{p-1}{2p} + \frac{1}{2p} + \frac{1}{2} = 1$, from (3.21) and applying the generalized Hölder's inequality, we obtain

$$|I_8| \leq C(\|v'_m + \phi'\|^2 + \|v''_m\|^2) + C(\|v_m + \phi\|^{p-1}_{\frac{2p}{p-1}}\|v'_m + \phi'\|_{2p}\|v''_m\|). \tag{3.23}$$

Since $p \leq \frac{n}{n-2}$ and Sobolev embedding theory (see [2]), we get

$$H^1_0(\Omega) \hookrightarrow L^{2p}(\Omega). \tag{3.24}$$

From (3.23) and (3.24), we obtain that

$$|I_8| \leq C(1 + \|v''_m\|^2 + \|\nabla v'_m\|^2). \tag{3.25}$$

Secondly, we estimate $I_9 = \int_{\Gamma_0} g(v'_m + \phi')(v''_m)^2 d\Gamma$.

Noticing that $g'(s) \geq 0$, we deduce that $I_9 \geq 0$.

Combining (3.21), (3.23) and (3.25), we get

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\|v''_m\|^2 + \overline{M(\|\nabla v + \nabla\phi\|^2)}\right\}\\&\leq \|\mathcal{F}'\|^2 + \frac{d}{dt}(\mathcal{G}', v'_m)_{\Gamma_0} + C(1 + \|v''_m\|^2 + \|\nabla v'_m\|^2).\end{aligned} \tag{3.26}$$

Integrating (3.26) over the interval $(0,t)$, and noticing that (3.20) and $v'_m(0) = 0$, it follows that

$$\|v''_m\|^2+\overline{M(\|\nabla v+\nabla\phi\|^2)} \leq C+(\mathcal{G}',v'_m)_{\Gamma_0}+C\int_0^t(\|v''_m\|^2+\|\nabla v'_m\|^2)ds. \tag{3.27}$$

Also, by Young's inequality, we have

$$(\mathcal{G}',v'_m)_{\Gamma_0} \leq \frac{C^2}{4\varepsilon}\|\mathcal{G}'\|^2_{\Gamma_0}+\varepsilon\|\nabla v'_m\|^2. \tag{3.28}$$

From (3.27), (3.28) and (A3), choosing $\varepsilon > 0$ sufficiently small enough and applying Gronwall's inequality, we get

$$\|v''_m\|^2+\|\nabla v'_m\|^2 \leq C. \tag{3.29}$$

Next, we shall prove existence of solutions and analyze the nonlinear terms f, g, h.

Analysis of f:

From (2.4) and the First Estimate, we have

$$\int_0^t\int_\Omega f(v_m+\phi)^{\frac{p+1}{p}}dxds \leq C\int_0^t\int_\Omega(1+|v_m+\phi|^{\frac{p+1}{p}})dxds \leq C. \tag{3.30}$$

The inequality (3.30) implies that

$$\{f(v_m+\phi)\} \text{ is bounded in } L^{\frac{p+1}{p}}(Q_T), \tag{3.31}$$

where $Q_T = \Omega \times (0,T)$.

From (3.29), we can find a subsequence v_m (still denote v_m) such that

$$v_m \to v \text{ strong in } L^2(Q_T), \tag{3.32}$$

then, $v_m \to v$ a.e in Q_T. Hence, we get

$$f(v_m+\phi) \to f(v+\phi) \text{ a.e in } Q_T. \tag{3.33}$$

Due to (3.31), (3.33) and applying Lions lemma (see [16]), we deduce

$$f(v_m+\phi) \to f(v+\phi) \text{ weakly in } L^{\frac{p+1}{p}}(Q_T). \tag{3.34}$$

Analysis of g:

From the First Estimate, we obtain

$$\{g(v'_m+\phi')\} \text{ is bounded in } L^{\frac{q+1}{q}}(\Sigma), \tag{3.35}$$

where $\Sigma = \Gamma_0 \times (0,T)$. So, there exists $\varrho \in L^{\frac{q+1}{q}}(\Sigma)$, such that

$$g(v'_m+\phi') \to \varrho \text{ weakly in } L^{\frac{q+1}{q}}(\Sigma). \tag{3.36}$$

From the First and Second Estimate, and Sobolev embedding theory, it is easy to see that

$$v'_m \to v \text{ in } L^2(0,\infty,L^2(\Omega)).$$

We also notice that

$$\begin{aligned}\int_{\Gamma_0}|g(v_m'+\phi')|^2d\Gamma &= \int_{|v_m'+\phi'|\le 1}|g(v_m'+\phi')|^2d\Gamma+\int_{|v_m'+\phi'|>1}|g(v_m'+\phi')|^2d\Gamma\\ &\le C+C\|v_m'+\phi'\|_{2q,\Gamma_0}^{2q}\\ &\le C+C\|\nabla v_m'+\nabla\phi'\|^{2q}\\ &\le C.\end{aligned}$$

Then, we deduce that

$$g(v_m'+\phi')\to\varrho \text{ weakly in } L^2(\Sigma). \tag{3.37}$$

Analysis of $M(\|\nabla u\|^2)$:

From now on we are interested in the convergence of the nonlinear term.

