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Some Results on δ-Semiperfect Rings and δ-Supplemented Modules

Cihat Abdioĝlu*
Department of Mathematics, Karamanoĝlu Mehmetbey University, Yunus Emre Campus, Karaman, Turkey
e-mail : cabdioglu@kmu.edu.tr

Serap Şahinkaya
Department of Mathematics, Gebze Institute of Technology, Çayırova Campus, 41400, Gebze- Kocaeli, Turkey
e-mail : ssahinkaya@gyte.edu.tr

Abstract. In [9], the author extends the definition of lifting and supplemented modules to δ-lifting and δ-supplemented by replacing "small submodule" with "δ-small submodule" introduced by Zhou in [13]. The aim of this paper is to show new properties of δ-lifting and δ-supplemented modules. Especially, we show that any finite direct sum of δ-hollow modules is δ-supplemented. On the other hand, the notion of amply δ-supplemented modules is studied as a generalization of amply supplemented modules and several properties of these modules are given. We also prove that a module M is Artinian if and only if M is amply δ-supplemented and satisfies Descending Chain Condition (DCC) on δ-supplemented modules and on δ-small submodules. Finally, we obtain the following result: a ring R is right Artinian if and only if R is a δ-semiperfect ring which satisfies DCC on δ-small right ideals of R.

1. Introduction

Throughout this paper, we will assume that R is an associative ring with unity and all modules are unital right R-modules.

We recall some basic notions related to our topic. A submodule N of a module M is called *small in* M, written $N \ll M$, if, whenever $M = L + N$ for any

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submodule L of M, we have $L = M$. A module M is called *lifting* if, for every submodule N of M, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B \ll M$ ([11]). In [13], the author defined the notion of δ-small submodules as follows. A submodule N of a module M is called a *δ-small submodule*, written as $N \ll_\delta M$, if, whenever $M = N + X$ with M/X is singular, we have $M = X$. Following Koşan [9], a module M is called *δ-lifting* if, for every $N \leq M$, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is δ-small in M. It is obvious that every lifting module is δ-lifting and every singular δ-lifting module is lifting.

Lemma 1.1.([9, Lemma 2.9]) *Assume that N and L are two submodules of the module M. Then the following conditions are equivalent :*

(1) *$M = N + L$ and $N \cap L$ is δ-small in L.*

(2) *$M = N + L$ and for any proper submodule K of L with L/K singular, $M \neq N + K$.*

A submodule L of M is called a *δ-supplement* of N in M if N and L satisfy the conditions in Lemma 1.1. A module M is called *δ-supplemented* if every submodule of M has a δ-supplement in M (see [9]). It is clear that every supplemented module is δ-supplemented and every singular δ-supplemented module is supplemented.

In Section 2, we give some properties of δ-supplements. We prove that any factor module of a δ-supplemented module is δ-supplemented and that any finite sum of δ-supplemented modules is δ-supplemented.

In Section 3, we give some results of decompositions and direct sums of δ-lifting modules. In particular, the main result in the third section shows that if $M = M_1 \oplus M_2$ is a direct sum of δ-lifting modules M_1 and M_2 such that M_i is M_j-projective (i=1,2), then M is a δ-lifting module.

In Section 4, we study the notion of amply δ-supplemented modules as a generalization of amply supplemented modules. Recall that a submodule N of M has *amply supplements in M* if, for every $L \subset M$ with $N+L = M$, there is a supplement L' of N with $L' \subset L$. Recall also that a module M is called *amply supplemented* if all submodules have amply supplements in M. We call a module M *amply δ-supplemented* if for any submodules N and K of M with $M = N + K$, K contains a δ-supplement of N in M. It is clear that every amply supplemented module is amply δ-supplemented and every singular amply δ-supplemented module is amply supplemented. It is proved in this section that if M is an amply δ-supplemented module such that every δ-supplement submodule of M is a direct summand, then M is δ-lifting. Recall that a ring R is δ-semiperfect if the module R_R is δ-supplemented (see [9, Theorem 3.3]). We also characterize δ-semiperfect rings in terms of amply δ-supplemented modules.

