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MULTIPLE PERIODIC SOLUTIONS OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS ACROSS RESONANCE

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Abstract. In this paper we study the existence of multiple periodic solutions of second-order ordinary differential equations. New results of multiplicity of periodic solutions are obtained when the nonlinearity may cross multiple consecutive eigenvalues. The arguments are proceeded by a combination of variational and degree theoretic methods.

1. Introduction and main results

This paper is devoted to the existence of multiple periodic solutions for the second-order nonlinear ordinary differential equation

$$\left\{ \begin{array}{l} -\ddot{x} = f(t,x), \\ x(0) = x(2\pi),\ \dot{x}(0) = \dot{x}(2\pi), \end{array} \right. \tag{1.1}$$

where $f \in C(\mathbb{R}^2, \mathbb{R})$, $f(t+T,x) = f(t,x)$ for all $(t,x) \in \mathbb{R}^2$, $T > 0$, $f(t,0) = 0$. Clearly, the problem (1.1) has a trivial solution $x = 0$. Our goal is to obtain multiple nontrivial periodic solutions of problem (1.1) when the nonlinearity f grows linearly and the "ratio" $\frac{2F(t,s)}{s^2}$ stays asymptotically at infinity between two consecutive eigenvalues of linear periodic boundary value problem

$$\left\{ \begin{array}{l} -\ddot{x} = \lambda x, \\ x(0) = x(2\pi),\ \dot{x}(0) = \dot{x}(2\pi), \end{array} \right. \tag{1.2}$$

where $F(t,s) \equiv \int_0^s f(t,\tau)d\tau$. The eigenvalues $\{k^2\}_{k\in\mathbb{N}}$ of the problem (1.2) are usually called resonant points of the problem (1.1).

Various works in the literature are devoted to resonant problems for ordinary or partial differential equations during the last several decades since the pioneer work Landesman and Lazer [21]. In these papers growth restrictions have been required on both the ratios $\frac{f(t,s)}{s}$ and $\frac{2F(t,s)}{s^2}$ to control the interference of the nonlinearity with the eigenvalues of the associated linear problem (see

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[4, 5, 8, 9, 10, 15, 17, 18, 20, 23, 26]). Among these works, Fabry and Fonda [15], Omari and Zanolin [26] established the solvability of (1.1) under the following so-called double-resonance conditions

$$k^2 \leq \liminf_{|s|\to\infty} \frac{f(t,s)}{s} \leq \limsup_{|s|\to\infty} \frac{f(t,s)}{s} \leq (k+1)^2 \tag{1.3}$$

for some $k \in \mathbb{N}$ and uniformly for a.e. $t \in [0, 2\pi]$ and some additional conditions imposed on f. Papageorgion and Staicu [27], Su and Zhao [28] studied the existence of multiple periodic solutions for problem (1.1) when double resonance occurs, where they assumed (1.3) and some generalization of the well-known Landesman-Lazer conditions (LL-conditions for short), see Landesman-Lazer [21], Fabry-Fonda [15] and Iannacci-Nkashama [20]. Under the additional assumption (1.6) or $f'(t,0) < 0$ with f being C^1, they both established the existence of at least two nontrivial solutions of (1.1). Recently, Barletta and Papageorgiou [2] used the variational methods together with Morse theory to obtain six nontrivial solutions of problem (1.1) when the nonlinearity is resonant both at infinity and at zero. In this paper, we shall show that if (1.3) with LL-conditions are replaced by the assumptions (1.4) and (1.5), problem (1.1) still admits at least two nontrivial solutions. Furthermore, if (1.4) and (1.5) are replaced by some stronger conditions (1.7) and (1.8), then we may obtain more nontrivial solutions. Note that in our results we don't require the ratio $\frac{f(t,s)}{s}$ stays asymptotically at infinity between two consecutive eigenvalues of (1.2), so the ratio $\frac{f(t,s)}{s}$ may cross resonant points $\{k^2\}$ asymptotically. Our method is a combination of variational method and degree theory.

Besides the above cited papers, for related works on resonant problems we also refer the interested reader to see [6, 7, 12, 13, 14, 16, 23, 29, 30] etc and the references therein.

Let $H^1_{[0,2\pi]}$ denote the Sobolev space

$$H^1_{[0,2\pi]} = \{x : [0, 2\pi] \to \mathbb{R} \mid x \text{ is absolutely continuous}, x(0) = x(2\pi), \int_0^{2\pi} |\dot{x}|^2 + |x|^2 dt < \infty\}$$

equipped with the usual inner product and norm

$$(x, y) = \int_0^{2\pi} (\dot{x}\dot{y} + xy)dt, \ \|x\| = (x, x)^{\frac{1}{2}}, \ \forall x, y \in H^1_{[0,2\pi]}.$$

Define the functional $I : H^1_{[0,2\pi]} \to \mathbb{R}$ by

$$I(x) = \int_0^{2\pi} [\frac{1}{2}|\dot{x}(t)|^2 - F(t, x)]dt.$$

Obviously, $I \in C^1(H^1_{[0,2\pi]}, \mathbb{R})$ (see [24]) and

$$[\nabla I(x)](y) = \int_0^{2\pi} [\dot{x}\dot{y} - f(t, x)y]dt, \ \forall x, y \in H^1_{[0,2\pi]}.$$

It is well known that x is a weak solution of (1.1) if and only if x is a critical point of the functional I.

Our main results are as follows.

Theorem 1.1. *Assume that* $f \in C([0, 2\pi] \times \mathbb{R}, \mathbb{R})$ *and the following conditions hold*:

(i) *there exist* $\eta_1, M, M_1 > 0$ *such that*

$$\eta_1 \le \frac{f(t,s)}{s} \le M \text{ for } |s| \ge M_1, \text{ a.e. } t \in [0, 2\pi]; \tag{1.4}$$

(ii)

$$k^2 \le l(t) \equiv \liminf_{|s|\to\infty} \frac{2F(t,s)}{s^2} \le \limsup_{|s|\to\infty} \frac{2F(t,s)}{s^2} \equiv r(t) \le (k+1)^2 \tag{1.5}$$

for a.e. $t \in [0, 2\pi]$ *and some* $k \in \mathbb{Z}^+$;

(iii) *there exists* $s_1 > 0$ *such that*

$$F(t,s) \le 0 \tag{1.6}$$

for a.e. $t \in [0, 2\pi]$ *and all* $s \in \mathbb{R}$ *with* $|s| \le s_1$.

