Commun. Korean Math. Soc. **30** (2015), No. 2, pp. 103–108
http://dx.doi.org/10.4134/CKMS.2015.30.2.103

A SUMMATION FORMULA FOR THE SERIES $_3F_2$ DUE TO FOX AND ITS GENERALIZATIONS

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Abstract. Fox [2] presented an interesting identity for $_pF_q$ which is expressed in terms of a finite summation of $_pF_q$'s whose involved numerator and denominator parameters are different from those in the starting one. Moreover Fox [2] found a very interesting and general summation formula for $_3F_2(1/2)$ as a special case of his above-mentioned general identity with the help of Kummer's second summation theorem for $_2F_1(1/2)$. Here, in this paper, we show how two general summation formulas for

$$ {}_3F_2\left[\begin{matrix} \alpha,\, \beta,\, \gamma; \\ \alpha - m,\, \frac{1}{2}(\beta+\gamma+i+1); \end{matrix} \;\frac{1}{2}\right], $$

m being a nonnegative integer and i any integer, can be easily established by suitably specializing the above-mentioned Fox's general identity with, here, the aid of generalizations of Kummer's second summation theorem for $_2F_1(1/2)$ obtained recently by Rakha and Rathie [7]. Several known results are also seen to be certain special cases of our main identities.

1. Introduction and preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{C}$, and $\mathbb{Z}_0^-$ denote the sets of positive integers, complex numbers, and nonpositive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The generalized hypergeometric series $_pF_q$ with p numerator parameters and q denominator parameters is defined by (see [1], [5, p. 73], [8, p. 40] and [9, pp. 71–75]):

$$ \begin{aligned} {}_pF_q\left[\begin{matrix} \alpha_1,\, \ldots,\, \alpha_p; \\ \beta_1,\, \ldots,\, \beta_q; \end{matrix}\; z\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \\ &= {}_pF_q(\alpha_1,\, \ldots,\, \alpha_p;\, \beta_1,\, \ldots,\, \beta_q;\, z), \end{aligned} \tag{1.1} $$

Received February 15, 2015.
2010 *Mathematics Subject Classification.* Primary 33B20, 33C20; Secondary 33B15, 33C05.
Key words and phrases. Gamma function, Pochhammer symbol, hypergeometric function, generalized hypergeometric function, Kummer's second summation theorem, Fox's identity.

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [9, p. 2 and p. 5]):

$$(\lambda)_n := \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases} \\ = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-) \tag{1.2}$$

and $\Gamma(\lambda)$ is the familiar Gamma function defined by

$$\Gamma(s) = \int_0^\infty e^{-x}\, x^{s-1}\, dx \quad (\Re(s) > 0). \tag{1.3}$$

Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume (for simplicity) that the variable z, the numerator parameters $\alpha_1, \ldots, \alpha_p$, and the denominator parameters $\beta_1, \ldots, \beta_q$ take on complex values, provided that no zeros appear in the denominator of (1.1), that is, that

$$(\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ j = 1, \ldots, q). \tag{1.4}$$

Whenever certain hypergeometric and generalized hypergeometric functions reduce to be expressed in terms of gamma functions, the results are very important from the application point of view. Therefore the classical summation theorems such as those of Kummer's first, second and third for the series ${}_2F_1$, and Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$ and others play key roles in the theory of generalized hypergeometric functions. It is well known that the above-mentioned summation theorems have been widely applied. Here we begin with recalling Kummer's second summation theorem:

$$ {}_2F_1\left[\begin{matrix} a,\ b; \\ \frac{1}{2}(a+b+1); \end{matrix}\ \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}. \tag{1.5}$$

In 1927, Fox [2] obtained the following interesting general result:

$$\begin{aligned} &{}_pF_q\left[\begin{matrix} \alpha_1,\ \ldots,\ \alpha_p; \\ \beta_1-m,\ \beta_2,\ \ldots,\ \beta_q; \end{matrix}\ x\right] \\ &= \frac{\Gamma(\beta_1)\,\Gamma(\beta_1-m)\,\Gamma(\beta_2)\cdots\Gamma(\beta_q)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\cdots\Gamma(\alpha_p)} \\ &\quad\times \sum_{r=0}^{m} \frac{x^r}{\Gamma(r+\beta_1-m)}\binom{m}{r}\frac{\Gamma(\alpha_1+r)\cdots\Gamma(\alpha_p+r)}{\Gamma(\beta_1+r)\cdots\Gamma(\beta_q+r)} \\ &\quad\times {}_pF_q\left[\begin{matrix} \alpha_1+r,\ \ldots,\ \alpha_p+r; \\ \beta_1+r,\ \ldots,\ \beta_q+r; \end{matrix}\ x\right] \quad (m \in \mathbb{N}). \end{aligned} \tag{1.6}$$

