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DUALITY OF Q_K-TYPE SPACES

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Abstract. For BMO, it is well known that $VMO^{**} = BMO$. In this paper such duality results of Q_K-type spaces are obtained which generalize the results by M. Pavlović and J. Xiao.

1. Introduction

Let $D = \{z : |z| < 1\}$ be the unit disk of complex plane $\mathbb{C}$ and $H(D)$ denote the class of functions analytic in D. For $a \in D$, $\varphi_a(z) = \frac{a-z}{1-\overline{a}z}$ is the Möbius map of D. Let $K : [0,\infty) \to [0,\infty)$ be a right-continuous and nondecreasing function. For $0 < p < \infty$ and $-2 < q < \infty$, the space $Q_K(p,q)$ consists of all functions $g \in H(D)$ such that

$$\|g\|^p_{Q_K(p,q)} = \sup_{a\in D} \int_D |g'(z)|^p (1-|z|^2)^q K(1-|\varphi_a(z)|^2) dA(z) < \infty,$$

where $dA(z)$ is the Euclidean area element on D. For $p \geq 1$, under the norm $\|g\| = |g(0)| + \|g\|_{Q_K(p,q)}$, $Q_K(p,q)$ is a Banach space. A $g \in H(D)$ is said to belong to $Q_{K,0}(p,q)$ space if $g \in Q_K(p,q)$ satisfying

$$\lim_{|a|\to 1} \int_D |g'(z)|^p (1-|z|^2)^q K(1-|\varphi_a(z)|^2) dA(z) = 0.$$

Spaces $Q_K(p,q)$ and $Q_{K,0}(p,q)$ are first introduced in [7], and it was proved that $Q_K(p,q) \subset B^{\frac{q+2}{p}}$. Setting $K(t) = t^s$, $s \geq 0$, $Q_K(p,q) = F(p,q,s)$. Hence, with different parameters, $Q_K(p,q)$ coincides with many classical function spaces such as $BMOA, Q_p$ and the Hardy space H^2. Note that $Q_K(p,q)$ generalizes the space $Q_K = Q_K(2,0)$ (see [1]).

An important tool in the study of $Q_K(p,q)$ spaces is the auxiliary function φ_K which is defined by

$$\varphi_K(s) = \sup_{0<t\leq 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty.$$

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It is clear to see that $\varphi_K(s)$ is nondecreasing and right-continuous on $(0, \infty)$. If $0 < s < 1$, $\varphi_K(s) < 1$, and if $s \geq 1$, $\varphi_K(s) \geq 1$ by the definition of K. We further assume that

$$\int_0^1 \varphi_K(t) \frac{dt}{t} < \infty \tag{1}$$

and

$$\int_1^\infty \varphi_K(t) \frac{dt}{t^2} < \infty. \tag{2}$$

The conditions (1) and (2) appeared firstly in [2]. We write that $A \lesssim B$ if there is a constant $c > 0$ such that $A \leq cB$. We write $A \approx B$ whenever $A \lesssim B \lesssim A$. We know that (2) implies that $K(2t) \approx K(t)$. We also know that $Q_K(p, q) = Q_{K_1}(p, q)$ for $K_1 = \inf(K(r), K(1))$ (see Theorem 3.1 in [7]) and so the function K can be assumed to be bounded.

2. Preliminaries

In the section, we will now state some preliminary results about the $Q_K(p, q)$ spaces that we will use later.

Lemma 2.1. *Let B_X denote the unit ball of the given Banach space $(X, \|\cdot\|_X)$ and co will denote the compact-open topology. Then $(B_{Q_K(p,q)}, co)$ is compact.*

Proof. By [6], for $g \in Q_K(p, q)$ and all $z \in D$, we have

$$|g(z)| \lesssim C(z)\|g\|_{B^{\frac{q+2}{p}}} \lesssim C(z)\|g\|_{Q_K(p,q)},$$

where

$$C(z) = \begin{cases} 1 & \text{for } p > q + 2, \\ -\log(1 - |z|) & \text{for } p = q + 2, \\ (1 - |z|)^{1 - \frac{q+2}{p}} & \text{for } p < q + 2. \end{cases} \tag{3}$$

Thus $B_{Q_K(p,q)}$ is relatively compact with respect to the compact-open topology by Montel's theorem. If $\{g_n\}$ is a sequence in $B_{Q_K(p,q)}$, we obtain

$$\begin{aligned} &\sup_{a \in D} \int_D \lim_{n\to\infty} |g_n'(z)|^p (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) dA(z) \\ &\leq \inf \lim_{n\to\infty} \|g_n\|_{Q_K(p,q)}^p \leq 1 \end{aligned}$$

by Fatou's Lemma. It follows that $(B_{Q_K(p,q)}, co)$ is co-closed and thus also co-compact. □

Lemma 2.2 ([9])**.** *Let $p \geq 1$ and $g \in Q_K(p, q)$, $g_r(z) = g(rz)$, K satisfies (2). Then $\|g - g_r\|_{Q_K(p,q)} \to 0$ as $r \to 1$ if and only if $g \in Q_{K,0}(p, q)$.*

Proof. See [9] Proposition 2.3.3. □

Lemma 2.3. *For $a \in D$ and $p > 1$, we have*

$$\int_D (1-|z|^2)^{-\frac{q}{p-1}}[K(1-|\varphi_a(z)|^2)]^{-\frac{1}{p-1}}dA(z) \lesssim \int_2^{+\infty} \frac{[\varphi_K(s)]^{\frac{1}{p-1}}}{s^{2-\frac{q}{p-1}}}ds.$$

