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VALUE DISTRIBUTION OF DIFFERENCE OPERATOR ON MEROMORPHIC FUNCTIONS

Jie Ding and Lian-Zhong Yang

Abstract. In this paper, we investigate the value distribution of the difference operator on meromorphic functions, and obtain a difference analogue of a theorem of Hayman on meromorphic functions.

1. Introduction

A function $f(z)$ is called meromorphic, if it is analytic in the complex plane except at isolated poles. It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory such as the characteristic function $T(r,f)$, proximity function $m(r,f)$, counting function $N(r,f)$, the first and second main theorem etc., (see [11], [15], [23]). In addition we use $\sigma(f)$ to denote the order of growth of the meromorphic $f(z)$ and $\lambda(f)$ to denote the exponent of convergence of the zeros of $f(z)$. The notation $S(r,f)$ denotes any quantity that satisfies the condition: $S(r,f)=o(T(r,f))$ as $r\to\infty$ possibly outside an exceptional set of r of finite logarithmic measure. Let $a(z)$ be a meromorphic function, we say that $a(z)$ is a small function of $f(z)$, if $T(r,a)=S(r,f)$.

In 1959, Hayman proved the following theorems.

Theorem A (see [10], Theorem 8)**.** *Let f be a nonconstant transcendental entire function, let $n\geq 3$ be an integer, and a be a non-zero constant. Then $f'(z)-af^n(z)$ assumes all finite values infinitely often.*

Theorem B (see [10], Theorem 9)**.** *Let f be a nonconstant transcendental meromorphic function, let $n\geq 5$ be an integer, and a be a non-zero constant. Then $f'(z)-af^n(z)$ assumes all finite values infinitely often.*

Recently, there have been significant results on Nevanlinna theory with respect to difference operators, see, the papers [7, 8] by Halburd and Korhonen;

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[5] by Chiang and Feng. Many papers (see [2], [4], [9], [11-20]) have focused on complex differences and given many difference analogues in value distribution theory of entire functions.

Liu and Laine [18] obtain the following theorem.

Theorem C (see [18], Theorem 1.1)**.** *Let f be a transcendental entire function of finite order, not of period c, where c is a non-zero constant, and let $s(z)$ be a nonzero small function of f. Then the difference polynomial $f^n(z) + f(z + c) - f(z) - s(z)$ has infinitely many zeros in the complex plane, provided that $n \geq 3$.*

In [3], Chen considered the difference counterpart of Theorem A and proved a difference analogue of Hayman's Theorem on entire function.

Theorem D (see [3], Theorem 1.1)**.** *Let f be a transcendental entire function of finite order, not of period c, and let $a(\neq 0)$, b, $c(\neq 0)$ be three complex numbers. Then $\Psi_n(z) = f(z+c) - f(z) - af^n(z)$ assumes all finite values infinitely often, provided that $n \geq 3$, and for every b one has $\lambda(\Psi_n(z) - b) = \sigma(f)$.*

Qi and Liu [22] investigated the existence of transcendental entire solutions of non-linear difference equations. As an application, they obtained the following result.

Theorem E (see [22], Theorem 2)**.** *Let f be a transcendental entire function of finite order, c be a non-zero constant, m and n be integers satisfying $n \geq m > 0$, and let λ, μ be two complex numbers such that $|\lambda| + |\mu| \neq 0$. If $n \geq 2$, then either $f^n(z)(\lambda f^m(z+c) + \mu f^m(z))$ assumes every non-zero finite values infinitely often or $f(z) = \exp\{\frac{\log t}{c} z\} g(z)$, where $t = (-\frac{\mu}{\lambda})^{\frac{1}{m}}$, and $g(z)$ is a periodic function with period c.*

The purpose of this paper is to study value distribution of meromorphic function with respect to differences. Our methods of proof are also different from those in the previous theorems.