Then the problem (1.1) *admits at least one nontrivial nonnegative solution and one nontrivial nonpositive solution.*

Theorem 1.2. *Assume that* $f \in C([0, 2\pi] \times \mathbb{R}, \mathbb{R})$ *with* $f(t, x) = g(x) + e(t)$ *and* (1.6) *holds. Suppose that the following conditions hold*:

(i) *there exist* $\eta_2, C, C_1 > 0$ *such that*

$$\eta_2 \le \frac{g(s)}{s} \le C \text{ for } |s| \ge C_1; \tag{1.7}$$

(ii) *there exist* $p, q > 0$ *such that*

$$k^2 < p \le l(t) \le r(t) \le q < (k+1)^2 \tag{1.8}$$

for a.e. $t \in [0, 2\pi]$ *and some* $k \in \mathbb{Z}^+$.

Then we have the following results:

(i) *if k is odd, then the problem* (1.1) *possesses at least two nontrivial solutions*; *if k is even, then the problem* (1.1) *possesses at least three nontrivial solutions, one of which is nonnegative, and another one is nonpositive*;

(ii) *if all the critical points of the functional I is nondegenerate, then the problem* (1.1) *admits at least four nontrivial solutions*;

(iii) *if* $g \in C^1(\mathbb{R}, \mathbb{R})$ *and there exists* $\beta > 0$ *such that*

$$g'(s) \le \beta < (k+1)^2 \tag{1.9}$$

for all $s \in \mathbb{R}$, *then the problem* (1.1) *admits at least four nontrivial solutions.*

This paper is organized as follows. In § 2, the proof of Theorem 1.1 is presented. In § 3, we give the proof of Theorem 1.2.

For convenience, we introduce some notations and definitions. $L^p(0, 2\pi)(1 < p < \infty)$ denotes the usual Sobolev space with inner product $\langle , \rangle_p$ and norm $\| \cdot \|_p$

respectively. $C^m[0,2\pi]$ denotes the space of m-times continuous differential real functions with norm

$$\|x\|_{C^m} = \max_{t\in[0,2\pi]} |x(t)| + \max_{t\in[0,2\pi]} |\dot{x}(t)| + \cdots + \max_{t\in[0,2\pi]} |x^{(m)}(t)|.$$

Note the eigenvalues $\{k^2\}_{k\in\mathbb{N}}$ are simple, and we denote $E(k^2)$ $(k \in \mathbb{N})$ the eigenspace corresponding to k^2. Let

$$V = E(0^2) \oplus E(1^2) \oplus \cdots \oplus E(k^2),\ W = \overline{E((k+1)^2) \oplus E((k+2)^2) \oplus \cdots}.$$

Then

$$H^1_{[0,2\pi]} = V \oplus W$$

and we can write $x \in H^1_{[0,2\pi]}$ as $x = v + w$, where

$$v(t) = a_0 + \sum_{i=1}^{k} (a_i \cos(it) + b_i \sin(it)) \in V,$$

$$w(t) = \sum_{i=k+1}^{\infty} (a_i \cos(it) + b_i \sin(it)) \in W.$$

2. Proof of Theorem 1.1

Let $f^+ : [0,2\pi] \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f^+(t,s) = \begin{cases} f(t,s) + s, & s \geq 0, \\ 0, & s < 0 \end{cases}$$

and $F^+(t,s) = \int_0^s f^+(t,\tau)d\tau$. Define the functional $I^+ : H^1_{[0,2\pi]} \to \mathbb{R}$ as follows

$$I_1^+(x) = \int_0^{2\pi} [\frac{1}{2}(|\dot{x}|^2 + |x|^2) - F^+(t,x)]dt.$$

By (1.4), $I_1^+ \in C^1(H^1_{[0,2\pi]}, \mathbb{R})$. We assume that the set of all critical points of I_1^+ is finite. In what follows, to obtain the critical points of I_1^+, we first show that the functional I_1^+ has a mountain pass geometry. Precisely, since $H^1_{[0,2\pi]}$ is embedded compactly into $C[0,2\pi]$, we can find $c_0 > 0$ such that

$$\|x\|_\infty \leq c_0\|x\| \text{ for all } x \in H^1_{[0,2\pi]}.$$

Set $\delta_0 = \frac{s_1}{c_0}$. Then, for all $x \in H^1_{[0,2\pi]}$ with $\|x\| \leq \delta_0$, we get

$$|x(t)| \leq c_0\|x\| \leq s_1,\ \text{a.e. } t \in [0,2\pi].$$

In view of (1.6), we have

$$F^+(t,x) \leq 0 \text{ for all } \|x\| \leq \delta_0,\ \text{a.e. } t \in [0,2\pi].$$

Hence, for $x \in \bar{B}_{\delta_0}$ with $B_{\delta_0} \equiv \{x \in H^1_{[0,2\pi]} \mid \|x\| < \delta_0\}$, it follows that

$$I_1^+(x) \geq \int_0^{2\pi} \frac{1}{2}|\dot{x}|^2 dt \geq 0 = I_1^+(0), \tag{2.1}$$

which implies 0 is a local minimizer of I_1^+. Since we assume that I_1^+ has finite critical points, we can find $\delta > 0$ such that

$$\inf_{\partial B_\delta} I_1^+ > 0 = I_1^+(0). \tag{2.2}$$

Note that

$$I_1^+(s) = -\int_0^{2\pi} F^+(t,s)dt \text{ for } s \in \mathbb{R}^+.$$

By (1.5) it follows that

$$I_1^+(s) \to -\infty \text{ as } s \to \infty.$$

Specially, there exists $x_0 \in H^1_{[0,2\pi]}$ with $\|x_0\| > \delta_0$ such that

$$I_1^+(x_0) < 0.$$

In what follows we come to prove that I_1^+ satisfies the (PS) condition. Assume that $\{x_n\}_{n\in\mathbb{N}} \subset H^1_{[0,2\pi]}$ is a sequence such that for some $M > 0$,

$$|I_1^+(x_n)| \le M, \ \nabla I_1^+(x_n) \to 0 \text{ as } n \to \infty. \tag{2.3}$$

It suffices to prove that $\{x_n\}_{n\in\mathbb{N}}$ is bounded in $H^1_{[0,2\pi]}$. Then a standard argument shows that $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence, which implies that I_1^+ satisfies the (PS) condition. On the contrary, suppose that $\{x_n\}_{n\in\mathbb{N}}$ is unbounded in $H^1_{[0,2\pi]}$, i.e.,

$$\|x_n\| \to \infty \text{ as } n \to \infty. \tag{2.4}$$

Let $z_n = \frac{x_n}{\|x_n\|}$. Then $\|z_n\| = 1$ and upon passing to a subsequence, there exists some $z_0 \in H^1_{[0,2\pi]}$ such that

$$z_n \rightharpoonup z_0 \text{ in } H^1_{[0,2\pi]}, \ z_n \to z_0 \text{ uniformly on } [0,2\pi]. \tag{2.5}$$

