Commun. Korean Math. Soc. **30** (2015), No. 2, pp. 109–121
http://dx.doi.org/10.4134/CKMS.2015.30.2.109

ON HORIZONTAL LIGHTLIKE HYPERSURFACES OF ROBERTSON-WALKER SPACETIMES

Ximin Liu and Quanxiang Pan

Abstract. In this paper, we investigate horizontal lightlike hypersurfaces of Robertson-Walker spacetimes. Some results involving the unique existence of the screen distribution and the symmetry of the induced Ricci curvature tensor of horizontal lightlike hypersurfaces are presented. We also obtain some properties concerning the symmetry and the parallelism of the second fundamental forms of such lightlike hypersurfaces.

1. Introduction

Recall that a generalized Robertson-Walker spacetime is a Lorentzian warped product of a negative definite one-dimensional base and a Riemnnian manifold as a fiber. This family includes classical Robertson-Walker spacetime (that is, the fiber has constant sectional curvature) and Lorentz-Minkowski spacetime. The generalized Robertson-Walker spacetimes are very important models both from mathematical and physical (cosmological) points of view, for more details see Bondi and Gold [3], Hawking and Ellis [8] and Hoyle [9]. Many authors have studied spacelike hypersurfaces of generalized Robertson-Walker spacetimes by requiring certain conditions on the mean curvatures of the hypersurfaces (see [1, 4, 14]). Recently, degenerate hypersurfaces of generalized Robertson-Walker spacetimes have been investigated by Kang [11]. In this paper, we shall study a special kind of lightlike hypersurfaces in Robertson-Walker spacetimes.

It is well known that the intersection of the normal bundle and the tangent bundle of a hypersurface of a semi-Riemannian manifold may be not trivial, it is more difficult and interesting to study the geometry of lightlike hypersurfaces than non-degenerate hypersurfaces. The two standard methods to deal with the above difficulties were developed by Kupeli [12], Duggal and Bejancu [6, 7], respectively. In general, the Ricci curvature tensor of a lightlike hypersurface induced from ambient space may be not symmetric. If the induced Ricci curvature tensor is not symmetric, then it has no physical and geometric meaning

Received October 7, 2014.
2010 *Mathematics Subject Classification.* 53B30, 53C15, 53C40, 53C50.
Key words and phrases. lightlike hypersurface, Robertson-Walker spacetime, symmetry, parallelism, induced Ricci curvature tensor.

and hence the scalar curvature of such a lightlike hypersurface has no way to study. In addition, the screen distribution of a lightlike hypersurface is not always unique, although it is canonically isomorphic to the factor vector bundle $TM^* = TM/Rad(TM)$ considered by Kupeli [12].

The object of this paper is to introduce the geometry of horiztontal lightlike hypersurfaces of Robertson-Walker spacetimes, providing some properties of such lightlike hypersurfaces satisfying certain reasonable geometric conditions. We obtain a sufficient condition such that the screen distribution of a horizontal lightlike hypersurface of Robertson-Walker spacetimes exists uniquely. We prove that the induced Ricci curvature tensor of a horiztontal lightlike hypersurfaces of Robertson-Walker spacetimes is symmetric. Moreover, the symmetry and parallelism of the local second fundamental forms of horizontal lightlike hypersurfaces and their screen distributions are studied respectively.

2. Preliminaries

2.1. Lightlike hypersurface

We first recall some fundamental formulas of lightlike hypersurfaces of semi-Riemannian manifolds, following Duggal and Bejancu [6] and Duggal and Jin [7], respectively.

A hypersurface (M, g) of dimension n immersed in a semi-Riemannian manifold $(\overline{M}, \overline{g})$ of dimension $(n+1)$ is called a lightlike hypersurface if the metric g induced from ambient space is degenerate and the index q $(1 \leq q \leq n)$ of g is a constant. It is well known [6] that the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ is of constant rank 1, where $TM^{\perp}$ is called the normal bundle of M in $\overline{M}$. Thus there exists a non-degenerate complementary distribution $S(TM)$ of $TM^{\perp}$ in TM, which is called the screen distribution and is of rank $(n-1)$. Then we have

$$TM = Rad(TM) \oplus_{orth} S(TM), \tag{2.1}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $(M, g, S(TM))$. We also denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$-module of smooth sections of any vector bundle E over M. From [6] we see that for any lightlike section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique lightlike section N of a unique vector bundle $\mathrm{tr}(TM)$ of rank 1 on $\mathcal{U}$ satisfying

$$\overline{g}(\xi, N) = 1, \quad \overline{g}(N, N) = \overline{g}(N, X) = 0, \ \forall\ X \in \Gamma(S(TM)).$$

In this case, the tangent bundle $T\overline{M}$ of $\overline{M}$ is decomposed as follows:

$$T\overline{M} = TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM). \tag{2.2}$$

We call $\mathrm{tr}(TM)$ and N the transversal vector bundle and the null transversal vector field of M with respect to the chosen screen distribution $S(TM)$ respectively.