We define

$$\psi_m(t)=\|\nabla v_m(t)\|^2,\ \ t\in[0,T]. \tag{3.38}$$

From the second estimate we obtain

$$|\psi_m(t)|\le C,\ \ \forall m\in N,\ \ t\in[0,T]. \tag{3.39}$$

Now, if $t_1,t_2\in[0,T]$, we can get

$$|\psi_m(t_1)-\psi_m(t_2)|\le\int_{t_1}^{t_2}|\psi_m'(s)|ds. \tag{3.40}$$

On the other hand, from the above estimates we deduce

$$\psi_m'(s)=2(\nabla v_m(s),\nabla v_m'(s))\le C. \tag{3.41}$$

From (3.40) and (3.41), it follows that

$$|\psi_m(t_1)-\psi_m(t_2)|\le C|t_1-t_2|. \tag{3.42}$$

Then, owing to (3.39) and (3.42) by Arezela-Ascoli's theorem [29], there exists a continuous function $\psi:[0,T]\to R$ such that

$$\psi_m(t)\to\psi(t) \tag{3.43}$$

uniformly in $[0,T]$ and, since (A3), we have

$$M(|\psi_m(t)|)\to M(|\psi(t)|) \tag{3.44}$$

uniformly in $[0,T]$.

Also, we note that from the first and second estimates and noticing that $\|v\|_{H^{\frac12}(\Gamma_0)}\le C|\nabla v|$ for all $v\in V$, we get

$$\begin{aligned}&\{v_m\} \text{ is bounded in } L^2(0,T;H^{\frac12}(\Gamma_0)),\\ &\{v_m'\} \text{ is bounded in } L^2(0,T;L^2(\Gamma_0)).\end{aligned} \tag{3.45}$$

Because of the injection $H^{\frac{1}{2}}(\Gamma_0) \hookrightarrow L^2(\Gamma_0)$ is continuous and compact, making use of Aubin-Lions theorem [17], there exists a subsequence of v_m (still denote v_m) such that

$$v_m \to v \text{ strongly in } L^2(0,T;L^2(\Gamma_0)). \tag{3.46}$$

From **Analysis of f** and above convergence, we get

$$v_m'' \to v'' \text{ weakly in } L^2(0,T,L^2(\Omega)). \tag{3.47}$$

So, we can pass to the limit in the approximate system given by (3.6) using standard arguments in order to obtain

$$u_{tt} - M(\|\nabla u\|^2) \triangle u + \alpha u_t + f(u) = 0 \text{ in } L^2(0,T,L^2(\Omega)). \tag{3.48}$$

Also, we have

$$\frac{\partial v}{\partial \nu} + \varrho = \mathcal{G} \text{ in } L^2(0,\infty,L^2(\Gamma_0)). \tag{3.49}$$

Indeed, we consider $w = v_m$ in (3.6) and integrate over $(0,T)$, then we get

$$\begin{aligned} &\int_0^T (v_m'', v_m)dt + \int_0^T M(\|\nabla v + \nabla\phi\|^2)|\nabla v_m|^2 dt + \alpha\int_0^T (v_m', v_m)dt \\ &+ \int_0^T (f(v_m+\phi), v_m)dt \\ = &\int_0^T (\mathcal{F}, v_m)dt + \int_0^T (\mathcal{G}, v_m)_{\Gamma_0} dt. \end{aligned} \tag{3.50}$$

Combining above estimate and convergence, we deduce that

$$v_m'' \to v'' \text{ weakly in } L^2(0,T,L^2(\Omega)). \tag{3.51}$$

Hence, we can pass to the limit in (3.50) to deduce that

$$v'' - M(\|\nabla v + \nabla\phi\|^2)\triangle v + \alpha v' + f(v+\phi) = \mathcal{F} \in L^2(0,\infty;H).$$

Next, we shall show $\varrho = g(v+\phi')$. For this goal, we shall use monotonicity arguments.