In view of (2.3), for all $\phi \in H^1_{[0,2\pi]}$, we have

$$\int_0^{2\pi} [\dot{z}_n\dot{\phi} + z_n\phi - \frac{f^+(t,x_n)}{\|x_n\|}\phi]dt \to 0 \text{ as } n \to \infty. \tag{2.6}$$

By (1.4), $\frac{f^+(t,x_n)}{\|x_n\|}$ remains bounded in $L^2[0,2\pi]$. Thus, for a subsequence $\frac{f^+(t,x_n)}{\|x_n\|}$ converges weakly in $L^2[0,2\pi]$ to some $\tilde{f}^+ \in L^2[0,2\pi]$ and by standard arguments based on (1.4), $\tilde{f}^+$ can be written as

$$\tilde{f}^+(t) = m(t)z_0(t), \tag{2.7}$$

where $m \in L^\infty(0,2\pi)$ satisfies

$$|m(t)| \le C \text{ for a.e. } t \in [0,2\pi].$$

Consequently, by (2.5)-(2.7), z_0 is a solution of

$$\begin{cases} -\ddot{z}_0 + z_0 = m(t)z_0 \text{ in } [0,2\pi], \\ z_0(0) = z_0(2\pi), \ \dot{z}_0(0) = \dot{z}_0(2\pi). \end{cases} \tag{2.8}$$

Define

$$\Omega_+ = \{t \in [0, 2\pi] \mid z_0(t) > 0\},\ \Omega_- = \{t \in [0, 2\pi] \mid z_0(t) < 0\},$$
$$\Omega_0 = \{t \in [0, 2\pi] \mid z_0(t) = 0\}.$$

Then we consider the three cases respectively.

(i) On Ω_+. For $t \in \Omega_+$, we have

$$x_n(t) = z_n(t)\|x_n\| \to +\infty \text{ as } n \to +\infty,$$

which implies that there exists $n_1 \in \mathbb{Z}^+$ such that $x_n(t) > M_1$ for all $t \in \Omega_+$ if $n \geq n_1$. Hence by (1.4),

$$\eta_1 + 1 \leq \frac{f^+(t, x_n(t))}{x_n(t)} \leq M + 1 \text{ for } n \geq n_1.$$

Then by standard arguments it follows that

$$\eta_1 + 1 \leq m(t) \leq M + 1 \text{ for } t \in \Omega_+.$$

(ii) On Ω_-. For $t \in \Omega_-$, we have

$$x_n(t) = z_n(t)\|x_n\| \to -\infty \text{ as } n \to +\infty,$$

which implies that there exists $n_2 \in \mathbb{Z}^+$ such that $x_n(t) < -M_1$ for all $t \in \Omega_-$ if $n \geq n_2$. Then we get

$$f^+(t, x_n(t)) \equiv 0 \text{ for } n \geq n_1.$$

Hence

$$m(t) \equiv 0 \text{ for } t \in \Omega_-.$$

(iii) On Ω_0. Firstly by virtue of (1.5) we can see that there exist $M_2 > 0$, $b_1 \in L^\infty(0, 2\pi)$ such that

$$|f(t, x_n(t))| \leq M_2|x_n(t)| + b_1(t). \tag{2.9}$$

For any given $t \in \Omega_0$, if $|x_n(t)| \leq M_1$, then by (2.9) we have

$$|f(t, x_n(t))| \leq M_2M_1 + b_1(t).$$

Since $\|x_n\| \to \infty$, we obtain that

$$\frac{f^+(t, x_n)}{\|x_n\|} \to 0;$$

if $|x_n(t)| \geq M_1$, then

$$0 \leq \frac{f^+(t, x_n)}{x_n} \leq M + 1.$$

Hence we have

$$0 \leq m(t) \leq M + 1 \text{ for } t \in \Omega_0.$$

Now, acting on (2.8) with z_0^- we can get $z_0 \geq 0$. Thus, $meas(\Omega_-) = 0$. In addition, by the former of (2.3), we obtain

$$\frac{|\nabla I_1^+(x_n) \cdot x_n|}{\|x_n\|^2} = \left|1 - \int_0^{2\pi} \frac{f^+(t, x_n)}{\|x_n\|} z_n dt\right| \to 0,$$

so that

$$\int_0^{2\pi} \frac{f^+(t,x_n)}{x_n} z_n^2 dt \to 1.$$

Then, by (2.5) and (2.7), we have

$$\int_0^{2\pi} m(t) z_0^2(t) dt = 1. \tag{2.10}$$

Hence $z_0 \not\equiv 0$. By modifying $m(t)$ on Ω_0 if necessary, we may assume that $\eta_1 + 1 \le m(t) \le M + 1$ on Ω_0. Then we have

$$\eta_1 + 1 \le m(t) \le M + 1 \quad \text{a.e. } t \in [0, 2\pi]. \tag{2.11}$$

Integrating (2.8) on $[0, 2\pi]$ we have $\int_0^{2\pi} (m(t) - 1) z_0(t) dt = 0$, which implies that

$$(m(t) - 1) z_0(t) = 0 \quad \text{a.e. in } [0, 2\pi]. \tag{2.12}$$

Acting on (2.8) with z_0 we get

$$0 \le \int_0^{2\pi} |\dot{z}_0(t)|^2 dt = \int_0^{2\pi} (m(t) - 1) z_0^2(t) dt = 0.$$

Together with (2.11) and (2.12) we can see $z_0 \equiv 0$ on $[0, 2\pi]$. A contradiction. Thus $\{x_n\}_{n\in\mathbb{N}}$ is bounded in $H^1_{[0,2\pi]}$ and standard arguments imply that I_1^+ satisfies the (PS) condition. Therefore, by the mountain pass theorem, I_1^+ has a critical point x^+ with $I_1^+(x^+) \ge I_1^+(x) \mid_{\partial B_\delta} > 0$. By the assumption of f^+ we can see that $x^+ \ge 0$ with $x^+ \not\equiv 0$ and x^+ is a nontrivial nonnegative critical point of I, which implies that (1.1) admits a nontrivial nonnegative weak solution x^+. Similar arguments show that (1.1) also admits a nontrivial nonpositive weak solution x^-. The proof is complete.

3. Proof of Theorem 1.2

Lemma 3.1. *If 0 is an isolated critical point, then there exists ρ_0 small such that*

$$\deg(\nabla I, B_\rho, 0) = 1 \textit{ for all } 0 < \rho \le \rho_0.$$

Proof. Similar arguments as in the proof of Theorem 1.1 imply that 0 is a strict local minimizer of I. Then the conclusion is obtained by Corollary 2 of Amann [1]. □

Lemma 3.2. *Under the assumption of Theorem 1.2, the problem* (1.1) *admits a nontrivial nonnegative solution x^+ and a nontrivial nonpositive solution x^-. Furthermore, there exist $r_1, d_1 > 0$ small such that*

$$\deg(\nabla I, B_r(x^+), 0) = -1 \textit{ for all } 0 < r \le r_1 \tag{3.1}$$

and

$$\deg(\nabla I, B_d(x^-), 0) = -1 \textit{ for all } 0 < d \le d_1, \tag{3.2}$$

where $B_r(x) = \{y \in H^1_{[0,2\pi]} \mid \|y - x\| < r\}$.