Let $\overline{\nabla}$ and P be the Levi-Civita connection of $\overline{M}$ and the projection morphism of TM on $S(TM)$ with respect to the decomposition (2.1). For any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $\xi \in \Gamma(Rad(TM))$, the Gauss and Weingarten formulas of M and $S(TM)$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.3}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X) N, \tag{2.4}$$

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \tag{2.5}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.6}$$

where ∇ and ∇^* are induced linear connection on TM and $S(TM)$ respectively, B and C are called the local second fundamental forms of M and $S(TM)$ respectively, $h := B \otimes N$ and $h^* := C \otimes \xi$ are called the second fundamental forms of M and $S(TM)$ respectively, A_N and A^*_ξ are linear shape operators on TM and $S(TM)$ respectively, and τ is a 1-form on M. Since that $\overline{\nabla}$ is torsion free, then ∇ is also torsion free and B is symmetric on M. Notice that B is independent of the choice of a screen distribution $S(TM)$ and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y) \tag{2.7}$$

for any $X, Y, Z \in \Gamma(TM)$, where $\eta(X) = \overline{g}(X, N)$. It follows from (2.7) that the induced connection ∇ on M is not a metric tensor, however, it is easy to verify that the induced connection ∇^* on $S(TM)$ is metric. The two local second fundamental forms B and C are related to their shape operators by

$$B(X, Y) = g(A^*_\xi X, Y), \quad \overline{g}(A^*_\xi X, N) = 0, \tag{2.8}$$

$$C(X, PY) = g(A_N X, PY), \quad \overline{g}(A_N X, N) = 0 \tag{2.9}$$

for any $X, Y \in \Gamma(TM)$. From the above two equations we see that A^*_ξ and A_N are $\Gamma(S(TM))$-valued shape operators related to B and C respectively, and A^*_ξ is self-adjoint on M and satisfies

$$A^*_\xi \xi = 0 \ (\Leftrightarrow B(X, \xi) = 0, \ \forall\ X \in \Gamma(TM)). \tag{2.10}$$

Denoted by $\overline{R}$, R and R^* the curvature tensor of semi-Riemannian connection $\overline{\nabla}$ of $\overline{M}$, the induced connection ∇ of M and the metric connection ∇^* on $S(TM)$, respectively, we obtain the following Gauss-Codazzi equations for M and $S(TM)$:

$$\begin{aligned} &\overline{R}(X, Y)Z \\ &= R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X \\ &\quad + \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z)\}N, \end{aligned} \tag{2.11}$$

$$\begin{aligned} &\overline{g}(R(X, Y)PZ, PW) \\ &= \overline{g}(R^*(X, Y)PZ, PW) + B(Y, PW)C(X, PZ) - B(X, PW)C(Y, PZ) \end{aligned} \tag{2.12}$$

for any $X, Y, Z, W \in \Gamma(TM)$, where we put $R^*(X,Y)PZ := \nabla^*_X \nabla^*_Y PZ - \nabla^*_Y \nabla^*_X PZ - \nabla^*_{[X,Y]} PZ$ and

$$(\nabla_X B)(Y,Z) = X(B(Y,Z)) - B(\nabla_X Y, Z) - B(X, \nabla_Y Z). \tag{2.13}$$

2.2. Robertson-Walker spacetime

Now we recall some basics on Robertson-Walker spacetimes following from O'Neill [13]. Let f be a positive smooth function defined on an open interval I of $\mathbb{R}$ and consider $L^{n+1} = I \times F$ endowed with the Lorentzian metric

$$\langle\ ,\ \rangle = -\pi_I^*(dt^2) + (f(\pi_I))^2 \pi_F^*(g_0), \tag{2.14}$$

where π_I and π_F denote the projections from L^{n+1} onto I and an n-dimensional Riemannian manifold F respectively, and g_0 is the usual Riemannian metric of F. A Lorentzian warped product manifold $(L^{n+1}, \langle\ ,\ \rangle)$ is also called a generalized Robertson-Walker spacetime (see [1]). When the fiber F of a Robertson-Walker spacetime L^{n+1} is of constant sectional curvature c, then L^{n+1} is called a Robertson-Walker spacetime, denoted by $(L_1^{n+1}(c,f), \overline{g})$. Let $\mathfrak{L}(I)$ and $\mathfrak{L}(F)$ be the set of horizontal and vertical lifts of vector fields on I and F to $I \times F$, respectively. Let $\partial_t \in \mathfrak{L}(I)$ denote the horizontal lift vector field to $I \times F$ of the standard vector field $\frac{d}{dt}$ on the base I. By a spacelike slice of $(L_1^{n+1}(c,f), \overline{g})$ we mean a hypersurface given by a fiber $S(t_0) := \pi^{-1}(t_0)$ with metric $f^2(t_0) g_c$. For any vector field X tangent to $L_1^{n+1}(c,f)$, we may put

$$X = \phi_X \partial_t + \hat{X}, \tag{2.15}$$

where $\phi_X = -\overline{g}(X, \partial_t)$ and $\hat{X}$ is the vertical component of X on the fiber. Thus we have the following two lemmas.

Lemma 2.1 ([13])**.** *Let $\overline{\nabla}$ be the Levi-Civita connection of $L_1^{n+1}(c,f)$. Then for any vector fields $X, Y \in \mathfrak{L}(F)$ we have*
(1) $\overline{\nabla}_{\partial_t} \partial_t = 0$,
(2) $\overline{\nabla}_{\partial_t} X = \overline{\nabla}_X \partial_t = (\ln f)' X$,
(3) $\overline{g}(\overline{\nabla}_X Y, \partial_t) = -\overline{g}(X,Y)(\ln f)'$.