Proof. Since $1-|\varphi_a(z)|^2 = \frac{(1-|z|^2)(1-|a|^2)}{|1-\overline{a}z|^2}$, it follows from the definition of $\varphi_K(s) = \sup_{0<t\le 1} \frac{K(st)}{K(t)}$ that $K(t) \ge \frac{K(st)}{\varphi_K(s)}$ and thus $[K(t)]^{-\frac{1}{p-1}} \le [\frac{\varphi_K(s)}{K(st)}]^{\frac{1}{p-1}}$. Therefore

$$\begin{aligned}
&\int_D (1-|z|^2)^{-\frac{q}{p-1}}[K(1-|\varphi_a(z)|^2)]^{-\frac{1}{p-1}}dA(z)\\
\le\ &\int_D (1-|z|^2)^{-\frac{q}{p-1}}[\frac{\varphi_K(\frac{|1-\overline{a}z|^2}{1-|z|^2})}{K(1-|a|^2)}]^{\frac{1}{p-1}}dA(z)\\
=\ &[K(1-|a|^2)]^{-\frac{1}{p-1}}\int_D (1-|z|^2)^{-\frac{q}{p-1}}[\varphi_K(\frac{|1-\overline{a}z|^2}{1-|z|^2})]^{\frac{1}{p-1}}dA(z)\\
\lesssim\ &\int_D (1-|z|^2)^{-\frac{q}{p-1}}[\varphi_K(\frac{2}{1-|z|})]^{\frac{1}{p-1}}dA(z)\\
\lesssim\ &\int_2^{+\infty} \frac{[\varphi_K(s)]^{\frac{1}{p-1}}}{s^{2-\frac{q}{p-1}}}ds.
\end{aligned}$$

□

Lemma 2.4 ([8])**.** *Let K satisfy* (1) *and $1 < p < \infty$. Then $g \in Q_K(p,q)$ if and only if $\int_D |g^{(n)}(z)|^p(1-|z|^2)^{np-p+q}K(1-|\varphi_a(z)|^2)dA(z) < \infty$ and $g \in Q_{K,0}(p,q)$ if and only if $\lim_{|a|\to 1}\int_D |g^{(n)}(z)|^p(1-|z|^2)^{np-p+q}K(1-|\varphi_a(z)|^2)dA(z) = 0$, respectively.*

Lemma 2.5 ([10])**.** *For all $z \in D$ and $t < 1$,*

$$\int_0^{2\pi} \frac{d\theta}{|1-ze^{-i\theta}|^t} \lesssim 1.$$

Proof. See [10] Theorem 1.7. □

Lemma 2.6. *If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=1}^{\infty} b_k z^k$, define the invertible linear operator $D^n : (H(D), co) \to (H(D), co)$ by*

$$D^n g(z) = \frac{1}{(n-1)!}\sum_{k=0}^{\infty}\frac{(k+n)!}{k!}b_{k+1}z^k, \quad n \in \mathbb{N},$$

then

$$\int_D \overline{f(z)}g'(z)dA(z) = \int_D \overline{f(z)}D^n g(z)(1-|z|^2)^{n-1}dA(z).$$

Lemma 2.7 ([10])**.** *For $\alpha > -1$, every analytic function in*

$$L^1(D, (1-|z|^2)^{\alpha}dA(z))$$

has the formula

$$f(z) = (\alpha + 1) \int_D f(\omega) \frac{(1 - |\omega|^2)^\alpha}{(1 - z\overline{w})^{\alpha+2}} dA(\omega).$$

Proof. See [10] Corollary 1.5. □

Lemma 2.8 (Riesz-Thorin convexity theorem)**.** *Assume T is a bounded linear operator from L_p to L_p and at the same time from L_q to L_q. Then it is also a bounded operator from L_r to L_r for any r between p and q.*

Now we introduce $R(p,q,K)$ spaces. Let $E_{k,j}$ be the pairwise disjoint sets given by

$$E_{k,j} = \left\{ z \in D : 1 - \frac{1}{2^k} \leq |z| \leq 1 - \frac{1}{2^{k+1}},\ \frac{\pi j}{2^{k+1}} \leq \arg z \leq \frac{\pi(j+1)}{2^{k+1}} \right\},$$

where $k = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots, 2^{k+2} - 1$, so that

$$\bigcup_{k=0}^{\infty} \bigcup_{j=0}^{2^{k+2}-1} E_{k,j} = D.$$

We denote $m = j - 1 + \sum_{i=1}^{k} 2^{i+1}$ so that

$$E_1 = E_{0,0}, \ldots,\ E_4 = E_{0,3},\ E_5 = E_{1,0}, \ldots,\ E_{12} = E_{1,7},\ E_{13} = E_{2,0}, \ldots.$$

Let a_m denote the center of E_m. The $R(p,q,K)$ consists of those functions $f \in H(D)$ for which $f(z) = \sum_{m=1}^{\infty} f_m(z)$, where each $f_m \in H(D)$ and

$$\sum_{m=1}^{\infty} \Big(\int_D |f_m(z)|^{\frac{p}{p-1}} (1 - |z|^2)^{-\frac{q}{p-1}} [K(1 - |\varphi_{a_m}(z)|^2)]^{-\frac{1}{p-1}} dA(z) \Big)^{\frac{p-1}{p}} < \infty.$$

The norm of $R(p,q,K)$ is given by

$$\begin{aligned} &\|f\|_{R(p,q,K)} \\ &= \inf \sum_{m=1}^{\infty} \Big(\int_D |f_m(z)|^{\frac{p}{p-1}} (1 - |z|^2)^{-\frac{q}{p-1}} [K(1 - |\varphi_{a_m}(z)|^2)]^{-\frac{1}{p-1}} dA(z) \Big)^{\frac{p-1}{p}}, \end{aligned}$$

where the infimum is taken over all such representations of f.