2. Some Properties of δ-Supplemented Submodules

We start with some general results on δ-small submodules which are taken from

[13, Lemmas 1.2 and 1.3].

Lemma 2.1. *Let M be an R-module.*

(1) *If $N \ll_\delta M$ and $M = X + N$, then $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \subseteq N$.*

(2) *If $K \ll_\delta M$ and $f : M \to N$ is a homomorphism, then $f(K) \ll_\delta N$. In particular, if $K \ll_\delta M \subseteq N$, then $K \ll_\delta N$.*

(3) *Let $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \ll_\delta M_1 \oplus M_2$ if and only if $K_1 \ll_\delta M_1$ and $K_2 \ll_\delta M_2$.*

Lemma 2.2. *Let A, B and C be submodules of an R-module M. If $M = A + B$, $B \leq C$, and $C/B \ll_\delta M/B$, then $(A \cap C)/(A \cap B) \ll_\delta M/(A \cap B)$.*

Proof. Let X be a submodule of M such that $A \cap B \leq X$, M/X is singular and $M/(A \cap B) = (A \cap C)/(A \cap B) + X/(A \cap B)$. Then $M = (A \cap C) + X = C + (A \cap X)$ by [4, Lemma 1.24]. Therefore $M/B = C/B + ((A \cap X) + B)/B$. Note that

$$\begin{aligned}
(M/B)/[((A \cap X) + B)/B] &\cong (C + (A \cap X))/(B + (A \cap X)) \\
&= ((A \cap C) + B + (A \cap X))/(B + (A \cap X)) \\
&\cong (A \cap C)/((A \cap C) \cap (B + (A \cap X))) \\
&= (A \cap C)/(A \cap C \cap X) \\
&\cong ((A \cap C) + X)/X \\
&= M/X.
\end{aligned}$$

Since M/X is singular, it follows that $M = (A \cap X) + B$. Now, since $M = A + X$ we get $M = X + (A \cap B) = X$. Hence, $(A \cap C)/(A \cap B) \ll_\delta M/(A \cap B)$. □

Proposition 2.3. *Let M be an R-module.*

(1) *Suppose that K and L are submodules of M such that $K \leq L$.*

   (a) *If L is a δ-supplement in M, then L/K is a δ-supplement in M/K.*

   (b) *If L has a δ-supplement H in M, then $(H + K)/K$ is a δ-supplement of L/K in M/K.*

(2) *Let $B \leq C \leq M$ be submodules of M. If C/B is a δ-supplement in M/B and B is a δ-supplement in M, then C is a δ-supplement submodule of M.*

(3) *Assume that $M = M_1 \oplus M_2$. If A is a δ-supplement of A' in M_1 and B is a δ-supplement of B' in M_2, then $A \oplus B$ is a δ-supplement of $A' \oplus B'$ in M.*

Proof. (1)(a) Let N be a submodule of M such that $L + N = M$ and $L \cap N \ll_\delta L$. Therefore $L/K + (N + K)/K = M/K$ and $[L \cap (N + K)]/K = [(L \cap N) + K]/K \ll_\delta M/K$ by Lemma 2.1(2).
(b) This can be proved by following the same method as in (a).
(2) Assume that C/B is a δ-supplement of X/B in M/B and B is a δ-supplement of Y in M. Then $M/B = (C/B) + (X/B)$ and $(C/B) \cap (X/B) \ll_\delta C/B$. Moreover,