Lemma 2.2 ([11])**.** *Let $\overline{R}$ be the curvature tensor of $L_1^{n+1}(c,f)$. Then for any vector fields X, Y, Z on $(L_1^{n+1}(c,f), \overline{g})$ we have*

$$\begin{aligned}\overline{R}(X,Y)Z = {} & \alpha\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\} \\ & + \beta\big\{\phi_X \phi_Z Y - \phi_Y \phi_Z X + \big[\phi_X \overline{g}(Y,Z) - \phi_Y \overline{g}(X,Z)\big]\partial_t\big\},\end{aligned} \tag{2.16}$$

where $\alpha = \frac{f'^2 + c}{f^2}$ and $\beta = \frac{ff'' - (f'^2 + c)}{f^2}$.

Remark 2.1. It follows from [13] that a Robertson-Walker spacetime $(L_1^{n+1}(c, f), \overline{g})$ is of constant sectional curvature $\overline{c}$ if and only if

$$\frac{f''}{f} = \overline{c} = \frac{f'^2 + c}{f^2}. \tag{2.17}$$

Thus, a Robertson-Walker spacetime $(L_1^{n+1}(c,f),\overline{g})$ is of constant sectional curvature if and only if $\beta=0$, as $\beta=0$ implies that α is a constant.

3. Lightlike hypersurfaces of Robertson-Walker spacetimes

Let $(M,g,S(TM))$ be a lightlike hypersurface of a Robertson-Walker spacetime $(L_1^{n+1}(c,f),\overline{g})$, then from decomposition (2.2) we have

$$\partial_t=\partial_t'+a\xi+bN, \tag{3.18}$$

where ∂_t' denotes the project component of the horizontal lift vector field ∂_t on screen distribution $S(TM)$. Using (2.15) we see that $a=-\phi_N$ and $b=-\phi_\xi$. It follows from (3.1) that $\overline{g}(\partial_t',\partial_t')=-1-2ab$. If we assume that $ab=0$, then it follows that $\overline{g}(\partial_t',\partial_t')=-1$. The screen distribution $S(TM)$ is a Riemannian distribution because of the metric of the ambient space $L_1^{n+1}(c,f)$ being Lorentzian, which means that $\overline{g}(\partial_t',\partial_t')\geq 0$, a contradiction. In this paper, we mainly discuss the properties of horizontal lightlike hypersurface which is defined as follows.

Definition 3.1. Let $(M,g,S(TM))$ be a lightlike hypersurface of a Robertson-Walker spacetime $(L_1^{n+1}(c,f),\overline{g})$. Then M is said to be
(1) a horizontal lightlike hypersurface if $\partial_t'=0$;
(2) a vertical lightlike hypersurface if $\partial_t'\neq 0$.

For a horizontal lightlike M of a Robertson-Walker spacetime $(L_1^{n+1}(c,f),\overline{g})$ and any $X\in\Gamma(S(TM))$, it follows from (2.15) and (3.1) that $\phi_X=0$. Then the screen distribution $S(TM)$ is tangent to a spacelike slice of $L_1^{n+1}(c,f)$.

Theorem 3.1. *A lightlike hypersurface of a Robertson-Walker spacetime $(L_1^{n+1}(c,f),\overline{g})$ is horizontal if and only if*

$$\phi_X=-b\eta(X) \tag{3.19}$$

for any $X\in\Gamma(TM)$, where $2ab+1=0$.

Proof. From (2.15) and (3.1) we have $-\phi_X=b\eta(X)+g(PX,\partial_t')$ for any $X\in\Gamma(TM)$. Notice that the screen distribution $S(TM)$ is non-degenerate, then we complete the proof. □

For any $X\in\Gamma(TM)$, from (2.15) we have $X=\phi_X\partial_t+\hat{X}$. Applying Lemma 2.1 and by a straightforward calculation we have

$$\overline{\nabla}_X\partial_t=(\ln f)'(X-\phi_X\partial_t). \tag{3.20}$$

Moreover, taking the covariant differentiation $\overline{\nabla}$ of relation (3.1) along any vector field $X\in\Gamma(TM)$ and using (2.3)–(2.6), we obtain

$$\begin{aligned}\overline{\nabla}_X\partial_t=(\nabla^*_X\partial_t'-aA^*_\xi X-bA_NX)+[C(X,\partial_t')+X(a)-a\tau(X)]\xi\\ +[B(X,\partial_t')+X(b)+b\tau(X)]N.\end{aligned} \tag{3.21}$$

Thus, we have the following result.

Lemma 3.1. *Let $(M, g, S(TM))$ be a horizontal lightlike hypersurface of a Robertson-Walker spacetime $(L_1^{n+1}(c, f), \overline{g})$. Then we have*

$$aA_\xi^* X + bA_N X + (\ln f)' PX = 0, \tag{3.22}$$

$$X(a) - a\tau(X) - \frac{1}{2}(\ln f)'\eta(X) = 0 \tag{3.23}$$

for any $X \in \Gamma(TM)$.