First of all, we notice that from the First and Second estimates and applying Aubin-Lions Theorem (see [17]), then we have

$$v_m' \to v' \text{ weakly in } L^2(0,T;H). \tag{3.52}$$

Considering $w = v_m'$ in (3.6), integrating over $[0,T]$, we deduce

$$v'' - M(\|\nabla v + \nabla\phi\|^2)\triangle v + \alpha v' + f(v+\phi) = \mathcal{F} \in L^2(0,\infty;H),$$

$$\frac{\partial v}{\partial \nu} + \varrho = \mathcal{G} \in L^2(0,\infty;L^2(\Gamma_0)).$$

Also, we have

$$\lim_{m\to\infty} \int_0^T (g(v_m'+\phi'), v_m'+\phi')dt = \int_0^T (\varrho, v'+\phi')_{\Gamma_0} dt. \tag{3.53}$$

Owing to function g is a non-decreasing monotone function, we have

$$\int_0^T \langle g(v'_m+\phi')-g(\psi), v'_m+\phi'-\psi\rangle dt \geq 0,\ \forall\psi\in L^{q+1}(\Gamma_0),$$

where $\langle\cdot,\cdot\rangle$ means the duality between Sobolev spaces $L^{\frac{q+1}{q}}(\Gamma_0)$ and $L^{q+1}(\Gamma_0)$. So, we deduce

$$\begin{aligned}&\int_0^T \langle g(v'_m+\phi'),\psi\rangle dt+\int_0^T\langle g(\psi), v'_m+\phi'-\psi\rangle dt\\ \leq\ &\int_0^T\langle g(v'_m+\phi'), v'_m+\phi'\rangle dt.\end{aligned}\tag{3.54}$$

From (3.54), we get

$$\begin{aligned}&\lim_{m\to\infty}\inf\int_0^T \langle g(v'_m+\phi'),\psi\rangle dt+\lim_{m\to\infty}\inf\int_0^T\langle g(\psi), v'_m+\phi'-\psi\rangle dt\\ \leq\ &\lim_{m\to\infty}\inf\int_0^T\langle g(v'_m+\phi'), v'_m+\phi'\rangle dt.\end{aligned}\tag{3.55}$$

Since

$$\|v_m\|_{q+1,\Gamma_0}\leq C\|\nabla v'_m\|\leq C,$$

then, it follows that

$$v'_m\to v' \text{ weakly star in } L^\infty(0,T;L^{q+1}(\Gamma_0)).\tag{3.56}$$

From (3.51), (3.53), (3.55) and (3.56), we obtain

$$\int_0^T\langle\varrho-g(\psi), v'_m+\phi'-\psi\rangle dt\geq 0.\tag{3.57}$$

Finally, we apply the monotone method to obtain $\varrho=g(v'+\phi')$.

Setting $\psi=(v'+\phi')+\lambda\xi$ in (3.57), where ξ is an arbitrary element of $L^{q+1}(\Gamma_0)$ and $\lambda>0$, we obtain

$$\int_0^T\langle\varrho-g(v'+\phi'+\lambda\xi),-\lambda\xi\rangle dt\geq 0.$$

So,

$$\int_0^T\langle\varrho-g(v'+\phi'+\lambda\xi),-\xi\rangle dt\leq 0,\ \forall\xi\in L^{q+1}(\Gamma_0).$$

As for the operator

$$g: L^{q+1}(\Gamma_0)\to L^{\frac{q+1}{q}}(\Gamma_0): v|\to g(v)$$

is hemi-continuous and we get

$$\int_0^T\langle\varrho-g(v'+\phi'),\xi\rangle dt\leq 0,\ \forall\xi\in L^{q+1}(\Gamma_0).$$

Also, we have

$$\int_0^T \langle \varrho - g(v' + \phi'), \xi \rangle dt \geq 0, \ \forall \xi \in L^{q+1}(\Gamma_0).$$

Hence,

$$\int_0^T \langle \varrho - g(v' + \phi'), \xi \rangle dt = 0, \ \forall \xi \in L^{q+1}(\Gamma_0),$$

which implies

$$\varrho = g(v' + \phi'). \tag{3.58}$$

Uniqueness:

Let u_1 and u_2 be two smooth solutions to problem (1.1), then, $z = u_1 - u_2$ satisfies

(3.59)
$$\begin{aligned}&(z'', w) + M(\|\nabla u_2\|^2)(\nabla z, \nabla w) + (M(\|\nabla u_2\|^2) - M(\|\nabla u_1\|^2))(\nabla u_1, \nabla w)\\ &+ (\alpha z', w) + (g(u_1') - g(u_2'), w)_{\Gamma_0} = (f(u_2) - f(u_1), w), \ \forall w \in V,\\ &z(0) = z'(0) = 0.\end{aligned}$$

Putting $w = z'(t)$ in (3.59), we deduce

(3.60)
$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\{\|z'\|^2 + M(\|\nabla u_2\|^2)(\nabla z, \nabla z') + (\alpha z', z') + (g(u_1') - g(u_2'), z')_{\Gamma_0}\\ &= (f(u_2) - f(u_1), w) + (M(\|\nabla u_1\|^2) - M(\|\nabla u_2\|^2))(\nabla u_1, \nabla z').\end{aligned}$$

Next, we estimate the terms in the right hand side of (3.55).