Remark 2.9. It is easy to check

$$1 \lesssim \frac{1 - |\varphi_a(z)|^2}{1 - |\varphi_{a_m}(z)|^2} \lesssim 1,\ a \in E_m,\ z \in D.$$

Remark 2.10. For $K(t) = t^s$, $0 < s < \infty$,

$$\begin{aligned} &R(p,q,K) \\ &= R(p,q,s) \\ &= \inf \sum_{m=1}^{\infty} \Big(\int_D |f_m(z)|^{\frac{p}{p-1}} (1 - |z|^2)^{-\frac{q}{p-1}} (1 - |\varphi_{a_m}(z)|^2)^{-\frac{s}{p-1}} dA(z) \Big)^{\frac{p-1}{p}}, \end{aligned}$$

were introduced in [3]. They show $R(p,q,s)$ is the dual of $F_0(p,q,s)$ as well as the predual of $F(p,q,s)$.

Remark 2.11. For $p>1$ and $\int_2^{+\infty}\frac{[\varphi_K(s)]^{\frac{1}{p-1}}}{s^{2-\frac{q}{p-1}}}ds<\infty$, it is easy to see

$$\|\cdot\|_{R(p,q,K)}\lesssim\|\cdot\|_{H_\infty}.$$

Indeed, let $f\in H_\infty$, then the representation of f can be chosen to be f itself, by Lemma 2.3, we get

$$\begin{aligned}&\Big(\int_D|f(z)|^{\frac{p}{p-1}}(1-|z|^2)^{-\frac{q}{p-1}}[K(1-|\varphi_{a_m}(z)|^2)]^{-\frac{1}{p-1}}dA(z)\Big)^{\frac{p-1}{p}}\\ \le{}&\|f\|_{H_\infty}\Big(\int_D(1-|z|^2)^{-\frac{q}{p-1}}[K(1-|\varphi_{a_m}(z)|^2)]^{-\frac{1}{p-1}}dA(z)\Big)^{\frac{p-1}{p}}\\ \lesssim{}&\|f\|_{H_\infty}\Big(\int_2^{+\infty}\frac{[\varphi_K(s)]^{\frac{1}{p-1}}}{s^{2-\frac{q}{p-1}}}ds\Big)^{\frac{p-1}{p}}<\infty\end{aligned}$$

for any center point a_m.

Proposition 2.12. *For $p>1$, if $f\in R(p,q,K)$, then*

$$|f(z)|\le(1-|z|)^{\frac{q+2}{p}-2}\|f\|_{R(p,q,K)}.$$

Proof. Fix $z\in D$. Using the inequality on page 39 in [10], which states that for $f\in H(D)$, $s\in R$ and $t\in(0,\infty)$,

$$(1-|z|^2)^s|f(z)|^t\lesssim\int_D(1-|\omega|^2)^{s-2}|f(\omega)|^tdA(\omega).$$

Let $s=2-\frac{q}{p-1}$, $t=\frac{p}{p-1}$, we obtain

$$|f_m(z)|^{\frac{p}{p-1}}\lesssim(1-|z|)^{\frac{q}{p-1}-2}\int_D|f_m(\omega)|^{\frac{p}{p-1}}(1-|\omega|^2)^{-\frac{q}{p-1}}dA(\omega)$$

and hence

$$\begin{aligned}&(1-|z|)^{2-\frac{q+2}{p}}|f(z)|\\ \le{}&(1-|z|)^{2-\frac{q+2}{p}}\sum_{m=1}^\infty(|f_m(z)|^{\frac{p}{p-1}})^{\frac{p-1}{p}}\\ \lesssim{}&(1-|z|)^{2-\frac{q+2}{p}}\sum_{m=1}^\infty[(1-|z|)^{\frac{q}{p-1}-2}\int_D|f_m(\omega)|^{\frac{p}{p-1}}(1-|\omega|^2)^{-\frac{q}{p-1}}dA(\omega)]^{\frac{p-1}{p}}\\ ={}&\sum_{m=1}^\infty(\int_D|f_m(\omega)|^{\frac{p}{p-1}}(1-|\omega|^2)^{-\frac{q}{p-1}}dA(\omega))^{\frac{p-1}{p}}\\ \lesssim{}&\sum_{m=1}^\infty(\int_D|f_m(\omega)|^{\frac{p}{p-1}}(1-|\omega|^2)^{-\frac{q}{p-1}}|[K(1-|\varphi_{a_m}(z)|^2)]^{-\frac{1}{p-1}}dA(\omega))^{\frac{p-1}{p}}.\end{aligned}$$

Taking infimum over all the representations of f, we finish the proof. □

Remark 2.13. It is easy to see the norm-topology of $R(p,q,K)$ is finer than the compact-open topology by Proposition 2.12. Furthermore we can verify that the normed space $R(p,q,K)$ is complete using the completeness criterion.

3. The $Q_{K,0}(p,q) - R(p,q,K)$ duality

To give the main theorem of this section, we need the following two lemmas and Theorem 3.1.