Proof. In this case we know $\partial_t' = 0$, then using decomposition (2.2) and comparing (3.3) with (3.4) prove the lemma. □

For a horizontal lightlike hypersurface of a Robertson-Walker spacetime, replacing X by ξ in (3.5) and using (2.10) we get $A_N\xi = 0$, which is equivalent to

$$C(\xi, PX) = 0, \ \forall \ X \in \Gamma(TM). \tag{3.24}$$

Applying Theorem 3.1 in (2.16) we have

$$\begin{aligned}
&\overline{R}(X,Y)Z \\
&= [\alpha g(Y,Z) - \beta b^2\eta(Y)\eta(Z)]PX - [\alpha g(X,Z) - \beta b^2\eta(X)\eta(Z)]PY \\
&\quad + (\alpha + \frac{\beta}{2})[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]\xi \\
&\quad - \beta b^2[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]N.
\end{aligned} \tag{3.25}$$

Lemma 3.2. *Let $(M, g, S(TM))$ be a horizontal lightlike hypersurface of a Robertson-Walker spacetime $(L_1^{n+1}(c, f), \overline{g})$. Then we have*

$$\begin{aligned}
&R(X,Y)Z \\
&= [\alpha g(Y,Z) - \beta b^2\eta(Y)\eta(Z) - \frac{1}{b}(\ln f)'B(Y,Z)]PX - \frac{a}{b}B(Y,Z)A_\xi^* X \\
&\quad - [\alpha g(X,Z) - \beta b^2\eta(X)\eta(Z) - \frac{1}{b}(\ln f)'B(X,Z)]PY + \frac{a}{b}B(X,Z)A_\xi^* Y \\
&\quad + (\alpha + \frac{\beta}{2})[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]\xi,
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
&(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\
&= -\beta b^2[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]
\end{aligned} \tag{3.27}$$

for any $X, Y, Z \in \Gamma(TM)$, where $\nabla_X B$ is given by (2.13).

Proof. The proof follows from (2.11), (3.5) and (3.8). □

For a lightlike hypersurface M of a semi-Riemannian manifold $\overline{M}$, the null sectional curvature (see [6]) is defined by

$$K(\xi, X) = \frac{g(R(X,\xi)\xi, X)}{g(X,X)} = g(R(X,\xi)\xi, X)$$

for a unit non-null vector field $X \in \Gamma(TM)$. Substituting $Y = Z = \xi$ into (3.9) gives $R(X,\xi)\xi = -\beta b^2 PX$ for any $X \in \Gamma(TM)$. Thus we obtain the following theorem.

Theorem 3.2. *The null sectional curvature of a horizontal lightlike hypersurface of a Robertson-Walker spacetime is* $-\beta b^2$.

Lemma 3.3. *Let* $(M, g, S(TM))$ *be a horizontal lightlike hypersurface of a Robertson-Walker spacetime* $(L_1^{n+1}(c,f), \overline{g})$*, then we have*

$$\begin{aligned}(3.28)\qquad & R^*(X,Y)Z \\ &= [\alpha g(Y,Z) - \frac{1}{a}(\ln f)'C(Y,Z)]X - [\alpha g(X,Z) - \frac{1}{a}(\ln f)'C(X,Z)]Y \\ &\quad - \frac{1}{a}[2bC(Y,Z) + (\ln f)'g(Y,Z)]A_N X \\ &\quad + \frac{1}{a}[2bC(X,Z) + (\ln f)'g(X,Z)]A_N Y,\end{aligned}$$

$$\begin{aligned}(3.29)\qquad & (\nabla_X C)(Y,Z) - (\nabla_Y C)(X,Z) + \tau(Y)C(X,Z) - \tau(X)C(Y,Z) \\ &= -\frac{1}{a^2 b}[X(a)C(Y,PZ) - Y(a)C(X,PZ)] - \frac{a}{b}X\left(\frac{(\ln f)'}{a}\right)g(Y,Z) \\ &\quad + \frac{a}{b}Y\left(\frac{(\ln f)'}{a}\right)g(X,Z) - \frac{1}{b}(\ln f)'[\tau(X)g(Y,Z) - \tau(Y)g(X,Z)],\end{aligned}$$

where $(\nabla_X C)(Y,Z) = X(C(Y,Z)) - C(\nabla_X Y, Z) - C(Y, \nabla^*_X Z)$ *for any* $X, Y, Z \in \Gamma(S(TM))$.

Proof. Notice that the screen distribution $S(TM)$ is non-degenerate, then (3.11) follows from (2.12), (3.5) and (3.9), and (3.12) follows from (3.5) and (3.10). □

Making using of the above result we may get the following classification theorem.

Theorem 3.3. *Let* $(M, g, S(TM))$ *be a horizontal lightlike hypersurface of a Robertson-Walker spacetime* $(L_1^{n+1}(c,f), \overline{g})$*. Then* M *is locally isometric to the product* $L_\xi \times M^*$*, where* L_ξ *is a lightlike curve tangent to the normal bundle* $TM^\perp$ *and* M^* *is a leaf of* $S(TM)$ *and its Riemannian curvature tensor* R^* *is given by* (3.11).