$M = B+Y$ and $B \cap Y \ll_\delta B$. Note that $C = C \cap (B+Y) = B+(C \cap Y)$. Since $(C \cap X)/B \ll_\delta C/B$, it follows from Lemma 2.2 that $(C \cap X \cap Y)/(B \cap Y) \ll_\delta C/(B \cap Y)$. As $B \cap Y \ll_\delta C$ we have $(C \cap X \cap Y) \ll_\delta C$. Since $X = X \cap (B+Y) = B+(X \cap Y)$, we see that $M = C + X = C + (X \cap Y)$. Therefore C is a δ-supplement of $X \cap Y$ in M.
(3) By assumption, we have $M_1 = A + A'$ and $A \cap A' \ll_\delta A$. Moreover, $M_2 = B + B'$ and $B \cap B' \ll_\delta B$. Then $M = (A \oplus B) + (A' \oplus B')$. By Lemma 2.1(3), $(A \cap A') \oplus (B \cap B') \ll_\delta A \oplus B$. Since $(A \oplus B) \cap (A' \oplus B') = (A \cap A') \oplus (B \cap B')$, it follows that $A \oplus B$ is a δ-supplement of $A' \oplus B'$ in M. □

Corollary 2.4. *Every factor module of a δ-supplemented module is δ-supplemented.*
Lemma 2.5. *Let M_1 and M_2 be submodules of M such that M_1 is δ-supplemented and $M_1 + M_2$ has a δ-supplement in M. Then M_2 has a δ-supplement in M.*

Proof. By assumption, there exits a submodule N of M such that $M_1+M_2+N = M$ and $(M_1+M_2) \cap N \ll_\delta N$. Moreover, since M_1 is δ-supplemented, $(M_2+N) \cap M_1$ has a δ-supplement in M_1. Then there exists $L \leq M_1$ such that $M_1 = (M_2+N) \cap M_1 + L$ and $(M_2 + N) \cap L \ll_\delta L$. Then we have $M = M_1 + M_2 + N = (M_2 + N) \cap M_1 + L + M_2 + N = M_2 + (L + N)$. Moreover, we have $M_2 \cap (L + N) \leq [(M_2 + L) \cap N] + [(M_2 + N) \cap L] \leq [(M_2 + M_1) \cap N] + [(M_2 + N) \cap L]$. Now, it follows that $M_2 \cap (L + N) \ll_\delta (L + N)$. Hence, $L + N$ is a δ-supplement of M_2 in M. □

Proposition 2.6. *Any finite sum of δ-supplemented modules is δ-supplemented.*

Proof. We prove it for two modules; the finite case can be proved similarly. Let M_1 and M_2 be two submodules of a module M such that $M = M_1 + M_2$ and M_1 and M_2 are δ-supplemented. It is easily seen that for every submodule N of M, $M_1 + (M_2 + N)$ has a δ-supplement in M. Hence by Lemma 2.5, $M_2 + N$ has a δ-supplement in M. Applying Lemma 2.5 again we conclude that N has a δ-supplement in M. □

Corollary 2.7. *Let M be a δ-supplemented module. Then every finitely M-generated module is δ-supplemented.*

Proof. By Corollary 2.4 and Proposition 2.6. □

Recall that an R-module M is said to be *hollow* (respectively δ-*hollow*) if every proper submodule of M is small (respectively δ-small) in M. It is clear that every hollow module is δ-hollow. In [3], the author called a module M δ-*local* if, $\delta(M) \ll_\delta M$ and $\delta(M)$ is a maximal submodule of M. Moreover, the author also shows in [3] that a local module needs not to be δ-local in general.

Proposition 2.8. *Let M be a δ-hollow module. Then $\delta(M) = M$ or M is a local and a δ-local module.*

Proof. Suppose that $\delta(M) \neq M$. Then $\delta(M) \ll_\delta M$ and $Rad(M) \neq M$ since $Rad(M) \leq \delta(M)$ (see [13, Lemma 1.5(1)]). Let N be a maximal submodule of M. By hypothesis, we have $N \ll_\delta M$. Therefore $N \leq \delta(M)$. It follows that $\delta(M) = N$. Then $Rad(M) = \delta(M)$ is the only maximal submodule of M. Consequently, M is a

δ-local module. On the other hand, it is easy to see that $\delta(M)$ is small in M. This implies that M is a local module. □

The proof of the following two results are clear.