Estimate for $J_1 = (f(u_2) - f(u_1), z')$

From (2.5), we get

$$\begin{aligned}|J_1| &\leq C \int_\Omega (|u_2|^{p-1} + |u_1|^{p-1})|z||z'|dx\\ &\leq C(\|u_2\|_{2p} + \|u_1\|_{2p})\|z\|_{2p}\|z'\|\\ &\leq C(\|\nabla z\|^2 + \|z'\|^2).\end{aligned} \tag{3.61}$$

Estimate for $J_2 = |M(\|\nabla u_1\|^2) - M(\|\nabla u_2\|^2)|$

Since M is C^1, we get

$$\begin{aligned}|J_2| &\leq C|\|\nabla u_1\|^2 - \|\nabla u_2\|^2|\\ &\leq C(\|\nabla u_1\| + \|\nabla u_2\|)(\|\nabla u_1\| - \|\nabla u_2\|)\\ &\leq C(\|\nabla z\|^2 + \|z'\|^2).\end{aligned} \tag{3.62}$$

From (A4), (3.60)-(3.62), observing that g is monotone function and making using of Gronwall's inequality, we deduce $\|\nabla z\| = \|z'\| = 0$ and so, $u_2 = u_1$.

Then, we complete the proof of existence and uniqueness of smooth solutions.

Existence of Weak Solutions

Consider

$$\{u^0, u^1\} \in V \times H \tag{3.63}$$

and set

$$D(-\triangle) = \{v \in V \cap H^2(\Omega); \tfrac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma_0\}.$$

Due to Sobolev space theory (see [4]), we get

$$D(-\triangle) \text{ is dense in } V \text{ and } H_0^1(\Omega) \cap H^2(\Omega) \text{ is dense in } H,$$

then, there exist $\{u_\mu^0\} \subset D(-\triangle)$ and $\{u_\mu^1\} \subset H_0^1(\Omega) \cap H^2(\Omega)$, such that

$$u_\mu^0 \to u^0 \text{ strongly in } V, \tag{3.64}$$

$$u_\mu^1 \to u^1 \text{ strongly in } H. \tag{3.65}$$

Furthermore, $\frac{\partial u^0}{\partial \nu} + g(u_\mu^1) = 0$ on Γ_0. Therefore, there exists $u_\mu : Q \to R$ smooth solution of problem (1.1) satisfying

$$\begin{cases} u_\mu'' - M(\|\nabla u_\mu\|^2)\triangle u_\mu + \alpha u_\mu' + f(u_\mu) = 0 \text{ in } L^2([0,\infty; H), \\ u_\mu = 0 \text{ on } \Gamma_1, \\ \frac{\partial u_\mu}{\partial \nu} + g(u_\mu') = 0 \text{ on } L^2([0,\infty; L^2(\Gamma_0)), \\ u_\mu(0) = u_\mu^0,\ u_\mu'(0) = u_\mu^1. \end{cases} \tag{3.66}$$

Repeating the same discussions, we get

$$\|u_\mu'\|^2 + \overline{M(\|\nabla u_\mu\|^2)} + \int_0^t \int_\Omega |u_\mu|^{p+1} dx dt + \int_0^t \int_{\Gamma_0} |g(u_\mu')|^{\frac{q+1}{q}} d\Gamma ds \le C, \tag{3.67}$$

and

$$\int_0^t \int_\Omega |f(u_\mu)|^{\frac{p+1}{p}} dx ds \le C, \tag{3.68}$$

$$\int_0^t \int_{\Gamma_0} |u_\mu'|^{q+1} d\Gamma ds \le C, \tag{3.69}$$

for all $t \in [0, T]$ and $\mu \in N$.

Setting $z_{\mu\sigma} = u_\mu - u_\sigma, \mu, \sigma \in N$, where u_μ, u_σ are smooth solutions of (3.66).

