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MINIMUM RANK OF THE LINE GRAPH OF CORONA $C_n \circ K_t$

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Abstract. The minimum rank $\mathrm{mr}(G)$ of a simple graph G is defined to be the smallest possible rank over all symmetric real matrices whose (i,j)-th entry (for $i \neq j$) is nonzero whenever $\{i,j\}$ is an edge in G and is zero otherwise. The corona $C_n \circ K_t$ is obtained by joining all the vertices of the complete graph K_t to each n vertex of the cycle C_n. For any t, we obtain an upper bound of zero forcing number of $L(C_n \circ K_t)$, the line graph of $C_n \circ K_t$, and get some bounds of $\mathrm{mr}(L(C_n \circ K_t))$. Specially for $t = 1, 2$, we have calculated $\mathrm{mr}(L(C_n \circ K_t))$ by the cut-vertex reduction method.

1. Introduction and preliminaries

Let S_n denote the set of real symmetric $n \times n$ matrices. A *graph* $G = (V, E)$ means a simple undirected graph (an edge is a two-element subset of vertices). For $A = (a_{i,j}) \in S_n$, the *graph of* A, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i,j\} \,|\, a_{i,j} \neq 0 \text{ and } i \neq j\}$. Note that the diagonal of A is ignored in determining $\mathcal{G}(A)$. The set of *symmetric matrices described by* G is $\mathcal{S}(G) = \{A \in S_n : \mathcal{G}(A) = G\}$. The *minimum rank* of the graph G is

$$\mathrm{mr}(G) = \min\{\mathrm{rank}\, A : A \in \mathcal{S}(G)\},$$

and the *maximum nullity* of the graph G is

$$\mathrm{M}(G) = \max\{\mathrm{null}(A) : A \in \mathcal{S}(G)\}.$$

A graph $G' = (V', E')$ is a *subgraph* of the graph $G = (V, E)$ if $V' \subseteq V$, $E' \subseteq E$. The subgraph $G[R]$ of $G = (V, E)$ *induced* by $R \subseteq V$ is the subgraph with vertex set R and edge set $\{\{i,j\} \in E \,|\, i, j \in R\}$.

Given a graph G, its *line graph* $L(G)$ is a graph for which each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint ("are adjacent ") in G.

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A *complete graph* is a graph $K_n = (\{v_1, \dots, v_n\}, E)$ such that $E = \{\{v_i, v_j\} : 1 \leq i < j \leq n\}$.

A subgraph G' of a graph G is a *clique* if G' has an edge between every pair of vertices of G' (i.e., G' is isomorphic to $K_{|G'|}$). A set of subgraphs of G, each of which is a clique and every edge of G is contained in at least one of these cliques, is called a *clique covering* of G. The *clique covering number* of G, denoted by $\text{cc}(G)$, is the smallest cardinality of a clique covering of G among all clique covering of G. The following observation is well known and straightforward.

Observation 1.1 ([4])**.**
(1) $\text{mr}(G) + \text{M}(G) = |G|$.
(2) *If* G' *is an induced subgraph of* G*, then* $\text{mr}(G') \leq \text{mr}(G)$.
(3) [6] *If* G' *is obtained from* G *by deleting a single vertex and each incident edge, then* $\text{mr}(G') \leq \text{mr}(G) \leq \text{mr}(G') + 2$.
(4) *If* G *is a graph,* $\text{mr}(G) \leq \text{cc}(G)$.

Theorem 1.2 ([1])**.**
(1) $\text{mr}(L(K_n)) = n - 2$.
(2) *If a graph* G *has* $n \geq 2$ *vertices and contains a Hamiltonian path, then* $mr(L(G)) = n - 2$.

A vertex v of a connected graph G is a *cut-vertex* if $G - v$ disconnected. More generally, v is a cut-vertex of a graph G if v is a cut-vertex of a component of G. The *rank-spread* of G at vertex v is $r_v(G) = \text{mr}(G) - \text{mr}(G - v)$. As noted in Observation 1.1(3), for any vertex v of G, we have $0 \leq r_v(G) \leq 2$.