Proposition 2.9. *Let K be a δ-hollow submodule of the module M. Then K is a δ-supplement of each proper submodule L of M such that $K + L = M$.*

Proposition 2.10. *Every δ-hollow module is δ-supplemented.*

Corollary 2.11. *Any finite sum of δ-hollow modules is δ-supplemented.*

Proof. It can be obtained by using Propositions 2.6 and 2.10. □

Recall that a module M is called *cofinitely* δ-supplemented if every submodule N of M such that M/N is finitely generated has a δ-supplement in M.

Also recall that a module is called *coatomic* if every proper submodule is contained in a maximal one.

Proposition 2.12. *Let M be a coatomic module. Then the following are equivalent:*

(i) *M is cofinitely δ-supplemented.*

(ii) *Every maximal submodule of M has a δ-supplement.*

(iii) *$M = \sum_{i\in I} H_i$ where each H_i is either simple or δ-local.*

Proof. The equivalences clearly hold if M is semisimple. So, assume that M is not semisimple.

(i)⇒(ii) Clear.

(ii)⇒(iii) Let K be the sum of all δ-supplement submodules of maximal submodules L of M with $Soc(M) \leq L$. By [3, Lemma 3.4], K is a sum of δ-local submodules of M. Suppose that $M \neq Soc(M) + K$. Then there is a maximal submodule N of M such that $Soc(M) + K \leq N$. By hypothesis, N has a δ-supplement H in M. Thus $H \leq K \leq N$ and $N = M$, a contradiction. It follows that $M = Soc(M) + K$, and the proof is complete.

(iii)⇒(i) It follows from [1, Proposition 2.5] and [3, Lemma 3.3]. □

Corollary 2.13. *If M is a coatomic δ-supplemented module, then $M = \sum_{i\in I} H_i$ where each H_i is either simple or δ-local.*

Proof. This is clear by Proposition 2.12. □

3. Decompositions and Direct Sums of δ-Lifting Modules

Following [13], a projective module P is called a *projective δ-cover* of a module M if there exists an epimorphism $f : P \to M$ with $Ker(f) \ll_\delta P$, and a ring R is called *δ-semiperfect* if every simple R-module has a projective δ-cover. In [9], it is proved that a ring R is δ-semiperfect if and only if the R-module R_R is δ-supplemented. The following example shows that a δ-lifting module need not be lifting.

Example 3.1. Let F be a field, $I = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ and $R = \{(x_1, x_2, ..., x_n, x, x, ...)|$ $n \in \mathbb{N},\ x_i \in \mathbb{M}_2(F),\ x \in I\}$. By [13, Example 4.3], the ring R is δ-semiperfect and $\delta(R) = \{(x_1, x_2, ..., x_n, x, x, ...)|\ n \in \mathbb{N},\ x_i \in \mathbb{M}_2(F),\ x \in J\}$ where $J = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Therefore the module R_R is δ-lifting by [13, Lemma 2.4 and Theorem 3.6].
On the other hand, by [13, Example 4.3], R is not semiregular. Hence R_R is not supplemented. Thus R_R is not lifting.

Lemma 3.2.(See [9, Lemma 2.3])

(1) *The following conditions are equivalent for a module M:*

   (a) *M is δ-lifting.*

   (b) *For every $N \leq M$, there exists a decomposition $N = A \oplus B$ such that A is a direct summand of M and $B \ll_\delta M$.*

(2) *If M is δ-lifting, then any direct summand of M is δ-lifting.*

Proposition 3.3. *Let M be an indecomposable module. Then M is δ-lifting if and only if M is δ-hollow.*

Proof. Let M be a δ-lifting indecomposable module. Let N be a proper submodule of M. Since M is δ-lifting, we have a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is δ-small in B for some submodules A and B of M. Since M is indecomposable and $N \neq M$, we have $A = 0$, and so $M = B$. Therefore $N \ll_\delta M$. Hence, M is δ-hollow. The converse is clear. □

Proposition 3.4.