Proof. Note that (1.7) and (1.8) imply that (1.4) and (1.5). Then by Theorem 1.1, (1.1) possesses a nontrivial nonnegative solution x^+ and a nontrivial non-positive solution x^-. The arguments in the proof of Theorem 1.1 show that x^+ and x^- are both of mountain pass type. Then, by Theorems 1 and 2 of [19], they both have local degree -1 and (3.1), (3.2) hold. □

Lemma 3.3. *Assume that $f \in C([0, 2\pi] \times \mathbb{R}, \mathbb{R})$ and* (1.7), (1.8) *hold. If x is a solution of* (1.1)*, then there exists a constant $C_2 > 0$ such that*

$$\|x\|_{C^1} \le C_2. \tag{3.3}$$

Proof. We assume by the contradiction that there exists a sequence of 2π-periodic solutions $\{x_n(t)\}_1^\infty$ of (1.1) corresponding to sequence $\{\lambda_n\}_1^\infty \subset [0, 1]$ such that

$$\|x_n\|_{C^1} \to \infty \text{ as } n \to \infty. \tag{3.4}$$

Define $z_n(t) = \frac{x_n(t)}{\|x_n\|_{C^1}}$. Then $\|z_n\|_{C^1} = 1$ and z_n satisfies

$$-\ddot{z}_n(t) = \frac{f(t, x_n)}{\|x_n\|_{C^1}}. \tag{3.5}$$

By (1.7), there must be some $M > 0$ such that

$$|\ddot{z}_n(t)| \le M, \ \forall t \in [0, 2\pi],$$

which implies that $\{z_n(t)\}$ and $\{\dot{z}_n(t)\}$ are uniformly bounded and equicontinuous on $[0, 2\pi]$. Then by the Arzela-Ascoli theorem, taking subsequences if possible, it follows that

$$z_n(t) \to \bar{z}(t), \dot{z}_n(t) \to \dot{\bar{z}}(t) \text{ as } n \to \infty \tag{3.6}$$

hold for some $\bar{z} \in C^1[0, 2\pi]$ with $\|\bar{z}(t)\|_{C^1} = 1$ on $[0, 2\pi]$. We prove now the following fact.

Claim 1. There exists $d > 0$ such that, for each n, there exists $t_n \in [0, 2\pi]$ such that

$$|x_n(t_n)| \le d. \tag{3.7}$$

Proof. Firstly, in view of (1.7), we can take $d > 0$ large enough such that

$$f(t, -d) < 0 < f(t, d), \ \forall t \in [0, 2\pi]. \tag{3.8}$$

Secondly, we shall show that there exists $d > 0$ satisfying (3.8) such that, for any n, the following conclusions hold.

$$\min_{t \in [0,2\pi]} x_n(t) \ne d, \ \max_{t \in [0,2\pi]} x_n(t) \ne -d. \tag{3.9}$$

We assume by contradiction that there is some $j \in \mathbb{Z}^+$ and $t_0 \in [0, 2\pi]$ such that $\min_{t \in [0,2\pi]} x_j(t) = x_j(t_0) = d$. By (3.8), it follows that there exist $\epsilon_j, \delta_j > 0$ such that

$$-\ddot{x}_j(t) = f(t, x_j(t)) \ge \epsilon_j$$

hold for $t \in [t_0 - \delta_j, t_0 + \delta_j]$. Thus, taking $t_1 \in [t_0 - \delta_j, t_0)$, by $\dot{x}_j(t_0) = 0$, we have

$$x_j(t_1) - x_j(t_0) = \int_{t_1}^{t_0} \ddot{x}_j(s)(s - t_1)ds < 0,$$

which is impossible. Thus the claim is right. □

By the boundedness of the sequence $\{t_n\}$, passing to a subsequence if possible, there exists $\hat{t} \in [0, 2\pi]$ such that

$$t_n \to \hat{t} \text{ as } n \to \infty. \tag{3.10}$$

Multiplying both sides of (3.5) by $\dot{z}_n(t)$ and integrating from t_n to t, we have

$$[\dot{z}_n(t_n)]^2 - [\dot{z}_n(t)]^2 = \frac{2G(x_n(t))}{x_n^2(t)}(z_n(t))^2 - \frac{2G(x_n(t_n))}{x_n}(z_n(t_n))^2 + \int_{t_n}^{t} \frac{e(s)}{\|x_n\|_{C^1}} \dot{z}_n(s)ds, \tag{3.11}$$

where $G(s) = \int_0^s g(\tau)d\tau$. By (3.7) it follows that

$$\lim_{n\to\infty} z_n(t_n) = \lim_{m\to\infty} \frac{x_n(t_n)}{\|x_n\|_{C^1}} = 0. \tag{3.12}$$

Taking a superior limit in (3.11), together with (3.6), (3.10) and (3.12), we can obtain that

$$[\dot{\bar{z}}(\hat{t})]^2 - [\dot{\bar{z}}(t)]^2 = \limsup_{n\to\infty} \frac{2G(x_n(t))}{x_n^2(t)} \cdot \bar{z}^2(t).$$

Now by (1.8) it follows that

$$[\dot{\bar{z}}(t)]^2 - [\dot{\bar{z}}(\hat{t})]^2 + q[\bar{z}(t)]^2 \geq 0. \tag{3.13}$$

Similarly, we can obtain that

$$[\dot{\bar{z}}(t)]^2 - [\dot{\bar{z}}(\hat{t})]^2 + p[\bar{z}(t)]^2 \leq 0. \tag{3.14}$$

Hence, by (3.13)-(3.14), for $t \in [0, 2\pi]$, we have

$$p[\bar{z}(t)]^2 \leq [\dot{\bar{z}}(\hat{t})]^2 - [\dot{\bar{z}}(t)]^2 \leq q[\bar{z}(t)]^2. \tag{3.15}$$

Claim 2. $\dot{\bar{z}}(t)$ has no zero accumulation points on $[0, 2\pi]$.