Theorem 1.3 ([3, 5](cut-vertex reduction))**.** *If* G *has a cut-vertex, the problem of computing the minimum rank of* G *can be reduced to computing minimum ranks of certain subgraphs. Specially, let* v *be a cut-vertex of* G*. For* $i = 1, \dots, h$*, let* $W_i \subseteq V(G)$ *be the vertices of the* i*-th component of* $G - v$ *and let* G_i *be the subgraph induced by* $\{v\} \cup W_i$*. Then*

$$r_v(G) = \min\left\{\sum_1^h r_v(G_i), 2\right\}$$

and thus

$$\text{mr}(G) = \sum_1^h \text{mr}(G_i - v) + \min\left\{\sum_1^h r_v(G_i), 2\right\}.$$

Let G be a graph for which each vertex colored either white or black. Vertices change color according to the *color-change rule*: if u is a black vertex and exactly one neighbor w of u is white, then change the color of w to black. When the color-change rule is applied to u to change the color of w, we say u *forces* w and write $u \to w$. Given a coloring of G, the *derived set* is the set of black vertices obtained by applying the color-change rule until no more color-changes are possible. The set Z is said to be a *zero forcing set* of G if all vertices of G will be turned black after finitely many applications of the color-change rule. The *zero forcing number* $Z(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V(G)$.

Theorem 1.4 ([1]). *For any graph G,* $\mathrm{M}(G) \leq Z(G)$.

In Theorem 2.2, we calculate the minimum rank of $L(C_n \circ K_t)$ for $t = 1, 2$ by the cut-vertex reduction method and Lemma 2.1. For the case of t greater than or equal to 3, the cut-vertex reduction method is not suitable, since $L(C_n \circ K_t)$ is complicated. So we obtain the upper bound of zero forcing number of $L(C_n \circ K_t)$ in Theorem 2.4, which is heavily used to obtain the lower bound of minimum rank of $L(C_n \circ K_t)$ in Theorem 2.5.

2. Minimum rank of the line graph of corona $C_n \circ K_t$

The *corona* of G with H, denoted $G \circ H$, is the graph of order $|G||H| + |G|$ obtained by taking one copy of G and $|G|$ copies of H, and joining all the vertices in the i-th copy of H to the i-th vertex of G. An *n-ciclo* of G with an edge e, denoted $C_n(G, e)$, is constructed from an n-cycle C_n and n copies of G by identifying each edge of C_n with the edge e in one copy of G. If a symbol for the graph identifies a specific edge, or if G is edge transitive (so it is not necessary to specify edge e), then the notation $C_n(G)$ is used. A vertex on C_n is called a *cycle vertex* [2].

Lemma 2.1. $\mathrm{mr}(L(P_2 \circ K_2)) = 4$.

Proof. Since $P_2 \circ K_2$ contains a Hamiltonian path as depicted in Figure 1, we have

$$\mathrm{mr}(L(P_2 \circ K_2)) = |P_2 \circ K_2| - 2 = 4$$

by Theorem 1.2(2). □

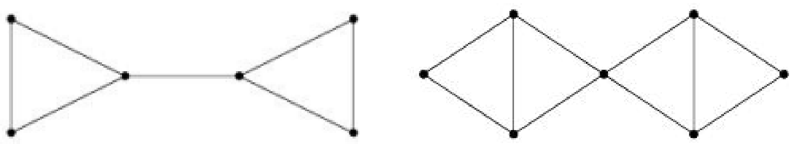

FIGURE 1. The graph $P_2 \circ K_2$ and its line graph.

We remark that each numbering of vertex i in Figure 3 and Figure 4 will be denoted by v_i in the proof of the following theorem to avoid a confusion.

Theorem 2.2. $\mathrm{mr}(L(C_n \circ K_t)) = tn$, *where* $t = 1, 2$.