Lemma 3.1. *Let $g \in H(D)$ be given by $g(z) = \sum_{k=1}^{\infty} b_k z^k$, K satisfy* (1) *and $1 < p < \infty$. Then $f \in Q_K(p,q)$ if and only if*

$$\sup_{a\in D}\Big(\int_D |D^n g(z)|^p (1-|z|^2)^{np-p+q} K(1-|\varphi_a(z)|^2)dA(z)\Big)^{\frac{1}{p}} < \infty,$$

$f \in Q_{K,0}(p,q)$ if and only if

$$\lim_{|a|\to 1}\Big(\int_D |D^n g(z)|^p (1-|z|^2)^{np-p+q} K(1-|\varphi_a(z)|^2)dA(z)\Big)^{\frac{1}{p}} = 0.$$

Proof. By Lemma 2.4 and the fact $D^1 g(z) = g'(z)$

$$D^n g(z) = \frac{1}{(n-1)!}\Big(z^{n-1}g^{(n)}(z) + \sum_{j=1}^{n-2} c_{n,j} z^{n-1-j} g^{(n-j)}(z) + nD^{n-1}(z)\Big)$$

the result follows by induction. □

Lemma 3.2. *For $p > \max\{1, 1+q\}$ and $n \in N$ with $n > 1 + \frac{-q-1}{p}$, we define the linear operator S on the set of Borel measurable function H on D by*

$$S(H)(\omega) = (1-|\omega|^2)^{\gamma}\int_D H(z)\frac{(1-|z|^2)^{n-1-\gamma}}{(1-\overline{z}\omega)^{n+1}}dA(z),$$

where $\omega \in D$ and $\gamma \in (\max\{0, -q-(n-1)(p-1)\}, \min\{n, p-1-q\})$. If the auxiliary function φ_K satisfies $\int_2^{+\infty} \frac{[\varphi_K(s)]^{\frac{1}{p-1}}}{s}ds < \infty$, then S maps $L^{\infty}(D, d\mu_a)$ and $L^1(D, d\mu_a)$ into $L^{\infty}(D, d\mu_a)$ and $L^1(D, d\mu_a)$, respectively, where

$$d\mu_a = \frac{(1-|z|^2)^{-\frac{q+p\gamma}{p-1}}}{[K(1-|\varphi_a(z)|^2)]^{\frac{1}{p-1}}}dA(z),\ a \in D.$$

Proof. Since $\gamma < n$ and $\gamma > 0$, then by Theorem 1.7 in [10], we get

$$\begin{aligned}\|S(H)\|_{L^{\infty}(D,d\mu_a)} &\le \|H\|_{L^{\infty}(D,d\mu_a)}(1-|\omega|^2)^{\gamma}\int_D \frac{(1-|z|^2)^{n-1-\gamma}}{|1-\overline{z}\omega|^{n+1}}dA(z)\\ &\lesssim \|H\|_{L^{\infty}(D,d\mu_a)}(1-|\omega|^2)^{\gamma}(1-|\omega|^2)^{-\gamma}\\ &= \|H\|_{L^{\infty}(D,d\mu_a)} < \infty.\end{aligned}$$

$$\|S(H)\|_{L^1(D,d\mu_a)}$$

$$\le \int_D (1-|\omega|^2)^{\gamma}\int_D |H(z)|\frac{(1-|z|^2)^{n-1-\gamma}}{|1-\overline{z}\omega|^{n+1}}dA(z)\frac{(1-|\omega|^2)^{-\frac{q+p\gamma}{p-1}}}{[K(1-|\varphi_a(\omega)|^2)]^{\frac{1}{p-1}}}dA(\omega)$$

$$= \int_D |H(z)|(1-|z|^2)^{n-1-\gamma} \int_D \frac{(1-|\omega|^2)^{-\frac{q+p\gamma}{p-1}+\gamma}}{|1-\overline{z}\omega|^{n+1}[K(1-|\varphi_a(\omega)|^2)]^{\frac{1}{p-1}}} dA(\omega)\, dA(z).$$

It suffices to show that

$$M(a) = \int_D \frac{(1-|\omega|^2)^{-\frac{q+p\gamma}{p-1}+\gamma}}{|1-\overline{z}\omega|^{n+1}[K(1-|\varphi_a(\omega)|^2)]^{\frac{1}{p-1}}} dA(\omega) \lesssim \frac{(1-|z|^2)^{-\frac{q+p\gamma}{p-1}+\gamma-n+1}}{[K(1-|\varphi_a(z)|^2)]^{\frac{1}{p-1}}}.$$

Fix $a, z \in D$, let $\lambda = \varphi_z(a)$, both $\varphi_a(\omega)$ and $\varphi_\lambda(\varphi_z(\omega))$ are Möbius transformations of D mapping a to zero. Therefore there is a unimodular constant $e^{i\theta}$ (which is $-\frac{1-a\overline{z}}{1-\overline{a}z}$) such that

$$\varphi_a(\omega) = e^{i\theta}\, \varphi_\lambda \circ \varphi_z(\omega).$$

Note $|\lambda|^2 = |\varphi_z(a)|^2 = |\varphi_a(z)|^2$, direct computation yields