Proof. Let $(M, g, S(TM))$ be a horizontal lightlike hypersurface of $(L_1^{n+1}(c,f)$. Since that B is symmetric on M, then from (3.5) we see that the second fundamental form C is symmetric on $S(TM)$. Also, it is known [6] that $S(TM)$ is integrable if and only if C is symmetric. Then we complete the proof. □

Denote by $\mathcal{S}^1$ the first derivative of screen distribution $S(TM)$ (see [2, 5]) defined as follows:

$$(3.30)\qquad \mathcal{S}^1 = \mathrm{Span}\big\{[X,Y]_p,\ X_p, Y_p \in S(TM_p)\big\},\ \forall\, p \in M,$$

where [,] denotes the Lie-bracket. For a horizontal lightlike hypersurface of a Robertson-Walker spacetime, (3.5) implies that A_N is symmetric on $S(TM)$. Then from [6] we see that $S(TM)$ is integrable and hence $\mathcal{S}^1$ is sub-bundle of $S(TM)$. It is known [2] that if $\mathcal{S}^1$ coincides with $S(TM)$, then there exists a unique screen distribution, up to an orthogonal transformation. Thus, we have the following result.

Theorem 3.4. *Let $(M, g, S(TM))$ be a horizontal lightlike hypersurface of a Robertson-Walker spacetime $(L_1^{n+1}(c, f), \overline{g})$. If $\mathcal{S}^1$ coincides with $S(TM)$, then there exists a unique screen distribution, up to an orthogonal transformation.*

For a lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold $(\overline{M}, \overline{g})$, we say that M is totally umbilical (see [6]) if on any coordinate neighborhood $\mathcal{U}$ in M, there exists a smooth function ρ such that

$$B(X, Y) = \rho g(X, Y), \ \forall\, X, Y \in \Gamma(TM). \tag{3.31}$$

In case $\rho = 0$ on $\mathcal{U}$, M is said to be totally geodesic. A screen distribution $S(TM)$ of M is called totally umbilical (see also [6]) if on any coordinate neighborhood $\mathcal{U}$ in M, there exists a smooth function γ such that

$$C(X, PY) = \gamma g(X, Y), \ \forall\, X, Y \in \Gamma(TM). \tag{3.32}$$

In case $\gamma = 0$ on $\mathcal{U}$, $S(TM)$ is said to be totally geodesic. From Lemma 3.1, we see that a horizontal lightlike hypersurface M of a Robertson-Walker spacetime is totally umbilical if and only if the screen distribution of M is totally umbilical. Then we get the following result.

Theorem 3.5. *Let $(M, g, S(TM))$ be a horizontal lightlike hypersurface of a Robertson-Walker spacetime $(L_1^{n+1}(c, f), \overline{g})$. Then the screen distribution $S(TM)$ is integrable. Moreover, M is totally umbilical if and only if any leaf M' of $S(TM)$ is so immersed in $L_1^{n+1}(c, f)$ as a codimension 2 spacelike submanifold.*

Proof. Consider any leaf M' of screen distribution $S(TM)$ immersed in $(L_1^{n+1}(c, f), \overline{g})$ as a codimension 2 spacelike submanifold, then it follows from (2.3) and (2.5) that

$$\overline{\nabla}_X Y = \nabla_X Y + h'(X, Y) \tag{3.33}$$

for any $X, Y \in \Gamma(TM')$, where $h'(X, Y) := C(X, Y)\xi + B(X, Y)N$ is called the second fundamental form of M' in $\overline{M}$. Thus our assertion follows from (3.5) and (3.16). □

4. Symmetries and parallelism

Let $\overline{M}$ be a semi-Riemannian manifold of dimension $(n + 1)$, the Ricci curvature tensor of $\overline{M}$ denoted by $\overline{\text{Ric}}$ is defined by $\overline{\text{Ric}}(X, Y) = \text{trace}\{Z \to$

$\overline{R}(X,Z)Y\}$ for any $X, Y \in \Gamma(T\overline{M})$. Locally, the Ricci curvature tensor is given by

$$\overline{\mathrm{Ric}}(X,Y) = \sum_{i=1}^{n+1} \epsilon_i \overline{g}(\overline{R}(X,E_i)Y,E_i), \tag{4.34}$$

where $\{E_1, E_2, \ldots, E_{n+1}\}$ is a local semi-orthonormal frame fields of $T\overline{M}$ and $\overline{g}(E_i,E_i) = \epsilon_i$. In particular, if $\overline{\mathrm{Ric}}(X,Y) = k\overline{g}(X,Y)$ for any $X, Y \in \Gamma(T\overline{M})$, then $\overline{M}$ is called an Einstein manifold, where k is a smooth function on $\overline{M}$. Note that if $\dim(\overline{M}) > 2$, then k is a constant. For $\dim(\overline{M}) = 2$, any semi-Riemannian manifold $\overline{M}$ is an Einstein manifold but k is not necessarily constant. The scalar curvature $\overline{r}$ of $\overline{M}$ is defined by

$$\overline{r} = \sum_{i=1}^{n+1} \epsilon_i \overline{\mathrm{Ric}}(E_i,E_i). \tag{4.35}$$