Theorem 1.1. *Let f be a non-constant meromorphic function of finite order, $s(z)$ be a small function of $f(z)$. Suppose that $P(z)$ is a polynomial, m is the cardinality of the set $\{z : P(z) = 0\}$ and $\deg(P(z)) - m > 3$, then $P(f(z)) + f(z + c) - s(z)$ has at least one zero. If f is a transcendental meromorphic function, then $P(f(z)) + f(z + c) - s(z)$ has infinitely many zeros.*

Zhang and Korhonen [24] researched the value distribution of q-difference of meromorphic function. Corresponding to Theorem 1.1, we give an analogue result in q-differences as follows.

Theorem 1.2. *Let f be a meromorphic function of zero order, $q \in \mathbb{C} \setminus \{0\}$, $s(z)$ and $P(z)$ satisfy the condition of Theorem 1.1. Then $P(f(z)) + f(qz) - s(z)$ has at least one zero. If f is a transcendental meromorphic function, then $P(f(z)) + f(qz) - s(z)$ has infinitely many zeros.*

Remark 1. Some ideas of this paper are based on [6].

2. Some lemmas

In order to prove our theorem, we need the following lemmas:

Lemma 2.1 is a difference analogue of the logarithmic derivative lemma, given by Halburd and Korhonen [8] and Chiang and Feng [5] independently.

Lemma 2.1 (see [8], Theorem 2.1). *Let $f(z)$ be a meromorphic function of finite order, and let $c \in \mathbb{C}$ and $\delta \in (0,1)$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O\left(\frac{T(r,f)}{r^{\delta}}\right) = S(r,f)$$

for all r outside of a possibly exceptional set with finite logarithmic measure.

Lemma 2.2 (see [23], Theorem 1.12). *Let $f(z)$ be a non-constant meromorphic function, and let $P(f) = a_0 f^n + a_1 f^{n-1} + \cdots + a_n$, where $a_0(\neq 0), a_1, \ldots, a_n$ are small functions of f. Then*

$$T(r, P(f)) = nT(r,f) + S(r,f).$$

Lemma 2.3 (see [1], Theorem 1.1). *Let f be a non-constant zero-order meromorphic function, and $q \in \mathbb{C}\backslash\{0\}$. Then*

$$m(r, \frac{f(qz)}{f(z)}) = S(r, f(z)).$$

Lemma 2.4 (see [24], Theorem 1.3). *Let f be a non-constant zero-order meromorphic function, and $q \in \mathbb{C}\backslash\{0\}$. Then*

$$N(r, f(qz)) = N(r, f(z)) + S(r, f(z))$$

on a set of lower logarithmic density 1.

Lemma 2.5. *Let $f(z)$ be a meromorphic function of finite order, $c \in \mathbb{C}$. Then*

$$N(r, f(z+c)) = N(r, f(z)) + S(r, f(z)).$$

Proof. Using Lemma 2.1 and the formula (12) in [13]

$$N(r, f(z+c)) \leq N(r+|c|, f) = N(r, f(z)) + S(r, f(z)). \tag{2.1}$$

Replacing $f(z)$ with $f(z-c)$, we have

$$N(r, f(z)) \leq N(r, f(z-c)) + S(r, f(z-c)) = N(r, f(z-c)) + S(r, f(z))$$

for every $c \in \mathbb{C}$, so we deduce that

$$N(r, f(z)) \leq N(r, f(z+c)) + S(r, f(z)). \tag{2.2}$$

From (2.1) and (2.2), we obtain that

$$N(r, f(z+c)) = N(r, f(z)) + S(r, f(z)).$$

Thus we completed the proof. □

3. Proof of theorems

Proof of Theorem 1.1. We define $\Phi(z) = P(f) + f(z+c) - s(z)$. Obviously, $\Phi(z) \not\equiv C$. If it is false, then $P(f) \equiv f(z+c) - s(z) + C$, so we have that

$$T(r, P(f)) = nT(r, f) + S(r, f) = T(r, f(z+c)) + S(r, f), \tag{3.1}$$

where $n = \deg(P) > 3$. Using Lemma 2.1 and Lemma 2.5, we deduce that

$$\begin{aligned} T(r, f(z+c)) &= m(r, f(z+c)) + N(r, f(z+c)) \\ &\leq m(r, f(z)) + m(r, \frac{f(z+c)}{f(z)}) + N(r, f(z)) + S(r, f(z)) \\ &= T(r, f(z)) + S(r, f(z)). \end{aligned} \tag{3.2}$$

The (3.1) and (3.2) imply $T(r, f(z)) = S(r, f(z))$, a contradiction, Therefore $\Phi(z) \not\equiv C$.