$$\begin{aligned}
&M(a)\\
&= \int_D \frac{(1-|\omega|^2)^{-\frac{q+p\gamma}{p-1}+\gamma}}{|1-\overline{z}\omega|^{n+1}[K(1-|\varphi_\lambda \circ \varphi_z(\omega)|^2)]^{\frac{1}{p-1}}} dA(\omega)\\
&= \int_D \frac{(1-|\varphi_z(u)|^2)^{-\frac{q+p\gamma}{p-1}+\gamma}|\varphi_z'(u)|^2}{|1-\overline{z}\varphi_z(u)|^{n+1}[K(1-|\varphi_\lambda(u)|^2)]^{\frac{1}{p-1}}} dA(u)\\
&= \int_D \frac{\left[\frac{(1-|u|^2)(1-|z|^2)}{|1-\overline{z}u|^2}\right]^{-\frac{q+p\gamma}{p-1}+\gamma}\ \frac{(1-|z|^2)^2}{|1-\overline{z}u|^4}}{\left|\frac{1-|z|^2}{1-\overline{z}u}\right|^{n+1}\ [K(\frac{(1-|u|^2)(1-|\lambda|^2)}{|1-\overline{\lambda}u|^2})]^{\frac{1}{p-1}}} dA(u)\\
&= \int_D \frac{(1-|z|^2)^{-\frac{q+p\gamma}{p-1}+\gamma-n+1}\ (1-|u|^2)^{-\frac{q+p\gamma}{p-1}+\gamma}}{|1-\overline{z}u|^{-2\frac{q+p\gamma}{p-1}+2\gamma-n+3}\ [K(\frac{(1-|u|^2)(1-|\lambda|^2)}{|1-\overline{\lambda}u|^2})]^{\frac{1}{p-1}}} dA(u)\\
&\le \frac{(1-|z|^2)^{-\frac{q+p\gamma}{p-1}+\gamma-n+1}}{[K(1-|\lambda|^2)]^{\frac{1}{p-1}}} \int_D \frac{(1-|u|^2)^{-\frac{q+p\gamma}{p-1}+\gamma}}{|1-\overline{z}u|^{-2\frac{q+p\gamma}{p-1}+2\gamma-n+3}}\ [\varphi_K(\frac{|1-\overline{\lambda}u|^2}{1-|u|^2})]^{\frac{1}{p-1}} dA(u)\\
&\le \frac{(1-|z|^2)^{-\frac{q+p\gamma}{p-1}+\gamma-n+1}}{[K(1-|\varphi_a(z)|^2)]^{\frac{1}{p-1}}} \int_D \frac{(1-|u|^2)^{-\frac{q+p\gamma}{p-1}+\gamma}}{|1-\overline{z}u|^{-2\frac{q+p\gamma}{p-1}+2\gamma-n+3}}\ [\varphi_K(\frac{2}{1-|u|})]^{\frac{1}{p-1}} dA(u)\\
&= \frac{(1-|z|^2)^{-\frac{q+p\gamma}{p-1}+\gamma-n+1}}{[K(1-|\varphi_a(z)|^2)]^{\frac{1}{p-1}}} \int_D \frac{(1-|u|^2)^{-\frac{q+p\gamma}{p-1}+\gamma+1}}{|1-\overline{z}u|^{-\frac{q+p\gamma}{p-1}+\gamma+1}} \frac{(1-|u|^2)^{-1}[\varphi_K(\frac{2}{1-|u|})]^{\frac{1}{p-1}}}{|1-\overline{z}u|^{-\frac{q+p\gamma}{p-1}+\gamma-n+2}} dA(u)\\
&\le \frac{(1-|z|^2)^{-\frac{q+p\gamma}{p-1}+\gamma-n+1}}{[K(1-|\varphi_a(z)|^2)]^{\frac{1}{p-1}}} \int_D \frac{(1-|u|^2)^{-1}[\varphi_K(\frac{2}{1-|u|})]^{\frac{1}{p-1}}}{|1-\overline{z}u|^{-\frac{q+p\gamma}{p-1}+\gamma-n+2}} dA(u)\\
&\lesssim \frac{(1-|z|^2)^{-\frac{q+p\gamma}{p-1}+\gamma-n+1}}{[K(1-|\varphi_a(z)|^2)]^{\frac{1}{p-1}}} \int_0^1 \frac{[\varphi_K(\frac{2}{1-|u|})]^{\frac{1}{p-1}}}{1-|u|^2} \left(\int_0^{2\pi} \frac{1}{|1-\overline{z}u|^{-\frac{q+p\gamma}{p-1}+\gamma-n+2}} d\theta\right) d|u|\\
&\lesssim \frac{(1-|z|^2)^{-\frac{q+p\gamma}{p-1}+\gamma-n+1}}{[K(1-|\varphi_a(z)|^2)]^{\frac{1}{p-1}}} \int_0^1 \frac{[\varphi_K(\frac{2}{1-|u|})]^{\frac{1}{p-1}}}{1-|u|^2} d|u|
\end{aligned}$$

$$\lesssim \frac{(1-|z|^2)^{-\frac{q+p\gamma}{p-1}+\gamma-n+1}}{[K(1-|\varphi_a(z)|^2)]^{\frac{1}{p-1}}} \int_2^{+\infty} \frac{[\varphi_K(s)]^{\frac{1}{p-1}}}{s} ds$$
$$\lesssim \frac{(1-|z|^2)^{-\frac{q+p\gamma}{p-1}+\gamma-n+1}}{[K(1-|\varphi_a(z)|^2)]^{\frac{1}{p-1}}},$$

where we use $-\frac{q+p\gamma}{p-1}+\gamma+1>0$ by $\gamma<p-1-q$ and also use Lemma 2.5 due to the fact $-\frac{q+p\gamma}{p-1}+\gamma-n+2<1$ by $\gamma>-q-(n-1)(p-1)$. $\square$

Theorem 3.3. *For $p>\max\{1,1+q\}$ and $n\in N$ with $n>1+\frac{-q-1}{p}$ and φ_K satisfies $\int_2^{+\infty}\frac{[\varphi_K(s)]^{\frac{1}{p-1}}}{s^{2-\frac{q}{p-1}}}ds<\infty$, we define the Banach spaces $X_m\subseteq H(D)$ and $Y_m\subseteq H(D)$ by*

$$\|g\|_{X_m}=(\int_D |D^n g(z)|^p(1-|z|^2)^{(n-1)p+q}K(1-|\varphi_{a_m}(z)|^2)dA(z))^{\frac{1}{p}}<\infty,$$
$$\|f\|_{Y_m}=(\int_D |f(z)|^{\frac{p}{p-1}}(1-|z|^2)^{-\frac{q}{p-1}}[K(1-|\varphi_{a_m}(z)|^2)]^{-\frac{1}{p-1}}dA(z))^{\frac{p-1}{p}}<\infty,$$

we assume $g(0)=0$, then $(X_m)^*\cong Y_m$ under the pairing

$$\langle f,g\rangle=\int_D \overline{f(z)}g'(z)dA(z).$$

Moreover, for every $f\in Y_m$, $\|f\|_{Y_m}\approx\sup_{g\in B_{X_m}}|\langle f,g\rangle|$ where the constants do not depend on m.