(1) *If M is a δ-lifting module, then $M/\delta(M)$ is a semisimple module.*

(2) *If M is a δ-lifting module, then any submodule N of M with $N \cap \delta(M) = 0$ is semisimple.*

(3) *If the module R_R is δ-lifting, then $M/\delta(M)$ is a semisimple module for every R-module M.*

Proof. (1) See [9, Lemma 2.12].
(2) Since $N \cong (N \oplus \delta(M))/\delta(M) \leq M/\delta(M)$ is semisimple by (1), then N is semisimple.
(3) Let M be an R-module. By hypothesis and (1), $R/\delta(R)$ is a semisimple ring. But, on the other hand $M\delta(R) = \delta(M)$ by [13, Theorem 1.8]. Thus, $M/\delta(M)$ is a semisimple module. □

In [9, Example 2.4], it is proved that if $R = \mathbb{Z}_8$, then the R-module $M = R \oplus (2R/4R)$ is not δ-lifting, although the R-modules R_R and $(2R/4R)_R$ are δ-lifting. The following result deals with a special case of a direct sum of two δ-lifting modules.

The following theorem may be seen in the literature but we want to give it here for the readers.

Theorem 3.5. *Let $M = M_1 \oplus M_2$. If M_1 and M_2 are δ-lifting modules such that M_i is M_j-projective (i=1,2), then M is a δ-lifting module.*

Proof. Let $N \leq M$. Since M_1 is δ-lifting, there exist submodules K and K' of M_1 such that $M_1 = K \oplus L$, $K \leq M_1 \cap (N + M_2)$ and $L \cap (N + M_2) \ll_\delta M_1$. Therefore $M = K \oplus L \oplus M_2 = N + (L \oplus M_2)$. On the other hand, since M_2 is δ-lifting, there exist submodules X and Y of M_2 such that $M_2 = X \oplus Y$, $X \leq M_2 \cap (N + L)$ and $Y \cap (N + L) \ll_\delta M_2$. Hence $M = (X \oplus K) \oplus (L \oplus Y)$ and $M = N + (L \oplus Y)$. Since M_i is M_j-projective, $X \oplus K$ is $(L \oplus Y)$-projective by [11, Propositions 4.32 and 4.33]. Then there exists a submodule N_1 of N such that $M = N_1 \oplus (L \oplus Y)$. Then we have $N \cap (L \oplus Y) \leq Y \cap (N + L) + L \cap (N + Y)$. But $Y \cap (N + L) \ll_\delta M_2$ and $L \cap (N + M_2) \ll_\delta M_1$. Thus, $N \cap (L \oplus Y) \ll_\delta M$ by Lemma 2.1. □

Corollary 3.6. *Let $M = M_1 \oplus M_2$. If M_1 and M_2 are δ-lifting projective modules, then M is δ-lifting.*

Proof. This is clear by Theorem 3.5. □

Corollary 3.7. *Let R be a δ-semiperfect ring. Then every free module of finite rank is δ-lifting.*

Proof. This is clear by Corollary 2.6. □

Theorem 3.8. (i) *If M is a δ-lifting module, then M has a decomposition $M = M_1 \oplus M_2$ such that M_1 is semisimple, M_2 is δ-lifting and $\delta(M_2)$ is an essential submodule of M_2.*

(ii) *If M is a δ-lifting module, then M has a decomposition $M = M_1 \oplus M_2$ such that M_1 and M_2 are δ-lifting modules, $\delta(M_1) = M_1$ and $\delta(M_2) \ll_\delta M_2$.*

Proof. (i) This is by [9, Proposition 2.13] and Lemma 3.2(2).
(ii) Since M is δ-lifting, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq \delta(M)$ and $\delta(M) \cap M_2 \ll_\delta M_2$. Now, $\delta(M) = \delta(M_1) \oplus \delta(M_2)$ implies that $\delta(M) \cap M_2 = \delta(M_2) \oplus (M_2 \cap \delta(M_1)) = \delta(M_2) \ll_\delta M_2$. On the other hand, $\delta(M) \cap M_1 = M_1 = \delta(M_1) \oplus (M_1 \cap \delta(M_2)) = \delta(M_1)$. Moreover, M_1 and M_2 are δ-lifting by Lemma 3.2(2). □