Proof. In fact, if not, we assume that there exists a sequence $\{\xi_i\} \subset [0, 2\pi]$ such that

$$\dot{\bar{z}}(\xi_i) = 0, \ \lim_{i\to\infty} \xi_i = \xi_0 \in [0, 2\pi].$$

Clearly, $\dot{\bar{z}}(\xi_0) = 0$. Taking $t = \xi_i$ in (3.15) and letting $i \to \infty$, it is not difficult to see that $\bar{z}(\xi_0) \neq 0$. Without loss of generality, we assume that $\bar{z}(\xi_0) > 0$. By the continuity of $\bar{z}(t)$, there exist $\epsilon, \delta > 0$ such that

$$\bar{z}(t) \geq \epsilon, \ \forall t \in [\xi_0 - \delta, \xi_0 + \delta].$$

Then, there exists $n_0 \in \mathbb{Z}^+$ such that, for $n \geq n_0$,

$$x_n(t) \geq C_1, \ \forall t \in [\xi_0 - \delta, \xi_0 + \delta]. \tag{3.16}$$

In view of (1.7), (1.8), we can see that

$$\frac{g(x_n(t))}{x_n(t)} \geq \eta_2, \ \forall t \in [\xi_0 - \delta, \xi_0 + \delta]. \tag{3.17}$$

Taking $\xi_*, \xi^* \in \{\xi_i\} \cap [\xi_0 - \delta, \xi_0 + \delta]$ and integrating from ξ_* to ξ^*, we obtain that

$$\dot{\tilde{z}}_n(\xi_*) - \dot{\tilde{z}}_n(\xi^*) = \int_{\xi_*}^{\xi^*} \frac{g(x_n(t)) + e(t)}{\|x_n\|_{C^1}} dt.$$

Taking $n \to \infty$, by (3.16)-(3.17) we get

$$0 \geq \eta_2 \epsilon(\xi^* - \xi_*) > 0.$$

A contradiction. □

Now we come to show that (3.15) has only a trivial 2π periodic solution. We assume by contradiction that (3.15) has a nontrivial 2π periodic solution $z_0(t)$. In (3.15), if $\dot{z}_0(\hat{t}) = 0$, then by (1.8) it follows that

$$k^2[z_0(t)]^2 < [\dot{z}_0(t)]^2 < (k+1)^2[z_0(t)]^2.$$

It is easily seen that $z_0(t) \equiv 0$ on $[0, 2\pi]$, which is contrary to that $\|z_0\|_{C^1} = 1$. If $\dot{z}_0(\hat{t}) \neq 0$, without loss of generality, we suppose that $\hat{t} = 0$ and $\dot{z}_0(0) > 0$. Assume that $z_1(t)$ and $z_2(t)$ are solutions of the following equations, respectively

$$[\dot{z}(t)]^2 - [\dot{z}(0)]^2 = -p[z(t)]^2, \ [\dot{z}(t)]^2 - [\dot{z}(0)]^2 = -q[z(t)]^2 \tag{3.18}$$

with

$$z_0(0) = z_1(0) = z_2(0), \ \dot{z}_1(0) \leq \dot{z}_0(0) \leq \dot{z}_2(0).$$

Then we have

$$z_1(t) \leq z_0(t) \leq z_2(t), \ t \in [0, t_1],$$

where t_1 is the first zero point of $z_0(t)$ in $(0, 2\pi]$. Similarly, if $z_1(t)$ and $z_2(t)$ are respectively solutions of (3.18) with

$$z_0(t_1) = z_1(t_1) = z_2(t_1), \ \dot{z}_1(t_1) \leq \dot{z}_0(t_1) \leq \dot{z}_2(t_1),$$

then we have

$$z_1(t) \leq z_0(t) \leq z_2(t), \ t \in [t_1, t_2],$$

where t_2 is the first zero point of z_0 in $(0, 2\pi]$. Since $\dot{z}_0(t)$ has no zero accumulation point in $[0, 2\pi]$, it follows that (3.18) are equivalent to the following equations

$$-\ddot{z}(t) = pz(t), \ -\ddot{z}(t) = qz(t)$$

respectively. Then there are positive constants $\rho_1, \rho_2, \rho_3, \rho_4$ such that

$$\rho_1 \sin(\sqrt{q}t) \leq z_0(t) \leq \rho_2 \sin(\sqrt{p}t), \ 0 < t \leq t_1,$$
$$\rho_3 \sin(\sqrt{q}(t - t_1)) \leq z_0(t) \leq \rho_4 \sin(\sqrt{p}(t - t_1)), \ t_1 \leq t \leq t_2,$$

which implies that

$$\frac{\pi}{\sqrt{q}} \le t_1 \le \frac{\pi}{\sqrt{p}},\ \frac{2\pi}{\sqrt{q}} \le t_2 \le \frac{2\pi}{\sqrt{p}}.$$

Hence, by virtue of the periodicity of $z_0(t)$, there exists some positive integer m such that

$$\frac{2m\pi}{k+1} < \frac{2m\pi}{\sqrt{q}} \le t_m = 2\pi \le \frac{2m\pi}{\sqrt{p}} < \frac{2m\pi}{k},$$

which implies that $k < m < k+1$, a contradiction. Therefore, (3.4) is impossible and (3.3) holds. This completes the proof. □

Denote $\alpha = \frac{k^2+(k+1)^2}{2}$. Then the following problem

$$\begin{cases} -\ddot{x} = \alpha x \text{ in } [0, 2\pi], \\ x(0) = x(2\pi),\ \dot{x}(0) = \dot{x}(2\pi) \end{cases}$$

has the only trivial solution 0. Define

$$\Phi(x) = \frac{1}{2}\int_0^{2\pi} [|\dot{x}|^2 - \alpha x^2(t)]dt,\ \forall x \in H^1_{[0,2\pi]}$$

and

$$N[x](y) = \int_0^{2\pi} (\alpha+1)xydx, \forall x, y \in H^1_{[0,2\pi]}.$$

Then $\Phi \in C^2(H^1_{[0,2\pi]})$ and N is a compact linear operator from $H^1_{[0,2\pi]}$ to $H^1_{[0,2\pi]}$. Note that

$$[\nabla\Phi(x)](y) = \int_0^{2\pi} [\dot{x}\dot{y} + xy]dt - \int_0^{2\pi} (\alpha+1)xydx,\ \forall x, y \in H^1_{[0,2\pi]}.$$

Then we have

$$\nabla\Phi = id - N.$$

Denote $B_R = \{x \in H^1_{[0,2\pi]} \mid \|x\| < R, R \in \mathbb{R}^+\}$. It is easily seen that 0 is the only critical point of Φ. Then $\nabla\Phi(x) \ne 0$ for all $x \in \partial B_R$ with $R > 0$. So $\nabla\Phi$ is well defined on B_R. Denote deg and $index_{LS}$ as Leray-Schauder degree and index respectively. id denotes the identical mapping. We have the following result.