Furthermore, we claim that

$$\frac{P'(f)f'}{P(f)} - \frac{\Phi'}{\Phi} \not\equiv 0.$$

Suppose contrary to the claim that $\frac{P'(f)f'}{P(f)} - \frac{\Phi'}{\Phi} \equiv 0$. By integration we obtain $\Phi(z) = aP(f(z))$, where a is a constant, hence $(a-1)P(f(z)) = f(z+c) - s(z)$.

If $a = 1$, we can deduce $T(r, f(z+c)) = T(r, s(z))$. This contradicts with the hypothesis.

Let $a \neq 1$. By the similar argument as the Case of $\Phi(z) \equiv C$, we get a contradiction. So the claim is true.

By a simple calculation we get that

$$P(f) = \frac{\frac{\Phi'}{\Phi}[f(z+c) - s(z)] - [f(z+c) - s(z)]'}{\frac{P'(f)f'(z)}{P(f)} - \frac{\Phi'}{\Phi}}. \tag{3.3}$$

From Lemma 2.1, Lemma 2.2 and the First Fundamental Theorem, we obtain that

$$\begin{aligned} T(r, P(f)) &= nT(r, f(z)) + S(r, f) \\ &= T\left(r, \frac{\frac{\Phi'}{\Phi}[f(z+c) - s(z)] - [f(z+c) - s(z)]'}{\frac{P'(f)f'(z)}{P(f)} - \frac{\Phi'}{\Phi}}\right) \\ &\leq m(r, f(z)) + N(r, \frac{\Phi'}{\Phi}[f(z+c) - s(z)] - [f(z+c) - s(z)]') \\ &\quad + m(r, \frac{\Phi'}{\Phi} - \frac{[f(z+c) - s(z)]'}{[f(z+c) - s(z)]}) + m(r, \frac{P'(f)f'(z)}{P(f)} - \frac{\Phi'}{\Phi}) \end{aligned} \tag{3.4}$$

$$+ N(r, \frac{P'(f)f'(z)}{P(f)} - \frac{\Phi'}{\Phi}) + S(r, f(z)).$$

In the following we will estimate $N(r, \frac{\Phi'}{\Phi}[f(z+c)-s(z)]-[f(z+c)-s(z)]')$ and

$$N(r, \frac{P'(f)f'(z)}{P(f)} - \frac{\Phi'}{\Phi}).$$

The poles of $\varphi_1(z) = \frac{\Phi'}{\Phi}[f(z+c)-s(z)]-[f(z+c)-s(z)]'$ come from the zeros of $\Phi(z)$, the poles of $P(f)$, the poles of $f(z+c)$ and the poles of $s(z)$. By hypothesis, $N(r,s) = S(r,f)$. In the following we always assume that z_0 is not a pole of s, let z_0 be a zero of $\Phi(z)$ or a pole of $P(f)$ but not a pole of $f(z+c)$, then z_0 is a simple pole of $\varphi_1(z)$; if z_0 is a common pole of $P(f)$ and $f(z+c)$, and the multiplicity is m and n, respectively, then z_0 is a pole of $\varphi_1(z)$ with the multiplicity no more than $n+1$; if z_0 is a pole of $f(z+c)$ but not a pole of $P(f)$, we obtain that z_0 is a simple pole of $\varphi_1(z)$ because of (3.3). Using Lemma 2.5, we can get that

$$\begin{aligned}
(3.5)\qquad & N(r, \frac{\Phi'}{\Phi}[f(z+c)-s(z)]-[f(z+c)-s(z)]') \\
&\le \overline{N}(r, \frac{1}{\Phi(z)}) + \overline{N}(r, P(f)) + N(r, f(z+c)) + S(r, f(z)) \\
&= \overline{N}(r, \frac{1}{\Phi(z)}) + \overline{N}(r, f(z)) + N(r, f(z)) + S(r, f(z)).
\end{aligned}$$