Proposition 3.9. *If M is a δ-lifting module such that $\delta(M)$ has a supplement in M, then we have a decomposition $M = M_1 \oplus M_2$ such that M_1 is a lifting module and M_2 is δ-lifting with $\delta(M_2) = M_2$.*

Proof. Assume that M is δ-lifting and let A be a supplement of $\delta(M)$ in M. Then we have a decomposition $M = M_1 \oplus M_2$ such that $A = M_1 \oplus (M_2 \cap A)$ and $M_2 \cap A \ll_\delta M_2$. Let N be a submodule of M_1. Since M_1 is a δ-lifting module by Lemma 3.2(2), we have a decomposition $M_1 = X \oplus Y$ such that $N = X \oplus (Y \cap N)$ and $Y \cap N \ll_\delta Y$. Since $A \cap \delta(M) \ll A$ and $Y \cap N \leq \delta(M) \cap A$, we obtain that $Y \cap N \ll A$. Hence $Y \cap N \ll Y$ by [11, Lemma 4.2(2)]. Therefore M_1 is a lifting module. Moreover, we have $M = \delta(M_2) + \delta(M_1) + M_1 = \delta(M_2) \oplus M_1$. This gives

$\delta(M_2) = M_2$. □

4. Amply δ-Supplemented Modules

In this section we study the notion of amply δ-supplemented modules. Several properties of this type of modules are proved. Recall that a module M is amply δ-supplemented if for any submodules N and K of M with $M = N + K$, K contains a δ-supplement of N in M. It is clear that every amply δ-supplemented module is δ-supplemented.

Lemma 4.1. *Let M be an R-module. If every submodule of M is δ-supplemented, then M is amply δ-supplemented.*

Proof. Let A and B be submodules of M such that $M = A + B$. Since A is δ-supplemented and $A \cap B \leq A$, there is a submodule $C \leq A$ such that $A \cap B + C = A$ and $A \cap B \cap C \ll_\delta C$. Therefore C + B = M. Since $C \cap B = C \cap B \cap A \ll_\delta C$, C is a δ-supplement of B in M. It follows that M is amply δ-supplemented. □

Proposition 4.2. *If M is an amply δ-supplemented module such that every δ-supplement submodule of M is a direct summand, then M is a δ-lifting module.*

Proof. Let N be a submodule of M. By assumption, N has a δ-supplement K and K has a δ-supplement L such that $L \leq N$ and L is a direct summand of M. Then $M = L \oplus T = L + K$ for some submodule T of M. Note that $N = L \oplus (N \cap T) = L + (N \cap K)$. Let π denote the canonical projection $\pi : L \oplus T \to T$. Then $\pi(N) = \pi(N \cap K) = N \cap T$. Since K is a δ-supplement of N, we have $N \cap K \ll_\delta K$. Hence $\pi(N \cap K) = N \cap T \ll_\delta T$ by Lemma 2.1(2). Consequently, M is a δ-lifting module by Lemma 3.2. □

Proposition 4.3. *Any epimorphic image of an amply δ-supplemented module is again amply δ-supplemented.*

Proof. Let M be an amply δ-supplemented module and let $f : M \to N$ be an epimorphism, where N is an R-module. Let $N = A + B$. Then $f^{-1}(N) = M = f^{-1}(A) + f^{-1}(B)$. Since M is an amply δ-supplemented module, there is a submodule $X \leq f^{-1}(B)$ such that $M = f^{-1}(A) + X$ and $f^{-1}(A) \cap X \ll_\delta X$. Hence $N = f(M) = A + f(X)$ and $A \cap f(X) = f(f^{-1}(A) \cap X) \ll_\delta f(X)$ by Lemma 2.1(2). This implies that $f(X)$ is a δ-supplement of A in M. Moreover, we have $f(X) \leq B$. Therefore N is amply δ-supplemented. □

Recall that a module M is called *π-projective* if for every two submodules N and L of M with $M = N + L$, there exists an endomorphism α of M such that $\alpha(M) \leq N$ and $(1 - \alpha)(M) \leq L$. It is well known that π-projective supplemented modules are amply supplemented. Next we prove an analogue for this result.