Lemma 3.4. *There exists $R_0 > 0$ such that*

$$\deg(\nabla\Phi, B_R, 0) = (-1)^k, \forall R \ge R_0.$$

Proof. From Theorem 2.8.1 in [25], it follows that

$$\begin{aligned} \deg(\nabla\Phi, B_R, 0) &= \deg(id - N, B_0, 0) \\ &= \text{index}_{LS}(id - N, 0) \\ &= (-1)^\beta, \end{aligned}$$

where

$$\beta = \sum_{\lambda_j > 1, \lambda_j \in \sigma(N)} \beta_j,\ \beta_j = \dim \cup_{i=1}^{\infty} \ker(\lambda_j \cdot id - N)^i.$$

Hence we just need to compute $\text{index}_{LS}(id - N, 0)$. If $Nx = \lambda x$ for some $\lambda \in \mathbb{R}$ and $x \neq 0$, then we have

$$(Nx, y) = \int_0^{2\pi} (\alpha+1)xy dt = \int_0^{2\pi} \lambda[\dot{x}\dot{y} + xy]dt,\ \forall x, y \in H^1_{[0,2\pi]},$$

which together with (1.1) implies that $\lambda = \frac{\alpha+1}{n^2+1}$ for some $n \in \mathbb{Z}^+$. By $k^2 < \alpha < (k+1)^2$, we can obtain that $\lambda > 1$ holds for all $n \leq k$, which implies that $\deg(\nabla\Phi, B_R, 0) = (-1)^k$ for all $R \geq R_0$. This completes the proof. □

Apply above lemma, we can obtain the following result.

Lemma 3.5. *Under the assumption of Theorem 1.2, there exists $R_1 > 0$ such that*

$$\deg(\nabla I, B_R, 0) = (-1)^k,\ \forall R \geq R_1.$$

Proof. Consider the following auxiliary problem

$$\begin{cases} -\ddot{x} = \lambda f(t,x) + (1-\lambda)\alpha x \equiv f_\lambda(t,x), \text{ a.e. } t \in [0,2\pi], \lambda \in [0,1], \\ x(0) = x(2\pi),\ \dot{x}(0) = \dot{x}(2\pi). \end{cases} \tag{3.19}$$

Define

$$I_\lambda(x) = \int_0^{2\pi} [\frac{1}{2}|\dot{x}(t)|^2 dt - F_\lambda(t,x)]dt,$$

where $F_\lambda(t,x) = \int_0^x f_\lambda(t,s)ds$. By (1.4)-(1.5) and the definition of α, we can see that the conditions of Lemma 3.3 hold, so there exists $R_1 > 0$ large enough such that, for all $R \geq R_1$, $\nabla I_\lambda |_{\partial B_R} (x) \neq 0$ uniformly for $\lambda \in [0,1]$. Then by the homotopy invariance of the Leray-Schauder degree and Lemma 3.4 , we obtain that

$$\begin{aligned} \deg(\nabla I, B_R, 0) &= \deg(\nabla I_1, B_R, 0) \\ &= \deg(\nabla I_0, B_R, 0) \\ &= \deg(\nabla \Phi, B_R, 0) \\ &= (-1)^k,\ \forall R \geq R_1. \end{aligned}$$

□

Proof of Theorem 1.2(i). Suppose that the set of all critical points of the functional I is finite. If k is odd, then by Lemma 3.2 we have obtained two nontrivial solutions. So we just prove the case of k is even. Choose $\rho \in (0, \rho_0)$, $r \in (0, r_1)$, $d \in (0, d_1)$ and $R \geq R_0$ such that

$$B_\rho \cap B_r(x^+) = \emptyset,\ B_\rho \cap B_d(x^-) = \emptyset,\ B_r(x^+) \cap B_d(x^-) = \emptyset \tag{3.20}$$

and

$$\bar{B}_\rho, \bar{B}_r(x^+), \bar{B}_d(x^-) \subset B_R.$$

Then from the additivity and excision properties of the Leray-Schauder degree, we have

$$\begin{aligned}\deg(\nabla I, B_R, 0) = &\ \deg(\nabla I, B_\rho, 0) + \deg(\nabla I, B_r(x^+), 0) + \deg(\nabla I, B_d(x^-), 0)\\ &+ \deg(\nabla I, B_R \setminus \overline{(B_\rho \cup B_r(x^+) \cup B_d(x^-))}, 0).\end{aligned}$$

In view of Lemmas 3.1, 3.2 and 3.5, we obtain

$$(-1)^k = 1 + (-1) + (-1) + \deg(\nabla I, B_R \setminus \overline{(B_\rho \cup B_r(x^+) \cup B_d(x^-))}, 0),$$

which together with the excision property of the Leray-Schauder degree implies that

$$\deg(\nabla I, B_R \setminus \overline{(B_\rho \cup B_r(x^+) \cup B_d(x^-))}, 0) = 1 + (-1)^k.$$

Then if k is even, it follows that

$$\deg(\nabla I, B_R \setminus \overline{(B_\rho \cup B_r(x^+) \cup B_d(x^-))}, 0) = 2.$$

Hence, by the existence property of the Leray-Schauder degree we can see that there exists $x^* \in B_R \setminus \overline{(B_\rho \cup B_r(x^+) \cup B_d(x^-))}$ such that $\nabla I(x^*) = 0$, which together with x^+, x^- gives the existence of at least three nontrivial solutions of problem (1.1). This completes the proof. □

Proof of Theorem 1.2(ii). Firstly we shall show that the functional I satisfies the hypotheses of the saddle point theorem. In fact, since f is continuous and satisfies (1.7) it follows that $I \in C^1(H^1_{[0,2\pi]}, \mathbb{R})$. By (1.8), there exist $\epsilon_1, \epsilon_2 > 0$ satisfying

$$k^2 < p - \epsilon_1 \le l(t) \le r(t) \le q + \epsilon_2 < (k+1)^2 \quad \text{a.e. } t \in [0, 2\pi] \tag{3.21}$$

and $a_{\epsilon_1}, b_{\epsilon_2} \in L^1(0, 2\pi)$ such that

$$(p - \epsilon_1)s^2 - a_{\epsilon_1}(t) \le 2F(t, s) \le (q + \epsilon_2)s^2 + b_{\epsilon_2}(t) \quad \text{a.e. } t \in [0, 2\pi], \forall s \in \mathbb{R}.$$

Then, we have

$$\begin{aligned}&\int_0^{2\pi} \frac{1}{2}[|\dot{x}|^2 - (q + \epsilon_2)x^2 + 2b_{\epsilon_2}(t)]dt\\ &\le I(x) \le \int_0^{2\pi} \frac{1}{2}[|\dot{x}|^2 - (p - \epsilon_1)x^2 + 2a_{\epsilon_1}(t)]dt.\end{aligned} \tag{3.22}$$

We claim that

$$I(w) \to \infty \text{ as } \|w\| \to \infty \text{ for } w \in W. \tag{3.23}$$