Using (1.5) and (1.6), Fox [2] found the following interesting summation formula:

$$(1.7)\quad {}_3F_2\left[\begin{matrix}\alpha,\ \beta,\ \gamma;\\ \alpha-m,\ \frac{1}{2}(\beta+\gamma+1);\end{matrix}\ \frac{1}{2}\right] = \frac{\Gamma(\alpha-m)\,\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\gamma+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\,\Gamma(\beta)\,\Gamma(\gamma)} \times 2^{\beta+\gamma-2}\sum_{r=0}^{m}\binom{m}{r}\frac{2^r\,\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}r\right)\,\Gamma\left(\frac{1}{2}\gamma+\frac{1}{2}r\right)}{\Gamma(r+\alpha-m)}\quad (m\in\mathbb{N}).$$

Recently a good deal of progress has been made in the direction of generalizing the classical summation theorems. In this regard, for example, we refer to two papers [3, 7]. Here, in this paper, we aim at establishing two general summation formulas for

$${}_3F_2\left[\begin{matrix}\alpha,\ \beta,\ \gamma;\\ \alpha-m,\ \frac{1}{2}(\beta+\gamma+i+1);\end{matrix}\ \frac{1}{2}\right]\quad (m\in\mathbb{N}_0)\,,$$

where i is any integer which, for simplicity, is partitioned into two sets of nonnegative and negative ones, by suitably specializing the above-mentioned Fox's general identity (1.6) with, here, the aid of generalizations of Kummer's second summation theorem for ${}_2F_1(1/2)$ (1.8) and (1.9). Several known results are also seen to be certain special cases of our main identities.

For our purpose, we need to recall the following known summation formulas due to Rakha and Rathie [7]:

$$(1.8)\quad {}_2F_1\left[\begin{matrix}a,\ b;\\ \frac{1}{2}(a+b+i+1);\end{matrix}\ \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}b\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)} \times\sum_{r=0}^{i}\binom{i}{r}(-1)^r\frac{\Gamma\left(\frac{1}{2}b+\frac{1}{2}r\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2}\right)}\quad (i\in\mathbb{N}_0)$$

and

$$(1.9)\quad {}_2F_1\left[\begin{matrix}a,\ b;\\ \frac{1}{2}(a+b-i+1);\end{matrix}\ \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}b\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}\sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma\left(\frac{1}{2}b+\frac{1}{2}r\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2}\right)}\quad (i\in\mathbb{N}_0)\,.$$

It is interesting to observe that the special case of both (1.8) and (1.9) when $i = 0$ immediately yields the Kummer's second summation theorem (1.5).

2. Summation formulas for ${}_3F_2(1/2)$

Here we establish two general summation formulas for ${}_3F_2(1/2)$ asserted by the following two theorems.

Theorem 1. *The following summation formula holds true*: *For* $m \in \mathbb{N}_0$,

$$
{}_3F_2\left[\begin{matrix} \alpha,\ \beta,\ \gamma; \\ \alpha - m,\ \frac{1}{2}(\beta+\gamma+i+1); \end{matrix}\ \frac{1}{2}\right] \tag{2.1}
$$

$$
= \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\alpha-m\right)\,\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\gamma+\frac{1}{2}i+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}\beta-\frac{1}{2}\gamma+\frac{1}{2}-\frac{1}{2}i\right)}{\Gamma\left(\beta\right)\,\Gamma\left(\gamma\right)\,\Gamma\left(\frac{1}{2}\beta-\frac{1}{2}\gamma+\frac{1}{2}i+\frac{1}{2}\right)}
$$

$$
\times \sum_{r=0}^{m}\binom{m}{r}\frac{1}{2^r\,\Gamma(\alpha+r-m)}\,\frac{\Gamma(\beta+r)\,\Gamma(\gamma+r)}{\Gamma\left(\frac{1}{2}\gamma+\frac{1}{2}r\right)\,\Gamma\left(\frac{1}{2}\gamma+\frac{1}{2}r+\frac{1}{2}\right)}
$$

$$
\times \sum_{s=0}^{i}\binom{i}{s}\frac{(-1)^s\,\Gamma\left(\frac{1}{2}\gamma+\frac{1}{2}r+\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}\beta-\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2}s+\frac{1}{2}\right)} \quad (i \in \mathbb{N}_0)\,.
$$