Proof. For $f(z)=\sum_{k=0}^\infty a_k z^k\in Y_m$ and $g(z)=\sum_{k=1}^\infty b_k z^k\in X_m$, by Lemma 2.6, $\int_D\overline{f(z)}g'(z)dA(z)=\int_D\overline{f(z)}D^n g(z)(1-|z|^2)^{n-1}dA(z)$. Using Hölder's inequality we obtain

$$\begin{aligned}|\langle f,g\rangle| &= |\int_D \overline{f(z)}D^n g(z)(1-|z|^2)^{n-1}dA(z)|\\ &\le (\int_D |D^n g(z)|^p(1-|z|^2)^{(n-1)p+q}K(1-|\varphi_{a_m}(z)|^2)dA(z))^{\frac{1}{p}}\\ &\quad\times(\int_D |f(z)|^{\frac{p}{p-1}}(1-|z|^2)^{-\frac{q}{p-1}}[K(1-|\varphi_{a_m}(z)|^2)]^{-\frac{1}{p-1}}dA(z))^{\frac{p-1}{p}}\\ &= \|g\|_{X_m}\|f\|_{Y_m}.\end{aligned}$$

It follows $Y_m\subseteq(X_m)^*$.

Conversely, let $L\in(X_m)^*$ and consider $T:X_m\to L^p$ given by

$$T(g)=D^n g(z)(1-|z|^2)^{(n-1)+\frac{q}{p}}[K(1-|\varphi_{a_m}(z)|^2)]^{\frac{1}{p}}.$$

Let $G=T(X_m)$. By Hahn-Banach theorem, $L\circ T^{-1}:G\to C$ can be extended (preserving the norm) to a bounded linear functional on L^p, denoted here by $\widetilde{L\circ T^{-1}}$. Therefore we can find $h_0\in L^{\frac{p}{p-1}}$ such that

$$(\widetilde{L\circ T^{-1}})(f)=\int_D f(z)\overline{h_0(z)}\,dA(z)\quad\text{for all } f\in L^p$$

and $\|L\circ T^{-1}\| = (\int_D |h_0(z)|^{\frac{p}{p-1}} dA(z))^{\frac{p-1}{p}}$. Especially for all $g\in X_m$ taking $f=T(g)\in L^p$, we have

$$\begin{aligned} L(g) &= \int_D T(g)\overline{h_0(z)}dA(z)\\ &= \int_D D^n g(z)(1-|z|^2)^{(n-1)+\frac{q}{p}}[K(1-|\varphi_{a_m}(z)|^2)]^{\frac{1}{p}}\overline{h_0(z)}dA(z)\\ &= \int_D D^n g(z)(1-|z|^2)^{n-1}\overline{h(z)}dA(z), \end{aligned}$$

where $\overline{h(z)} = \overline{h_0(z)}(1-|z|^2)^{\frac{q}{p}}[K(1-|\varphi_{a_m}(z)|^2)]^{\frac{1}{p}}$ and

$$\begin{aligned} \|L\| &= (\int_D |h_0(z)|^{\frac{p}{p-1}}dA(z))^{\frac{p-1}{p}}\\ &= (\int_D |h(z)|^{\frac{p}{p-1}}(1-|z|^2)^{-\frac{q}{p-1}}[K(1-|\varphi_{a_m}(z)|^2)]^{-\frac{1}{p-1}}dA(z))^{\frac{p-1}{p}}. \end{aligned}$$

We now claim that $D^n g\in L^1(D,(1-|z|^2)^{n-1}dA(z))$ whenever $g\in X_m$.

$$\begin{aligned} &\int_D |D^n g(z)|(1-|z|^2)^{n-1}dA(z)\\ &= \int_D |D^n g(z)|(1-|z|^2)^{(n-1)+\frac{q}{p}}[K(1-|\varphi_{a_m}(z)|^2)]^{\frac{1}{p}}(1-|z|^2)^{-\frac{q}{p}}\\ &\qquad [K(1-|\varphi_{a_m}(z)|^2)]^{-\frac{1}{p}}dA(z)\\ &\le \|g\|_{X_m}(\int_D (1-|z|^2)^{-\frac{q}{p-1}}[K(1-|\varphi_{a_m}(z)|^2)]^{-\frac{1}{p-1}}dA(z))^{\frac{p-1}{p}}\\ &\lesssim \|g\|_{X_m}(\int_2^{+\infty}\frac{[\varphi_K(s)]^{\frac{1}{p-1}}}{s^{2-\frac{q}{p-1}}}ds)^{\frac{p-1}{p}} < \infty. \end{aligned}$$

Thus by Lemma 2.7

$$D^n g(z) = n\int_D D^n g(\omega)\frac{(1-|\omega|^2)^{n-1}}{(1-z\overline{w})^{n+1}}dA(\omega).$$