Theorem 4.4. *Let M be a π-projective module. If M is δ-supplemented, then M is amply δ-supplemented.*

Proof. Let $M = N + K$. Then there exists $\alpha \in End(M)$ such that $\alpha(M) \leq N$ and

$(1-\alpha)(M) \in K$. Since M is δ-supplemented, there exists a submodule $L \leq M$ such that $M = N+L$ and $N \cap L \ll_\delta L$. Clearly, $(1-\alpha)(L) \leq K$ and $M = N+(1-\alpha)(L)$. Since $N \cap L \ll_\delta L$, then $N \cap (1-\alpha)(L) = (1-\alpha)(N \cap L) \ll_\delta (1-\alpha)(L)$. So M is amply δ-supplemented. □

Corollary 4.5. *Let $M_1, M_2, ..., M_k$ be submodules of a projective module M such that $M = \overset{k}{\underset{i=1}{\oplus}} M_i$. The following are equivalent:*

(i) *M is amply δ-supplemented.*

(ii) *For every i $(1 \leq i \leq k)$, M_i is amply δ-supplemented.*

Proof. (i)⇒(ii) By Proposition 4.3.

(ii)⇒(i) Since for every $1 \leq i \leq k$, M_i is amply δ-supplemented, it follows from Proposition 2.6 that $M = \overset{k}{\underset{i=1}{\oplus}} M_i$ is δ-supplemented. By Theorem 4.4, M is amply δ-supplemented. □

Proposition 4.6. *Let M be an amply δ-supplemented module. Assume that for every submodule K of M such that $K = K_1 \cap K_2$ where K_1 and K_2 are δ-supplement submodules in M with $M = K_1 + K_2$, every homomorphism $\beta : M \to M/K$ can be lifted to a homomorphism $\gamma : M \to M$. Then M is π-projective.*

Proof. Let A and B be submodules of M with $M = A + B$. Since M is an amply δ-supplemented module, there exist two submodules $B' \leq B$ and $A' \leq A$ such that $M = A + B' = A' + B'$, $A \cap B' \ll_\delta B'$ and $A' \cap B' \ll_\delta A'$. Now, we consider the homomorphism $\beta : M \to M/(A' \cap B')$ defined by $\beta(a' + b') = b' + A' \cap B'$, where $a' \in A'$ and $b' \in B'$. By hypothesis, β can be lifted to a homomorphism $\alpha : M \to M$. Moreover, we have $\alpha(M) \leq B'$ and $(1-\alpha)(M) \leq A'$. Hence M is π-projective. □

Let M be a module and $B \leq A \leq M$. If $A/B \ll M/B$ then B is called a *coessential submodule* of A in M. If A has no proper coessential submodule in M, then A is called *coclosed* in M (see [8]).

If $A/B \ll_\delta M/B$ and A/B is singular, then B will be called a *δ-coessential submodule* of A. If A has no proper δ-coessential submodule in M, then A is called δ-coclosed in M (see [3]). Clearly, every δ-coclosed submodule is coclosed.
Note that every δ-supplement submodule of a module M is δ-coclosed by [3, Corollary 2.6].

Let $K \leq N \leq M$. The submodule K is said to be a *δ-coclosure* of N in M if K is a δ-coessential submodule of N and K is δ-coclosed in M.

Proposition 4.7. *Let M be a δ-lifting module. Then every singular δ-coclosed submodule of M is a direct summand.*

Proof. Let N be a singular δ-coclosed submodule of M. Since M is δ-lifting, there exist submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2$, $M_1 \leq N$ and

$N \cap M_2 \ll_\delta M$. Therefore $N = M_1 \oplus (N \cap M_2)$ and $N \cap M_2 \ll_\delta N$ by [3, Corollary 2.6]. It follows that $N = M_1$ since N/M_1 is singular. □

Lemma 4.8. *Let M be an amply δ-supplemented module. Then every submodule N of M has a δ-coclosure in M.*

Proof. The proof is clear.