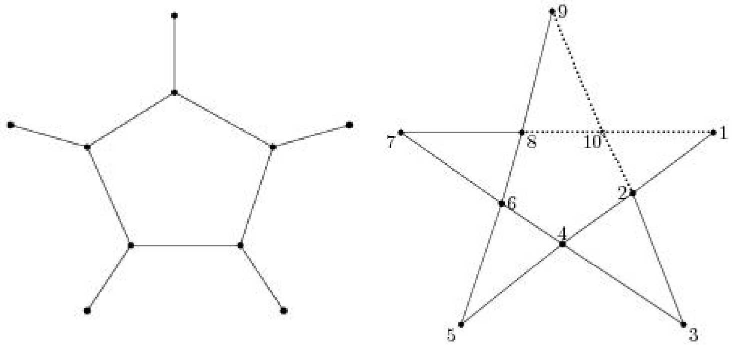

FIGURE 2. The graph $C_5 \circ K_1$ and its line graph.

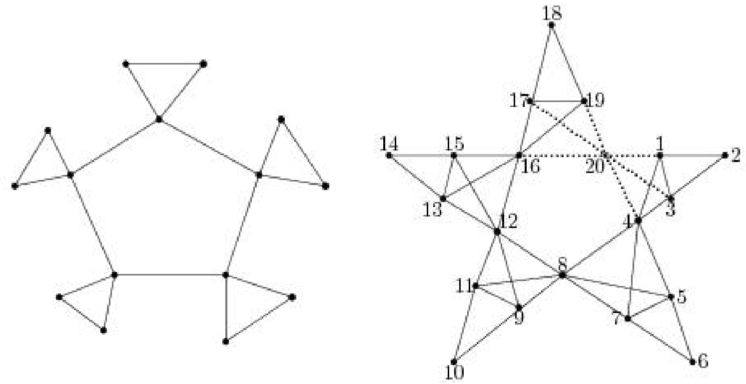

FIGURE 3. The graph $C_5 \circ K_2$ and its line graph.

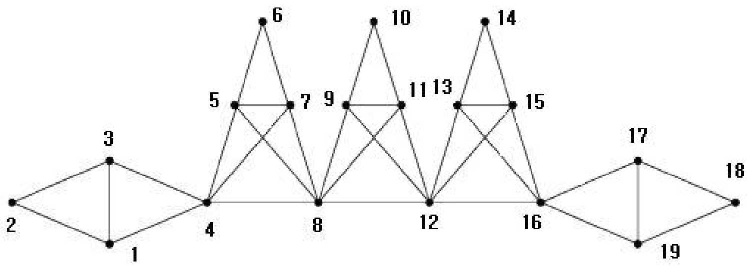

FIGURE 4. The graph $G^5 := L(C_5 \circ K_2) - v_{20}$.

Proof. (Case 1) $t = 1$: See Figure 2 for the case of $n = 5$. The line graph of $C_n \circ K_1$ has an induced subgraph P_{n+1}. So we get $\text{mr}(P_{n+1}) = n \leq \text{mr}(L(C_n \circ K_1))$

by Observation 1.1(2). Moreover, we note that the clique covering number of $L(C_n \circ K_1)$ is n. Hence by Observation 1.1(4) we now have the result.

(Case 2) $t = 2$: By Observation 1.1(4), $\mathrm{mr}(L(C_n \circ K_2)) \leq \mathrm{cc}(L(C_n \circ K_2)) = 2n$. And by Observation 1.1(2), it is enough to show that the minimum rank of the induced subgraph $L(C_n \circ K_2) - v_{4n}$ of $L(C_n \circ K_2)$ is precisely $2n$.