Repeating the same discussions used in the existence and uniqueness of strong solutions, we deduce that there exists $u : Q \to R$ such that

$$u_\mu \to u \text{ in } L^\infty([0,T]; V), \tag{3.70}$$

$$u_\mu' \to u' \text{ in } L^\infty([0,T]; H). \tag{3.71}$$

Furthermore, from (3.67), (3.70) and (2.6), we get

$$u_\mu' \to u' \text{ weakly in } L^{q+1}(\Sigma), \tag{3.72}$$

$$f(u_\mu) \to \eta \text{ weakly in } L^{\frac{p+1}{p}}(Q_T), \tag{3.73}$$

$$g(u_\mu') \to \chi \text{ weakly in } L^{\frac{q+1}{q}}(\Sigma). \tag{3.74}$$

From (3.66) and Lion's Lemma (see [17]), it is easy to obtain that $\eta = f(u)$.

Furthermore, we have

$$\begin{cases} u_{tt} - M(\|\nabla u\|^2) \triangle u + \alpha u_t + f(u) = 0, \\ u(x,0) = u_0, \;\; u_t(x,0) = u_1. \end{cases} \tag{3.75}$$

Next, we shall prove $\chi = g(u')$.

Indeed, multiplying the first equation (3.66) by u_μ and integrating over Ω, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\{\|u'_\mu\|^2 + \overline{M(\|\nabla u_\mu\|^2)}\} + \alpha\|u'_\mu\|^2 \\ &+ (f(u_\mu), u'_\mu) + (g(u'_\mu), u'_\mu)_{\Gamma_0} = 0. \end{aligned}$$

Also, we have

$$\begin{aligned} &\frac{1}{2}\|u'_\mu\|^2 + \frac{1}{2}\overline{M(\|\nabla u_\mu\|^2)} + \alpha\int_0^t \|u'_\mu\|^2 + \int_0^t (f(u_\mu(s)), u'_\mu(s))ds \\ &+ \int_0^t (g(u'_\mu(s)), u'_\mu(s))_{\Gamma_0} ds \\ = &\frac{1}{2}\|u^1_\mu\|^2 + \frac{1}{2}\|\nabla u^0_\mu\|^2 + \frac{1}{2}\overline{M(\|\nabla u^0_\mu\|^2)}. \end{aligned} \tag{3.76}$$

From (3.65), (3.66), (3.68), (3.69), (3.71), (3.73) and (3.76), we deduce

$$\begin{aligned} &\lim_{\mu\to\infty}\int_0^t (g(u'_\mu(s)), u'_\mu(s))_{\Gamma_0} ds \\ = &\frac{1}{2}\|u^1\|^2 + \frac{1}{2}\|\nabla u^0\|^2 - \frac{1}{2}\|u'\|^2 - \frac{1}{2}\overline{M(\|\nabla u\|^2)} + \alpha\|u'\|^2 \\ &- \int_0^t (f(u(s)), u'(s))ds + \frac{1}{2}\overline{M(\|\nabla u^0_\mu\|^2)}. \end{aligned} \tag{3.77}$$

At the same time, assuming that ω is a weak solution to problem

$$\begin{cases} \omega'' - M(\|\nabla\omega\|^2)\triangle\omega + \alpha\omega' + f(\omega) = 0 \text{ in } \Omega\times(0,\infty), \\ \omega = 0 \text{ on } \Gamma_1\times(0,\infty), \\ \frac{\partial\omega}{\partial\nu} + \chi = 0 \text{ on } \Gamma_0\times(0,\infty), \\ \omega(0) = u^0, \; \omega'(0) = u^1, \end{cases} \tag{3.78}$$

then, we have

$$\begin{aligned} &\int_0^t (\chi(s)), \omega'(s))_{\Gamma_0} ds \\ = &\frac{1}{2}\|u^1\|^2 + \frac{1}{2}\|\nabla u^0\|^2 - \frac{1}{2}\|\omega'\|^2 + \overline{M(\|\nabla\omega\|^2)} - \alpha\|\omega'\|^2 \\ &- \int_0^t (f(\omega(s)), \omega'(s))ds. \end{aligned} \tag{3.79}$$

Since u is a weak solution to problem (3.78), hence, from (3.77) and (3.79), we get

$$\lim_{\mu\to\infty}\int_0^t (g(u'_\mu(s)), u'_\mu(s))_{\Gamma_0} ds = \int_0^t (\chi(s)), u'(s))_{\Gamma_0} ds.$$

According to the above arguments, it is easy to show that $\chi = g(u')$.

Remark 3.1. For the uniqueness of weak solutions, we require a regularization procedure using standard arguments (see [17]).

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