If $g\in X_m$, then

$$\begin{aligned} L(g) &= \int_D D^n g(z)\overline{h(z)}(1-|z|^2)^{n-1}dA(z)\\ &= \int_D n\int_D D^n g(\omega)\frac{(1-|\omega|^2)^{n-1}}{(1-z\overline{w})^{n+1}}dA(\omega)\overline{h(z)}(1-|z|^2)^{n-1}dA(z)\\ &= \int_D D^n g(\omega)(1-|\omega|^2)^{n-1} n\int_D \overline{h(z)}\frac{(1-|z|^2)^{n-1}}{(1-z\overline{w})^{n+1}}dA(z)\,dA(\omega). \end{aligned}$$

Let $\overline{f_0(\omega)} = n\int_D \overline{h(z)}\frac{(1-|z|^2)^{n-1}}{(1-z\overline{w})^{n+1}}dA(z)$, f_0 is analytic, thus it remains to show $\|f_0\|_{Y_m}\lesssim \|L\|$ where the constant does not depend on m. It suffices to show

$$\|f_0\|_{Y_m}^{\frac{p}{p-1}} \lesssim \int_D |h(z)|^{\frac{p}{p-1}}(1-|z|^2)^{-\frac{q}{p-1}}[K(1-|\varphi_{a_m}(z)|^2)]^{-\frac{1}{p-1}}dA(z).$$

Fix $\gamma\in(\max\{0,-q-(n-1)(p-1)\},\min\{n,p-1-q\})$, the interval is non-empty by $n>1+\frac{-q-1}{p}$. Define the functions F and H by

$$F(\omega)=f_0(\omega)(1-|\omega|^2)^\gamma,\ H(z)=h(z)(1-|z|^2)^\gamma.$$

Then it reduces to show

$$\begin{aligned}&\int_D|F(\omega)|^{\frac{p}{p-1}}(1-|\omega|^2)^{-\frac{q+p\gamma}{p-1}}[K(1-|\varphi_{a_m}(\omega)|^2)]^{-\frac{1}{p-1}}dA(\omega)\\ \lesssim&\int_D|H(z)|^{\frac{p}{p-1}}(1-|z|^2)^{-\frac{q+p\gamma}{p-1}}[K(1-|\varphi_{a_m}(z)|^2)]^{-\frac{1}{p-1}}dA(z),\end{aligned}$$

where

$$\begin{aligned}F(\omega)&=(1-|\omega|^2)^\gamma n\int_D h(z)\frac{(1-|z|^2)^{n-1}}{(1-z\overline{w})^{n+1}}dA(z)\\&=n(1-|\omega|^2)^\gamma\int_D H(z)\frac{(1-|z|^2)^{n-1-\gamma}}{(1-z\overline{w})^{n+1}}dA(z).\end{aligned}$$

Since $2-\frac{q}{p-1}>1$, $\int_2^{+\infty}\frac{[\varphi_K(s)]^{\frac{1}{p-1}}}{s^{2-\frac{q}{p-1}}}ds\le\int_2^{+\infty}\frac{[\varphi_K(s)]^{\frac{1}{p-1}}}{s}ds$. Thus we can use Lemma 2.8 and Lemma 3.2 to prove the above inequality and note that the constants obtained are independent of m. $\square$

Theorem 3.4. *Let $p>\max\{1,1+q\}$, K satisfy (1) and φ_K satisfies*

$$\int_2^{+\infty}\frac{[\varphi_K(s)]^{\frac{1}{p-1}}}{s^{2-\frac{q}{p-1}}}ds<\infty.$$

Then $R(p,q,K)\cong Q_{K,0}(p,q)^*$ under the pairing

$$\langle f,g\rangle=\int_D\overline{f(z)}g'(z)dA(z).$$

That is, every $f\in R(p,q,K)$ induces a bounded linear functional $\langle f,\cdot\rangle$: $Q_{K,0}(p,q)\to C$. Conversely, if $L\in Q_{K,0}(p,q)^*$, then there exists $f\in R(p,q,K)$ such that $L(g)=\langle f,g\rangle$ for all $g\in Q_{K,0}(p,q)$. Moreover, for every $f\in R(p,q,K)$, $\|f\|_{R(p,q,K)}\approx\sup_{g\in B_{Q_{K,0}(p,q)}}|\langle f,g\rangle|$.

Proof. Let $f\in R(p,q,K)$ and $g\in Q_{K,0}(p,q)$. Given $\epsilon>0$, there exists a representation of f such that

$$\begin{aligned}&\sum_{m=1}^\infty\int_D|\overline{f_m(z)}g'(z)|dA(z)\\ \le&\|g\|_{Q_{K,0}(p,q)}\sum_{m=1}^\infty\left(\int_D|f_m(z)|^{\frac{p}{p-1}}(1-|z|^2)^{-\frac{q}{p-1}}[K(1-|\varphi_{a_m}(z)|^2)]^{\frac{1}{p-1}}dA(z)\right)^{\frac{p-1}{p}}\\ \le&\|g\|_{Q_{K,0}(p,q)}(\|f\|_{R(p,q,K)}+\epsilon).\end{aligned}$$

It follows $|\langle f,g\rangle|\le\|g\|_{Q_{K,0}(p,q)}\|f\|_{R(p,q,K)}$ and thus $f\in Q_{K,0}(p,q)^*$.