For our convenience, we denote $L(C_n \circ K_2) - v_{4n}$ by G^n. Let us take the vertex v_{4n-4} as a cut-vertex, then we have two induced subgraphs $G_1^n := G^n[\{v_1, v_2, \ldots, v_{4n-4}\}]$ and $G_2^n := G^n[\{v_{4n-4}, v_{4n-3}, v_{4n-2}, v_{4n-1}\}]$. By cut-vertex reduction method of Theorem 1.3, we have $\mathrm{mr}(G^n)$ is precisely $\mathrm{mr}(G_1^n - v_{4n-4}) + \mathrm{mr}(G_2^n - v_{4n-4}) + \min\{2, r_{v_{4n-4}}(G_1^n) + r_{v_{4n-4}}(G_2^n)\}$. By G_1^{n-i} we denote $G^n[\{v_1, v_2, \ldots, v_{4(n-i-1)}\}]$ and by G_2^{n-i} we denote $G^n[\{v_{4(n-i-1)}, v_{4(n-i-1)+1}, v_{4(n-i-1)+2}, v_{4(n-i-1)+3}\}]$, where $1 \leq i \leq n-3$. And by G^{n-i} we denote $G_1^{n-i+1} - v_{4(n-i)}$, where $1 \leq i \leq n-3$, then since G^{n-i} has a cut-vertex $v_{4(n-i-1)}$, G_1^{n-i} and G_2^{n-i} are two induced subgraphs of G^{n-i}. Then by Theorem 1.3, we get $\mathrm{mr}(G^{n-i}) = \mathrm{mr}(G_1^{n-i} - v_{4(n-i-1)}) + \mathrm{mr}(G_2^{n-i} - v_{4(n-i-1)}) + \min\{2, r_{v_{4(n-i-1)}}(G_1^{n-i}) + r_{v_{4(n-i-1)}}(G_2^{n-i})\}$.

Since $G^{n-1} = G_1^n - v_{4(n-1)}$, the graph G^{n-1} has a cut-vertex $v_{4(n-2)}$ and two induced subgraphs $G_1^{n-1} = G^n[\{v_1, v_2, \ldots, v_{4(n-2)}\}]$ and $G_2^{n-1} = G^n[\{v_{4(n-2)}, v_{4(n-2)+1}, v_{4(n-2)+2}, v_{4(n-2)+3}\}]$. Then we get

$$\begin{aligned}\mathrm{mr}(G^{n-1}) = &\mathrm{mr}(G_1^{n-1} - v_{4(n-2)}) + \mathrm{mr}(G_2^{n-1} - v_{4(n-2)}) \\ &+ \min\{2, r_{v_{4(n-2)}}(G_1^{n-1}) + r_{v_{4(n-2)}}(G_2^{n-1})\}.\end{aligned}$$

Continuing in this way, we have

$$\mathrm{mr}(G^3) = \mathrm{mr}(G_1^3 - v_8) + \mathrm{mr}(G_2^3 - v_8) + \min\{2, r_{v_8}(G_1^3) + r_{v_8}(G_2^3)\}.$$

Note that $\mathrm{mr}(G_1^3 - v_8) = \mathrm{mr}(G^3[\{v_1, v_2, \ldots, v_7\}]) = \mathrm{mr}(L(P_2 \circ K_2)) = 4$ by Lemma 2.1, $\mathrm{mr}(G_2^3 - v_8) = \mathrm{mr}(G^3[\{v_9, v_{10}, v_{11}\}]) = \mathrm{mr}(K_3) = 1$, $r_{v_8}(G_1^3) = \mathrm{mr}(G_1^3) - \mathrm{mr}(G_1^3 - v_8) = 0$ and $r_{v_8}(G_2^3) = \mathrm{mr}(G_2^3) - \mathrm{mr}(G_2^3 - v_8) = 2 - 1 = 1$. Hence $\mathrm{mr}(G^3) = 4 + 1 + 1 = 6$. Moreover $\mathrm{mr}(G_2^{n-i} - v_{n-i+1}) = \mathrm{mr}(K_3) = 1$, $r_{v_{n-i+1}}(G_1^{n-i}) = 0$ and $r_{v_{n-i+1}}(G_2^{n-i}) = 1$, for $1 \leq i \leq n-3$. So we have $\mathrm{mr}(G^n) = 4 + (n-2)(1+1) = 2n$. □