We deal with the poles of $s(z)$ as above. The zeros of $\Phi(z)$, the poles of $P(f(z))$, the poles of $f(z+c)$ and the zeros of $P(f(z))$ compose the poles of $\varphi_2(z) = \frac{P'(f)f'(z)}{P(f)} - \frac{\Phi'}{\Phi}$. If z_0 is a zero of $\Phi(z)$, zero of $P(f(z))$ or pole of $f(z+c)$, then z_0 is a simple pole of $\varphi_2(z)$; if z_0 is a pole of $P(f(z))$ but not a pole of $f(z+c)$, using the Laurent series, we can get that $\varphi_2(z)$ is analytic at z_0. Therefore, we conclude that

$$\begin{aligned}
(3.6)\qquad & N(r, \frac{P'(f)f'(z)}{P(f)} - \frac{\Phi'}{\Phi}) \\
&\le \overline{N}(r, \frac{1}{\Phi(z)}) + \overline{N}(r, \frac{1}{P(f)}) + \overline{N}(r, f(z+c)) + S(r, f(z)) \\
&= \overline{N}(r, \frac{1}{\Phi(z)}) + \sum_{i=1}^{m} \overline{N}(r, \frac{1}{f(z)-a_i}) + \overline{N}(r, f(z)) + S(r, f(z)),
\end{aligned}$$

where $a_1, a_2, \ldots, a_m$ are the solutions of equation $P(z) = 0$.

Combining (3.4), (3.5) and (3.6), we have that

$$\begin{aligned}
(3.7)\qquad & nT(r, f(z)) \\
&\le 2m(r, \frac{\Phi'}{\Phi}) + m(r, f(z)) + m(r, \frac{f'(z+c)}{f(z+c)}) + \sum_{i=1}^{m} \overline{N}(r, \frac{1}{f(z)-a_i})
\end{aligned}$$

$$+ 2\overline{N}(r, \frac{1}{\Phi(z)}) + 2\overline{N}(r, f(z)) + N(r, f(z)) + S(r, f(z)).$$

From (3.2) and Lemma 2.2, we deduce that $T(r, \Phi(z)) = O(T(r, f(z))$. Therefore, we get that

$$m(r, \frac{f'(z+c)}{f(z+c)}) = S(r, f(z)),\ m(r, \frac{\Phi'}{\Phi}) = S(r, \Phi(z)) = S(r, f(z)). \tag{3.8}$$

By (3.8) and the First Fundamental Theorem, we can simplify (3.7) to be

$$(n-m)T(r, f(z)) \leq 3T(r, f(z)) + 2\overline{N}(r, \frac{1}{\Phi(z)}) + S(r, f(z)). \tag{3.9}$$

Because of $n - m > 3$, we deduce that

$$T(r, f(z)) \leq C\overline{N}(r, \frac{1}{P(f) + f(z+c) - s(z)}) + S(r, f(z)).$$

If $P(f) + f(z+c) - s(z)$ has no zero, then $T(r, f(z)) = S(r, f(z))$, a contradiction.

When f is transcendental, by the same reason, we conclude that $P(f) + f(z+c) - s(z)$ has infinite many zeros.

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We define $\Phi(z) = P(f(z)) + f(qz) - s(z)$. The following process is almost literally the same as the proof of Theorem 1.1, with Lemmas 2.3 and 2.4 replacing Lemmas 2.1 and 2.5.

This completes the proof of Theorem 1.2. □

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Jie Ding
School of Mathematics
Taiyuan University of Technology
Taiyuan, Shanxi, 030024, P. R. China
E-mail address: jieding1984@gmail.com

Lian-Zhong Yang
School of Mathematics
Shandong University
Jinan, Shandong, 250100, P. R. China
E-mail address: lzyang@sdu.edu.cn