Since for $w \in W$,

$$\int_0^{2\pi} \dot{w}^2(t)dt \ge (k+1)^2 \int_0^{2\pi} w^2(t)dt,$$

we get

$$\|w\|^2 = \int_0^{2\pi} [|\dot{w}|^2 + w^2]dt \le (1 + \frac{1}{(k+1)^2}) \int_0^{2\pi} \dot{w}^2(t)dt. \tag{3.24}$$

Then, we can obtain by (3.22) and (3.24) that

$$\begin{aligned} I(w) &\geq \frac{1}{2}[1-\frac{q+\epsilon_2}{(k+1)^2}]\int_0^{2\pi}|\dot{w}|^2dt - 2\|b_{\epsilon_2}\|_{L^2(0,2\pi)} \\ &\geq \delta_1\|w\|^2 - 2\|b_{\epsilon_2}\|_{L^2(0,2\pi)}, \end{aligned}$$

where $\delta_1 = \frac{1}{2}[1-\frac{q+\epsilon_2}{(k+1)^2}]\frac{1}{1+\frac{1}{(k+1)^2}}$. By (3.21), $\delta_1 > 0$. Thus I satisfies (3.23).

Now we claim that

$$I(v) \to -\infty \text{ as } \|v\| \to \infty \text{ for } v \in V. \tag{3.25}$$

Write

$$V = V_0 \oplus V_1,$$

where $V_0 = E(0^2)$, $V_1 = E(1^2) \oplus E(2^2) \oplus \cdots \oplus E(k^2)$. Then we can write for each $v \in V$ as $v = v_0 + v_1$, where $v_0 \in V_0$, $v_1 \in V_1$. Note for $v_1 \in V_1$,

$$\int_0^{2\pi} v_1^2(t)dt \leq \int_0^{2\pi} \dot{v}_1^2(t)dt \leq k^2\int_0^{2\pi} v_1^2(t)dt.$$

We have

$$(1+\frac{1}{k^2})\int_0^{2\pi}\dot{v}_1^2(t)dt \leq \|v_1\|^2 = \int_0^{2\pi}[\dot{v}_1^2(t)+v_1^2(t)]dt \leq 2\int_0^{2\pi}\dot{v}_1^2(t)dt. \tag{3.26}$$

Then we obtain by (3.21), (3.22) and (3.26) that, for $v = v_0 + v_1 \in V$,

$$\begin{aligned} I(v) &= \int_0^{2\pi}\frac{1}{2}[|\dot{v}|^2 - \frac{1}{2}(p-\epsilon_1)v^2 + a_{\epsilon_1}(t)]dt. \\ &= \frac{1}{2}\int_0^{2\pi}|\dot{v}_1|^2dt - \frac{1}{2}(p-\epsilon_1)[\int_0^{2\pi}v_1^2dt + \int_0^{2\pi}v_0^2dt] + \|a_{\epsilon_1}\|_{L^2(0,2\pi)} \\ &\leq \frac{1}{2}(1-\frac{p-\epsilon_1}{k^2})\int_0^{2\pi}|\dot{v}_1|^2dt - \frac{1}{2}(p-\epsilon_1)\int_0^{2\pi}v_0^2dt + \|a_{\epsilon_1}\|_{L^2(0,2\pi)} \\ &\leq \frac{1}{4}(1-\frac{p-\epsilon_1}{k^2})\|v_1\|^2 - \frac{1}{2}(p-\epsilon_1)\|v_0\|^2 + \|a_{\epsilon_1}\|_{L^2(0,2\pi)} \\ &\leq -\delta_2\|v\|^2 + \|a_{\epsilon_1}\|_{L^2(0,2\pi)}, \end{aligned}$$

where $\delta_2 = \frac{1}{4}(\frac{p-\epsilon_1}{k^2}-1) > 0$. Hence, as $\|v\| \to \infty$, $I(v) \to -\infty$.

By (1.7) and (1.8), we can show as in the proof of Theorem 1.1 that I satisfies the (PS) condition. Consequently, the functional I satisfies the hypotheses of the saddle point theorem. Hence, using Lemma 1.1 of [22], if I has finite critical points which are all nondegenerate, then there exists a critical x^* with Morse index equal to $\dim V = k+1 \geq 2$. Moreover, using that 0 is a local minimum of I and $I(0) = 0$, it follows that x^* is nontrivial. On the other hand, by Lemma 3.2, I has a nontrivial nonnegative solution x^+ and a nontrivial nonpositive solution x^-. Since x^+ and x^- are all nondegenerate and of mountain pass type, by [19] it follows that x^+ and x^- are all of Morse index less that or equal to 1. Then

$$x^* \neq x^+, \ x^* \neq x^-.$$

Now, similar as the arguments in the proof of Theorem 1.2(i), we can take $\tau > 0$ small enough and $\rho \in (0, \rho_0)$, $r \in (0, r_1)$, $d \in (0, d_1)$ and $R \geq R_0$ such that (3.20) holds and

$$B_\tau(x^*) \cap B_r(x^+) = \emptyset, \ B_\tau(x^*) \cap B_d(x^-) = \emptyset, \ B_\tau(x^*) \cap B_\rho = \emptyset$$

and

$$\bar{B}_\tau(x^*), \bar{B}_\rho, \bar{B}_r(x^+), \bar{B}_d(x^-) \subset B_R.$$

By the additivity and excision properties of the Leray-Schauder degree, we have

$$\begin{aligned} \deg(\nabla I, B_R, 0) = \ & \deg(\nabla I, B_\tau(x^*), 0) + \deg(\nabla I, B_\rho, 0) + \deg(\nabla I, B_r(x^+), 0) \\ & + \deg(\nabla I, B_d(x^-), 0) \\ & + \deg(\nabla I, B_R \setminus (\overline{B_\tau(x^*) \cup B_\rho \cup B_r(x^+) \cup B_d(x^-)}), 0). \end{aligned}$$

Note that all the critical points of I are nongenerate, we can see that

$$|\deg(\nabla I, B_\tau(x^*), 0)| = 1.$$

Then, using Lemmas 3.1, 3.2, 3.4 and the excision property of the Leray-Schauder degree, we obtain

$$\deg(\nabla I, B_R \setminus (\overline{B_\rho \cup B_r(x^+) \cup B_d(x^-)}), 0) \neq 0.$$

So, by the existence property of the Leray-Schauder degree it follows that there exists $x_4 \in B_R \setminus (\overline{B_\tau(x^*) \cup B_\rho \cup B_r(x^+) \cup B_d(x^-)})$ such that $\nabla I(x_4) = 0$. Thus problem (1.1) has at least four nontrivial solutions: x^+, x^-, x^*, x_4. □

Before proving Theorem 1.2(iii), we recall a global version of the Lyapunov-Schmidt method.