Now let us consider the minimum rank of $L(C_n \circ K_t)$ for any t, not just for $t = 1, 2$ as in the above theorem, in which we applied the cut-vertex reduction method, as well as Lemma 2.1 for the proof. The case of $t = 2$ is treated in [2], where $L(C_n \circ K_2)$ is called the *full house ciclo*. Various bounds of minimum rank, maximum nullity and zero forcing number are obtained to find their exact values in [2]. For the case of t greater than or equal to 3 the cut-vertex reduction is too complicated, hence we calculate the upper bound of zero forcing number of $L(C_n \circ K_t)$ by the ordinary color change rule in Theorem 2.4 and obtain the lower and upper bound of minimum rank of $L(C_n \circ K_t)$ for any t in Theorem 2.5.

The *t,k-pineapple* (with $t \geq 3, k \geq 2$) is $P_{t,k} = K_t \cup K_{1,k}$ such that $K_t \cap K_{1,k}$ is the vertex of $K_{1,k}$ of degree k. Note that $L(C_n \circ K_t)$ is the ciclo $C_n(G)$, where G is the line graph of the $t+1, 2$-*pineapple* $P_{t+1,2}$. So we can draw the Figure 5 for the case $t = 3$.

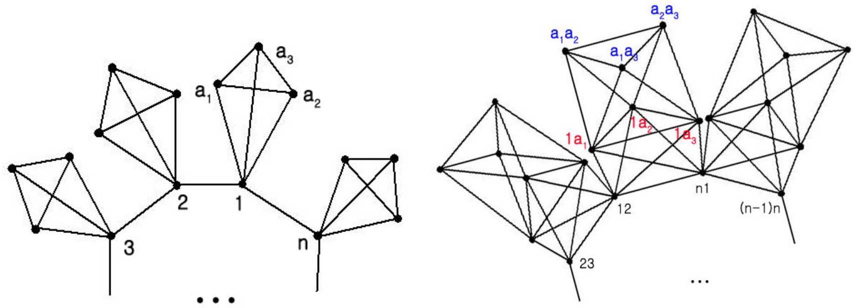

FIGURE 5. The graph $C_n \circ K_3$ and its line graph.

As shown in Figure 5 for the special case of $t = 3$, each vertex of $L(C_n \circ K_t)$ must belong to one of the following 3 types:

(1) n vertices $12, 23, \ldots, (n-1)n, n1$ obtained from all edges of C_n;
(2) $t \times n$ vertices $ia_1, ia_2, \ldots, ia_t$ obtained from all edges which connect the i-th K_t-copy to each vertex i of C_n, where $i = 1, 2, \ldots, n$;
(3) $\binom{t}{2}$ vertices $a_1a_2, a_1a_3, \ldots, a_{t-1}a_t$ obtained from $\binom{t}{2}$ edges of each K_t-copy, so all together $\binom{t}{2} \times n$ vertices of this type.

Note that all vertices of each type has the identical degree. In fact, the degree of the first type is $2(t+1)$, the second type is $(t-1) + (t-1) + 2 = 2t$ and third type is $2(t-1)$.