Let M be a lightlike hypersurface of a Robertson-Walker spacetime $(L_1^{n+1}(c, f), \overline{g})$, it follows from [11] that the relation between the Ricci curvature tensor $\overline{\mathrm{Ric}}$ of $L_1^{n+1}(c,f)$ and the induced Ricci curvature tensor Ric of M is given by

$$\mathrm{Ric}(X,Y) = \overline{\mathrm{Ric}}(X,Y) - B(X,Y)\mathrm{tr}A_N + g(A_\xi^* Y, A_N Y) + \overline{g}(\overline{R}(\xi,Y)X,N) \tag{4.36}$$

for any $X, Y \in \Gamma(TM)$, where $\mathrm{tr}A_N$ denotes the trace of the shape operator A_N on $S(TM)$ (as $A_N\xi = 0$) and $\mathrm{Ric}(X,Y) := \mathrm{trace}\{Z \to R(X,Z)Y\}$. If the induced Ricci curvature tensor of a lightlike hypersurface M of dimension n is symmetric, then the scalar curvature r of M is defined by [7] as follows:

$$r = \mathrm{Ric}(\xi,\xi) + \sum_{i=1}^{n-1} \mathrm{Ric}(e_i,e_i), \tag{4.37}$$

where $\{e_1, e_2, \ldots, e_{n-1}\}$ is a local orthonormal frame fields of $S(TM)$.

Theorem 4.1. *Let $(M, g, S(TM))$ be a horizontal lightlike hypersurface of a Robertson-Walker spacetime $(L_1^{n+1}(c,f), \overline{g})$. Then the induced Ricci curvature tensor is symmetric and is given by*

$$\begin{aligned}\mathrm{Ric}(X,Y) = & -\left((n-1)\alpha + \frac{1}{2}\beta\right) g(X,Y) + (n-1)\beta b^2 \eta(X)\eta(Y) \\ & - \left(\mathrm{tr}A_N + \frac{1}{b}(\ln f)'\right) B(X,Y) - \frac{a}{b} g(A_\xi^* X, A_\xi^* Y)\end{aligned} \tag{4.38}$$

for any $X, Y \in \Gamma(TM)$. Moreover, the scalar curvature of M is given by

$$\begin{aligned} r = {} & (n-1)\left((b^2 - \frac{1}{2})\beta - (n-1)\alpha\right) + \frac{1}{b}\left((n-2)(\ln f)' + a\mathrm{tr}A_\xi^*\right)\mathrm{tr}A_\xi^* \\ & - \frac{a}{b}\mathrm{tr}(A_\xi^*)^2, \end{aligned} \tag{4.39}$$

where $\mathrm{tr}A_\xi^* := \mathrm{trace}\{X \to A_\xi^* X\}$ for $X \in \Gamma(TM)$.

Proof. Substituting $X = \xi$ and $Z = X$ into (2.16) and using (2.11) we obtain

$$\overline{g}(\overline{R}(\xi, Y)X, N) = (\alpha + \frac{1}{2}\beta)g(X, Y),\ \forall\ X, Y \in \Gamma(TM). \tag{4.40}$$

The Ricci curvature tensor $\overline{\text{Ric}}$ of $(L_1^{n+1}(c, f)$ is given by [11] as follows:

$$\overline{\text{Ric}}(X, Y) = -n\alpha\overline{g}(X, Y) + \beta\{(n-1)\phi_X\phi_Y - \overline{g}(X, Y)\} \tag{4.41}$$

for any $X, Y \in \Gamma(L_1^{n+1}(c, f))$. Applying Theorem 3.1 and Lemma 3.1 in (4.8) gives an equation, using the resulting equation and (4.7) in (4.3) we obtain (4.5). It follows from [11] that $\text{Ric}(\xi, \xi) = (n-1)\beta b^2$, then using (4.5) in (4.4) we obtain (4.6). □

Theorem 4.2. *Let* $(M, g, S(TM))$ *be a horizontal lightlike hypersurface of a Robertson-Walker spacetime* $(L_1^{n+1}(c, f), \overline{g})$*. Then the following assertions are true*:

(1) *each* 1*-form* τ *is closed, that is,* $d\tau = 0$ *on any* $\mathcal{U} \subset M$;
(2) *the transversal connection of* M *is flat, that is,* $R^{\perp} = 0$;
where the curvature tensor $R^{\perp}$ *of* $\text{tr}(TM)$ *is defined by* $R^{\perp}(X, Y)N = \nabla^{\perp}_X\nabla^{\perp}_Y N - \nabla^{\perp}_X\nabla^{\perp}_Y N - \nabla^{\perp}_{[X,Y]}N$ *for any* $X, Y \in \Gamma(TM)$*.*

Proof. From [10], the transversal connection $\nabla^{\perp}$ on transversal bundle $\text{tr}(TM)$ is defined by $\nabla^{\perp}_X N = \tau(X)N$ for any $X \in \Gamma(TM)$. If $\nabla^{\perp}$ vanishes identically, then the transversal connection $\nabla^{\perp}$ of M is said to be flat. D. H. Jin in [10] proved that the induced Ricci curvature Ric on M is symmetric if and only if the 1-form τ is closed, which is also equivalent to $R^{\perp} = 0$. Thus, the proof follows from Theorem 4.1 and Theorem 3.6 of [10]. □

Definition 4.1. A horizontal lightlike hypersurface $(M, g, S(TM))$ of a Robertson-Walker spacetime $(L_1^{n+1}(c, f), \overline{g})$ is said to be Ricci semi-symmetric if

$$R(X, Y) \cdot \text{Ric} = 0 \tag{4.42}$$

for any $X, Y \in \Gamma(TM)$, where curvature tensor R acts on Ricci curvature tensor as a derivation.