To prove that every $L \in Q_{K,0}(p,q)^*$, there is an $f \in R(p,q,K)$ such that $L(g) = \langle f, g\rangle$, $g \in Q_{K,0}(p,q)$, we consider $A = \{a_m : m \geq 1\}$. Noticing

$$1 \lesssim \frac{1 - |\varphi_a(z)|^2}{1 - |\varphi_{a_m}(z)|^2} \lesssim 1,\ a \in E_m,\ z \in D,$$

we get that the supremum in the definition of the $Q_K(p,q)$-norm can be taken over A. Suppose now that each X_m consists of those functions g holomorphic on D which $g(0) = 0$ and

$$\|g\|_{X_m} = (\int_D |D^n g(z)|^p (1-|z|^2)^{(n-1)p+q} K(1 - |\varphi_{a_m}(z)|^2) dA(z))^{\frac{1}{p}} < \infty.$$

Denote by X the direct c_0-sum of X_m, that is, the space of holomorphic functions $\{g_m\}_1^\infty$ on D such that $g_m \in X_m$ for every $m = 1, 2, 3, \dots$ and $\lim_{m\to\infty} \|g_m\|_{X_m} = 0$. The norm in X is given by

$$\|\{g_m\}_1^\infty\|_X = \sup_{m=1,2,3,\dots} \|g_m\|_{X_m}.$$

It is clear that the space $Q_{K,0}$ with this new norm is a normed subspace of X. Define Y to be l^1-sum of the spaces Y_m, the dual of X is isometrically isomorphic to the l^1-sum of the spaces X_m^*. By Theorem 3.3, we can replace X_m^* by Y_m and so the dual of X is equal to $Y : X^* \cong Y$, with the paring

$$\sum_{m=1}^\infty \langle f_m, g_m\rangle,\ f_m \in Y_m,\ g_m \in X_m.$$

Let $L \in Q_{K,0}(p,q)^*$. Due to the fact that $X^* \cong Y$ and $Q_{K,0}(p,q) \subset X$, using a Hahn-Banach extension of L to X we obtain $f_m \in Y_m$ such that

$$L(g) = \sum_{m=1}^\infty \langle f_m, g\rangle,\ g \in Q_{K,0}(p,q)$$

holds and the norm $\|L\|$ of L obeys

$$\sum_{m=1}^\infty \|f_m\|_{Y_m} \lesssim \|L\|.$$

Finally, if $f(z) = \sum_{m=1}^\infty f_m(z)$, then this series converges uniformly on compact sets of D by Proposition 2.12. Thus

$$f \in H(D), \quad \|f\|_Y \leq \sum_{m=1}^\infty \|f_m\|_{Y_m} \quad \text{and} \quad L(f) = \langle f, g\rangle.$$

Thus we have proved $Q_{K,0}(p,q)^* \subset R(p,q,K)$. □

4. The $E(p,q,K) - Q_K(p,q)$ duality

Lemma 4.1. *Let* $p > \max\{1, 1+q\}$ *and* φ_K *satisfies* $\int_2^{+\infty} \frac{[\varphi_K(s)]^{\frac{1}{p-1}}}{s^{2-\frac{q}{p-1}}} ds < \infty$. *Then the polynomials are dense in* $R(p,q,K)$.

Proof. Combining Theorem 3.3 and the proof of Lemma 4.2 in [5], the result follows directly. □

The following result is an immediate consequence of Lemma 4.1, Lemma 2.2 and Theorem 3.4.

Proposition 4.2. *Let* $q \geq 0$, $p > 1+q$, $f \in R(p,q,K)$, K *satisfies* (1), (2) *and* φ_K *satisfies* $\int_2^{+\infty} \frac{[\varphi_K(s)]^{\frac{1}{p-1}}}{s^{2-\frac{q}{p-1}}} ds < \infty$. *Then* $\|f - f_r\|_{R(p,q,K)} \to 0$ *as* $r \to 1$.

Remark. If $q \geq 0, 1+q < p < 2$, then $\frac{\varphi_K(s)}{s^2} \leq \frac{[\varphi_K(s)]^{\frac{1}{p-1}}}{s^{2-\frac{q}{p-1}}}$. It follows that the condition $\int_2^{+\infty} \frac{[\varphi_K(s)]^{\frac{1}{p-1}}}{s^{2-\frac{q}{p-1}}} ds < \infty$ implies condition (2).

Let $E(p,q,K) = \{L \in Q_K(p,q)^* : L|_{(B_{Q_K(p,q)},co)}$ is continous$\}$ be the subspace of $Q_K(p,q)^*$. Since $(B_{Q_K(p,q)},co)$ is compact by Lemma 2.1. Hence, by the Dixmier-Ng theorem in [4], we have:

Theorem 4.3. $E(p,q,K)$ *space is a Banach space and*

$$J : Q_K(p,q) \to E(p,q,K)^*$$

is an isometric isomorphism.

Theorem 4.4. *For* $q \geq 0$, $p > 1+q$, K *satisfies* (1), (2) *and* φ_K *satisfies*

$$\int_2^{+\infty} \frac{[\varphi_K(s)]^{\frac{1}{p-1}}}{s^{2-\frac{q}{p-1}}} ds < \infty,$$

the restriction map $E(p,q,K) \to Q_{K,0}(p,q)^*$ *is an isometric isomorphism.*

Proof. Combining Proposition 4.2, Theorem 3.4 and the proof of Theorem 4.2 in [3], we can obtain the result. □

Corollary 4.5. *Let* $q \geq 0$, $p > 1+q$, K *satisfies* (1), (2) *and* φ_K *satisfies*

$$\int_2^{+\infty} \frac{[\varphi_K(s)]^{\frac{1}{p-1}}}{s^{2-\frac{q}{p-1}}} ds < \infty,$$

then $Q_{K,0}(p,q)^{**}$ *is isometrically isomorphic to* $Q_K(p,q)$ *and* $R(p,q,K)$ *is isomorphic to* $E(p,q,K)$.

Proof. Combining Theorem 4.3 and Theorem 4.4, we can get $Q_{K,0}(p,q)^{**}$ is isometrically isomorphic to $Q_K(p,q)$. $R(p,q,K)$ is isomorphic to $E(p,q,K)$ by Theorem 3.4 and Theorem 4.4. □

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