Theorem 2.3. *For $t \geq 2$, the zero forcing number $Z(L(P_{t+1,2}))$ of the line graph of the $t+1, 2$-pineapple $P_{t+1,2}$ is less than or equal to $|L(P_{t,2})| = \binom{t}{2} + 2$.*

Proof. We claim the set $Z = \{12, 1a_1, a_1a_2, \ldots, a_{t-1}a_t\}$ is a zero forcing set. Indeed, each vertex of type (3) is adjacent to two vertices of type (2). Since all vertices of type (3) and a vertex $1a_1$ of type (2) are black, the other vertices $1a_2, 1a_3, \ldots, 1a_t$ of type (2) are forced. And each vertex of type (2) is adjacent two vertices of type (1). Since all vertices of type (2) and a vertex 12 of type (1) are black, the other vertex $n1$ is forced. So the set Z is a zero forcing set and $Z(L(P_{t+1,2})) \leq |Z| = \binom{t}{2} + 2$. □

Theorem 2.4. *The zero forcing number $Z(L(C_n \circ K_t))$ of line graph of $C_n \circ K_t$ is less than or equal to $n(\binom{t}{2} + 1)$.*

Proof. Our line graph $L(C_n \circ K_t)$ is the union of n copies of G of Theorem 2.3. Each G-copy includes one of the n-cycle edges $\{12, 23\}$, $\{23, 34\}$, ...,

$\{n1, 12\}$. We force the first G-copy, say G_1, with the edge $\{12, 23\}$ of C_n, then the second G-copy, G_2, with the edge $\{23, 34\}$ of C_n, and continue this way to end up forcing the last G-copy, G_n, with the edge {n1,12} of C_n. We claim that $Z(L(C_n \circ K_t)) = Z(G) + (n-2) \times \{Z(G) - 1\} + \{Z(G) - 2\}$. Indeed, since $G_i \cap G_{i+1}$ is the single vertex $(i+1)(i+2)$ which is a cycle vertex (of type (3)), where $i = 1, 2, \ldots, n-1$. We have that $Z(G_1 \cup G_2)$ is precisely $Z(G_1) + \{Z(G_2) - 1\}$ and that $Z(G_1 \cup G_2 \cup \cdots \cup G_{n-1}) = Z(G_1) + \{Z(G_2) - 1\} + \cdots + \{Z(G_{n-1}) - 1\} = Z(G) + (n-2)\{Z(G) - 1\}$. When we reach to force the last G_n, since we have already made two cycle vertices n1,12 of G_n into black, the zero forcing number $Z(L(C_n \circ K_t))$ goes up only $Z(G_1) - 2$ from $Z(G_1 \cup \cdots \cup G_{n-1})$. Thus we obtain the result. □

The following theorem is a generalized version of Theorem 2.2 for any t.

Theorem 2.5. *For $n \geq 3$, $nt \leq \mathrm{mr}(L(C_n \circ K_t)) \leq nt + n - 2$ and $\mathrm{mr}(L(C_n \circ K_t)) - \mathrm{mr}(L(K_n)) \leq nt$.*

Proof. To obtain the upper bound of $\mathrm{mr}(L(C_n \circ K_t))$, it is well known, that if H is a subgraph of G (not-necessarily induced), then $L(H)$ is an induced subgraph of $L(G)$. Since $|C_n \circ K_t| = n(t+1)$, $L(C_n \circ K_t)$ is an induced subgraph of $L(K_{n(t+1)})$. So we get

$$\mathrm{mr}(L(C_n \circ K_t)) \leq \mathrm{mr}(L(K_{n(t+1)})) = n(t+1) - 2$$

by Observation 1.1(2) and Theorem 1.2(1).

Now for the lower bound of $\mathrm{mr}(C_n \circ K_t)$, we have

$$\mathrm{mr}(L(C_n \circ K_t)) = |L(C_n \circ K_t)| - M(L(C_n \circ K_t))$$

by Observation 1.1(1). Then by Theorem 1.4, we have

$$\mathrm{mr}(L(C_n \circ K_t)) \geq |L(C_n \circ K_t)| - Z(L(C_n \circ K_t)).$$

Note that $|L(C_n \circ K_t)| = n\left(\binom{t}{2} + t + 1\right)$. Therefore

$$\mathrm{mr}(L(C_n \circ K_t)) \geq (t+1)n + n\binom{t}{2} - n\left(\binom{t}{2} + 1\right) = nt$$

by Theorem 2.4. □

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