Theorem 4.3. *Let* $(M, g, S(TM))$ *be a horizontal lightlike hypersurface of a Robertson-Walker spacetime* $(L_1^{n+1}(c, f), \overline{g})$*. If* M *is Ricci semi-symmetric, then either* $L_1^{n+1}(c, f)$ *is of constant sectional curvature* α *or the Ricci tensor of* M *satisfies* (4.12)*.*

Proof. The condition of Ricci semi-symmetry means that

$$\text{Ric}(R(X, Y)Z, W) + \text{Ric}(Z, R(X, Y)W) = 0,\ \forall\ X, Y, Z, W \in \Gamma(TM). \tag{4.43}$$

It follows from (3.9) that $R(\xi, Y)Z = \beta b^2\eta(Z)PY + (\alpha + \frac{\beta}{2})g(Y, Z)\xi$ for any $Y, Z \in \Gamma(TM)$. Also, from (4.5) we see that $\text{Ric}(X, \xi) = (n-1)\beta b^2\eta(X)$ for any $X \in \Gamma(TM)$. Thus, substituting $X = Z = \xi$ into (4.10) we get

$$2\beta b^2\text{Ric}(PY, W) + (n-1)\beta b^2(2\alpha + \beta)g(Y, W) = 0 \tag{4.44}$$

for any $Y, W \in \Gamma(TM)$. If $\beta = 0$, from Remark 2.1 we know that $L_1^{n+1}(c, f)$ is of constant sectional curvature α. Otherwise, if $\beta \neq 0$ holds on an open subset $\mathcal{U} \subseteq M$, then we have

$$\text{Ric}(X, Y) = (n-1)\left(\beta b^2 \eta(X)\eta(Y) - (\alpha + \frac{\beta}{2})g(X, Y)\right) \tag{4.45}$$

for any $X, Y \in \Gamma(TU)$. Thus we complete the proof. □

Definition 4.2. Let $(M, g, S(TM))$ be a horizontal lightlike hypersurface of a Robertson-Walker spacetime $(L_1^{n+1}(c, f), \overline{g})$. The local second fundamental form B of M is said to be parallel with respect to ∇ on M if it satisfies $\nabla_X B = 0$ for any $X \in \Gamma(TM)$.

Theorem 4.4. *Let $(M, g, S(TM))$ be a horizontal lightlike hypersurface of a Robertson-Walker spacetime $(L_1^{n+1}(c, f), \overline{g})$. If the local second fundamental form B of M is parallel, then $L_1^{n+1}(c, f)$ is of constant sectional curvature α.*

Proof. Assume that B is parallel with respect to ∇ on M, then using $\nabla B = 0$ and $X = \xi$ in (3.10) gives

$$\tau(\xi)B(Y, Z) = -\beta b^2 g(Y, Z) \tag{4.46}$$

for any $Y, Z \in \Gamma(TM)$. Now we suppose that $\tau(\xi) = 0$, it is easy to see from (4.13) that $\beta = 0$ and hence $L_1^{n+1}(c, f)$ is of constant sectional curvature α. If $\tau(\xi) \neq 0$ holds on an open subset $\mathcal{U} \subseteq M$, then it follows from (4.14) that $B(Y, Z) = \kappa g(Y, Z)$, where we set $\kappa = \frac{-\beta b^2}{\tau(\xi)}$. Taking covariant differentiation ∇ along the arbitrary vector field $X \in \Gamma(TU)$ on both sides of $B(Y, Z) = \kappa g(Y, Z)$, and making use of (2.7) then we obtain

$$\begin{aligned}(\nabla_X B)(Y, Z) &= X(\kappa)g(Y, Z) + \kappa(\nabla_X g)(Y, Z) \\ &= X(\kappa)g(Y, Z) + \kappa^2 g(X, Y)\eta(Z) + \kappa^2 g(X, Z)\eta(Y)\end{aligned} \tag{4.47}$$

for any $X, Y, Z \in \Gamma(TU)$. Substituting $Z = \xi$ into (4.15) and using (2.10) imply $\kappa^2 g(X, Y) = 0$ for any $X, Y \in \Gamma(TU)$, which means that $\kappa = 0$ and hence $B = 0$. Using $B = 0$ in (3.10) we see that $\beta = 0$, that is, the Robertson-Walker spacetime $L_1^{n+1}(c, f)$ is of constant sectional curvature α. Thus we complete the proof. □

Definition 4.3. Let $(M, g, S(TM))$ be a horizontal lightlike hypersurface of a Robertson-Walker spacetime $(L_1^{n+1}(c, f), \overline{g})$. M is said to be semi-parallel if the second fundamental form h of M satisfies $R(X, Y) \cdot h = 0$ for any $X, Y \in \Gamma(TM)$.