Lemma 3.6 ([3]). *Let H be a real separable Hilbert space. Let V and W be closed subspaces of H such that $H = V \oplus W$. Assume that $J \in C^1(H, \mathbb{R})$. If there are $\mu > 0$ and $\tau > 1$ such that*

$$[\nabla J(v + w_1) - \nabla J(v + w_2)](w_1 - w_2) \geq \mu \|w_1 - w_2\|^\tau \text{ for all } v \in V, w_1, w_2 \in W,$$

then we have the following.

(i) *there exists $\psi \in C(V, W)$ such that*

$$J(v + \psi(v)) = \min_{w \in W} J(v + w).$$

Moreover, $\psi(x)$ is the unique member of W such that

$$[\nabla J(v + \psi(v)](w) = 0 \text{ for all } w \in W;$$

(ii) *if we define $\bar{J}(v) = J(v + \psi(v))$, then $\bar{J} \in C^1(V, \mathbb{R})$ and*

$$[\nabla \bar{J}(v)](v_1) = [\nabla J(v + \psi(v)](v_1) \text{ for all } v, v_1 \in V;$$

(iii) *An element $v \in V$ is a critical point of $\bar{J}$ if and only if $v + \psi(v)$ is a critical point of J;*

(iv) *Let* $dim X < \infty$ *and* P *be the projection onto* V *across* W*. Let* $S \subset V$ *and* $D \subset H$ *be open bounded regions such that*

$$\{v + \psi(v) \mid v \in S\} = D \cap \{v + \psi(v) \mid v \in V\}.$$

If $\nabla \bar{J}(v) \neq 0$ *for* $v \in \partial S$*, then*

$$\deg(\nabla \bar{J}, S, 0) = \deg(\nabla J, D, 0);$$

(v) *If* $x_0 = v_0 + w_0$ *is a critical point of mountain pass type of* J*, then* v_0 *is a critical point of mountain pass type of* $\bar{J}$*.*

Proof of Theorem 1.2(iii)*.* Define the functional $\eta(w) : W \to \mathbb{R}$ by

$$\eta(w) = \int_0^1 [|\dot{w}|^2 - \beta w^2]dt, \forall w \in W.$$

Clearly, by $\int_0^{2\pi} |\dot{w}(t)|^2 dt \geq (k+1)^2 \|w\|_2^2$ for all $w \in W$, we have

$$\eta(w) \geq \int_0^1 [(k+1)^2 - \beta] w^2 dt \geq 0, \ \forall w \in W. \tag{3.27}$$

We claim that there exist $\mu, M > 0$ such that

$$\eta(w) \geq \mu \|w\|^2, \ \|w\| \geq M. \tag{3.28}$$

Indeed, we assume, by contradiction, that (3.28) doesn't hold. Then there exists a sequence $\{w_n\} \subset W$ with $\|w_n\| \to \infty$ such that $\frac{\eta(w_n)}{\|w_n\|^2} \to 0$ as $n \to \infty$. Let $y_n = \frac{w_n}{\|w_n\|}$. Then $\|y_n\| = 1$. Passing, if necessary, to a subsequence we assume that $y_n \rightharpoonup y_0 \in W$ weakly and $y_n \to y_0$ in $C[0, 2\pi]$. By $y_n, y_0 \in W$ it follows that

$$0 \leq \eta(y_0) \leq \liminf \eta(y_n) = 0, \tag{3.29}$$

which implies that

$$0 = \eta(y_0) \geq \int_0^{2\pi} [\dot{y}_0^2 - (k+1)^2 y_0^2] dt \geq 0.$$

Thus y_0 is an eigenfunction corresponding to $(k+1)^2$. Note that (1.9), (3.27) and (3.29) imply that $y_0 \equiv 0$. A contradiction. Thus (3.28) holds. Now by the mean value theorem and (1.9) it follows that

$$\begin{aligned}[\nabla I(v+w) - \nabla I(v+w_1)](w - w_1) &\geq \int_0^{2\pi} [|\dot{w} - \dot{w}_1|^2 - \beta(w - w_1)^2] dt \\ &\geq \mu \|w - w_1\|^2.\end{aligned}$$

Then by Lemma 3.6 there exists $\psi : V \to W$ such that

$$I(v + \psi(v)) = \min_{w \in W} I(v + w).$$

Moreover, $\psi(v)$ is the unique element of W such that

$$[\nabla I(v + \psi(v))](w) = 0 \text{ for all } w \in W.$$

Define $\bar{I} : V \to \mathbb{R}$ by

$$\bar{I}(v) = I(v + \psi(v)).$$

Then $\bar{I}$ is of class C^1, and

$$[\nabla \bar{I}(v)](v_1) = [\nabla I(v + \psi(v))(x)](x_1) \text{ for all } v, v_1 \in V.$$

By (3.25), $I(v) \to -\infty$ as $\|v\| \to \infty$. Then in view of $\bar{I}(v) \leq I(v)$, we can obtain that

$$\bar{I}(v) \to -\infty \text{ as } \|v\| \to \infty.$$

Since V is of finite dimension, there exists $v_0 \in V$ such that

$$\bar{I}(v_0) = \max_{v \in V} I(v + \psi(v)). \tag{3.30}$$

Then $x_0 = v_0 + \psi(v_0)$ is a critical point of I, i.e., $\nabla I(x_0) = 0$. Suppose that v_0 is an isolated critical point of $\bar{I}$, so x_0 is an isolated critical point of I. By (3.30), v_0 is a strictly local maximum of the functional $\bar{I}$. Then there exists $\hat{v}$ in some neighborhood S_0 of v_0 such that $\bar{I}(\hat{v}) < \bar{I}(v_0)$, i.e.,

$$I(\hat{v} + \psi(\hat{v})) < I(v_0 + \psi(v_0)),$$

which means that x_0 can't be local minimum of the functional I. Thus x_0 is nontrivial. On the other hand, if we denote $x^+ = v^+ + \psi(v^+)$, then by Lemma 3.2 we can see that v^+ is a critical point of mountain pass type of $\bar{I}$, which implies $x_0 \neq x^+$. Similarly, $x_0 \neq x^-$. Furthermore, denoting $B_\sigma(x_0) = \{x \in H^1_{[0,2\pi]} \mid \|x - x_0\| \leq \sigma\}$, by (3.30) there exists $\sigma_0 > 0$ small such that

$$\deg(\nabla \bar{I}, B_\sigma(x_0) \mid_V, 0) = (-1)^k \text{ for all } 0 < \sigma \leq \sigma_0. \tag{3.31}$$

Hence by Lemma 3.6 it follows that

$$\deg(\nabla I, B_\sigma(x_0), 0) = (-1)^k \text{ for all } 0 < \sigma \leq \sigma_0. \tag{3.32}$$

Now similar arguments as in the proof of Theorem 1.2(ii) implies that there exists at least a nontrivial critical point x_4 of I that different from x_0, x^+, x^-. This completes the proof. □

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