Theorem 2. *The following summation formula holds true*: *For* $m \in \mathbb{N}_0$,

$$
{}_3F_2\left[\begin{matrix} \alpha,\ \beta,\ \gamma; \\ \alpha - m,\ \frac{1}{2}(\beta+\gamma-i+1); \end{matrix}\ \frac{1}{2}\right] \tag{2.2}
$$

$$
= \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\alpha-m\right)\,\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\gamma-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\beta\right)\,\Gamma\left(\gamma\right)}
$$

$$
\times \sum_{r=0}^{m}\binom{m}{r}\frac{1}{2^r\,\Gamma(\alpha+r-m)}\,\frac{\Gamma(\beta+r)\,\Gamma(\gamma+r)}{\Gamma\left(\frac{1}{2}\gamma+\frac{1}{2}r\right)\,\Gamma\left(\frac{1}{2}\gamma+\frac{1}{2}r+\frac{1}{2}\right)}
$$

$$
\times \sum_{s=0}^{i}\binom{i}{s}\frac{\Gamma\left(\frac{1}{2}\gamma+\frac{1}{2}r+\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}\beta-\frac{1}{2}i+\frac{1}{2}s+\frac{1}{2}\right)} \quad (i \in \mathbb{N}_0)\,.
$$

Proof. We prove only (2.1). In the general result (1.6) due to Fox, if we set $p = 3$, $q = 2$, $x = \frac{1}{2}$, $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = \gamma$, $\beta_1 = \alpha - m$ and $\beta_2 = \frac{1}{2}(\beta+\gamma+i+1)$, then, for $i \in \mathbb{N}_0$, it reduces to the following form:

$$
{}_3F_2\left[\begin{matrix} \alpha,\ \beta,\ \gamma; \\ \alpha - m,\ \frac{1}{2}(\beta+\gamma+i+1); \end{matrix}\ \frac{1}{2}\right] \tag{2.3}
$$

$$
= \sum_{r=0}^{m}\binom{m}{r}\frac{1}{2^r\,\Gamma(\alpha+r-m)}\,\frac{\Gamma(\beta+r)\,\Gamma(\gamma+r)}{\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\gamma+\frac{1}{2}i+\frac{1}{2}+r\right)}
$$

$$\times\ {}_2F_1\left[\begin{matrix}\beta+r,\,\gamma+r;\\ \frac{1}{2}(\beta+\gamma+i+1)+r;\end{matrix}\ \frac{1}{2}\right].$$

Now, we observe that the $_2F_1$ appearing on the right-hand side of (2.3) can be evaluated with the help of (1.8). After a little simplification, we are easily led to the right-hand side of our desired formula (2.1). This completes the proof of (2.1).

A same argument as above will be easily seen to establish the formula (2.2) with, this time, instead of (1.8), the aid of (1.9). □

3. Special cases

Here some known results are shown to be special cases of our main results.

(1) Setting $i = 0$ in (2.1) immediately yields Fox's summation formula (1.7).
(2) The special cases of (2.1) and (2.2) when $m = 0$ are seen to give the identities (1.8) and (1.9).
(3) In (2.1), if we take $m = 1$ and $i = 1$, we get a result recently obtained by Rathie and Pogany [6].
(4) Setting $i = 0, 1, \dots, 5$ in (2.1), we get the results which are equivalent to those presented earlier by Kim and Rathie [4].
(5) Setting $i = 0, 1, \dots, 5$ in (2.2), we also get the results which are equivalent to those obtained earlier by Kim and Rathie [4].

We conclude this section by remaking that, similarly, many other interesting summation identities can also be considered as certain special cases of our main results (2.1) and (2.2).

Acknowledgements. This research was supported by the Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology of the Republic of Korea (Grant No. 2010-0011005). Also this work was supported by Dongguk University Research Fund.

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