Theorem 4.5. *Let $(M, g, S(TM))$ be a horizontal lightlike hypersurface of a Robertson-Walker spacetime $(L_1^{n+1}(c,\ f), \overline{g})$. If M is semi-parallel, then $L_1^{n+1}(c, f)$ is of constant sectional curvature α.*

Proof. By the definition of semi-parallelism of M we know

$$h(R(X,Y)Z,W)+h(Z,R(X,Y)W)=0,\ \forall\ X,Y,Z,W\in\Gamma(TM). \tag{4.48}$$

Substituting $Y=Z=\xi$ into (3.9) and using (2.10) we obtain $R(X,\xi)\xi=-\beta b^2PX$ for any $X\in\Gamma(TM)$. Thus, it follows from (4.16) and (2.10) that

$$h(R(X,\xi)\xi,W)=-\beta b^2h(X,W)=0,\ \forall\ X,W\in\Gamma(S(TM)), \tag{4.49}$$

which means that either $\beta=0$ or M is totally geodesic on an open subset $\mathcal{U}$. Then the proof follows from Theorem 4.4. □

Definition 4.4. Let $(M,g,S(TM))$ be a horizontal lightlike hypersurface of a Robertson-Walker spacetime $(L_1^{n+1}(c,f),\overline{g})$. M is said to be semi-symmetric if the curvature tensor R satisfies $R(X,Y)\cdot R=0$ for any $X,Y\in\Gamma(TM)$.

Theorem 4.6. *Let $(M,g,S(TM))$ be a horizontal lightlike hypersurface of a Robertson-Walker spacetime $(L_1^{n+1}(c,f),\overline{g})$. If M is semi-symmetric, then $L_1^{n+1}(c,f)$ is of constant sectional curvature α.*

Proof. By the definition of semi-symmetry of M we see that

$$\begin{aligned}&R(X,Y)R(Z,W)U-R(R(X,Y)Z,W)U-R(Z,R(X,Y)W)U\\&-R(Z,W)R(X,Y)U=0\end{aligned} \tag{4.50}$$

for any $X,Y,Z,W,U\in\Gamma(TM)$. Substituting $Y=Z=\xi$ into (3.9) and making use of (2.10) we obtain $R(X,\xi)\xi=-\beta b^2PX$ for any $X\in\Gamma(TM)$. Similarly, we get $R(\xi,Y)\xi=\beta b^2PY$ for any $Y\in\Gamma(TM)$. Also, it follows from (3.9) that $R(\xi,\xi)Z=0$ for any $Z\in\Gamma(TM)$. Thus, substituting $Y=Z=W=U=\xi$ into (4.18) implies that

$$2(\beta b^2)^2PX=0,\ \forall\ X\in\Gamma(TM), \tag{4.51}$$

which means that $\beta=0$ since $b\neq0$. Thus we complete the proof. □

Acknowledgements. This work is supported by National Natural Science Foundation of China (No. 11371076 and 11431009).

References

[1] L. J. Alías, A. Romero, and M. Sánchez, *Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes*, Gen. Relativity Gravitation **27** (1995), no. 1, 71–84.

[2] C. Atindogbe and K. L. Duggal, *Conformal screen on lightlike hypersurfaces*, Int. J. Pure Appl. Math. **11** (2004), no. 4, 421–442.

[3] H. Bondi and T. Gold, *On the generation of magnestism by fliud motion*, Monthly Not. Roy. Astr. Soc. **108** (1948), 252–270.

[4] A. Caminha and H. F. de Lima, *Complete vertical graph with constant mean curvature in semi-Riemannian warped products*, Bull. Belg. Math. Soc. Simon Stevin **16** (2009), no. 1, 91–105.

[5] K. L. Duggal, *A Report on canonical null curves and screen distributions for lightlike geometry*, Acta Appl. Math. **95** (2007), no. 2, 135–149.

[6] K. L. Duggal and A. Bejancu, *Lightlike Submainfolds of Semi-Riemannian Mainfolds and Applications*, Kluwer Academic Publishers, Dordrecht, 1996.

[7] K. L. Duggal and B. Sahin, *Differential Geometry of Lightlike Submanifolds*, Birkhauser Veralag AG, Basel, Boston, Berlin, 2010.
[8] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime*, Cambridge Univ. Press, Cambridge, 1973.
[9] F. Hoyle, *A new model for the expanding universe*, Monthly Not. Roy. Astr. Soc. **108** (1948), 372–382.
[10] D. H. Jin, *Geometry of lightlike hypersurfaces of an indefinite Sasakian manifold*, Indian J. Pure Appl. Math. **41** (2010), no. 4, 569–581.
[11] T. H. Kang, *On lightlike hypersurfaces of a GRW space-time*, Bull. Korean Math. Soc. **49** (2012), no. 4, 863–874.
[12] D. N. Kupeli, *Singular Semi-Riemnnian Geometry*, Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, 1996.
[13] B. O'Neill, *Semi-Riemannian Geometry with Application to Relativity*, Academic Press, New York, London, 1983.
[14] Y. Wang and X. Liu, *On complete spacelike hypersurfaces in a semi-Riemannian warped product*, J. Appl. Math. **2013** (2013), Art. ID 757041, 8 pp.

Ximin Liu
School of Mathematical Sciences
Dalian University of Technology
Dalian 116024, Liaoning, P. R. China
E-mail address: ximinliu@dlut.edu.cn

Quanxiang Pan
School of Mathematical Sciences
Dalian University of Technology
Dalian 116024, Liaoning, P. R. China
E-mail address: panquanxiang@dlut.edu.cn