A module M is called *weakly δ-supplemented* if for every submodule $N \leq M$, there exists a submodule $K \leq M$ such that $M = N + K$ and $N \cap K \ll_\delta M$. It is clear that every δ-supplemented module is weakly δ-supplemented. □

Proposition 4.9. *A module M is amply δ-supplemented if and only if M is weakly δ-supplemented and every submodule of M has a δ-coclosure in M.*

Proof. ($\Rightarrow$) This is clear by Lemma 4.8.
($\Leftarrow$) Suppose that M is weakly δ-supplemented and every submodule of M has a δ-coclosure in M. Let A and B be two submodules of M such that $M = A + B$. Since M is weakly δ-supplemented, there exists a submodule C of M such that $(A \cap B) + C = M$ and $(A \cap B) \cap C \ll_\delta M$. Then $(A \cap B) + (C \cap B) = B$. Thus $A + (C \cap B) = M$ by [4, Lemma 1.24]. Let N be a δ-coclosure of $C \cap B$ in M. Then $(C \cap B)/N$ is singular, N is δ-coclosed in M and $(C \cap B)/N \ll_\delta M/N$. On the other hand, we have $[(A + N)/N] + (C \cap B)/N = M/N$ and $M/(A + N) \cong (C \cap B)/[(C \cap B) \cap (A + N)]$. Hence $M/(A + N) \cong (C \cap B)/[N + (A \cap B) \cap C]$ is a factor module of $(C \cap B)/N$. So $M/(A+N)$ is singular. It follows that $M = A+N$. Since $A \cap N \leq (A \cap B) \cap C \ll_\delta M$, we get $A \cap N \ll_\delta N$ by [3, Corollary 2.6]. Consequently, N is a δ-supplement of A in M with $N \leq B$. Therefore M is amply δ-supplemented. □

The next result gives a characterization of δ-semiperfect rings in terms of δ-supplemented modules. It is taken from [13, Theorem 3.6] and [9, Theorem 3.3].

Lemma 4.10. *The following are equivalent for a ring R:*

(1) *R is a δ-semiperfect ring.*

(2) *$R/\delta(R)$ is a semisimple ring and idempotents lift modulo $\delta(R)$.*

(3) *There exists a complete orthogonal set of idempotents $e_1, ..., e_n$ such that, for each i, either e_iR is a simple R-module or e_iR has a unique essential maximal submodule.*

(4) *Every finitely generated R-module is δ-supplemented.*

(5) *Every finitely generated projective R-module is δ-lifting.*

(6) *Every finitely generated projective R-module is δ-supplemented.*

(7) *R_R is δ-supplemented.*

It is well-known that a ring R is semiperfect if and only if R_R is supplemented if and only if R_R is amply supplemented.

Corollary 4.11. *The following are equivalent for a ring R:*

(1) *R is δ-semiperfect.*

(2) *R_R is amply δ-supplemented.*

(3) *Every finitely generated module is amply δ-supplemented.*

Proof. (1)⇔(2) is follows from Theorem 4.4 and Lemma 4.10.
(2)⇒(3) By Proposition 4.3 and Corollary 4.5.
(3)⇒(2) Clear. □

Theorem 4.12. *Let M be an R-module. Then M is Artinian if and only if M is amply δ-supplemented and satisfies DCC on δ-supplement submodules and on δ-small submodules.*

Proof. The necessity is clear. Conversely, assume that M is amply δ-supplemented module which satisfies DCC on δ-supplement submodules and on δ-small submodules. By [10, Proposition 2.6], $\delta(M)$ is an Artinian module. We next show that $M/\delta(M)$ is an Artinian module. Let $\delta(M) \leq N_1 \leq N_2 \leq \cdots$ be an ascending chain of submodules of M. Since M is amply δ-supplemented, there exists a descending chain $K_1 \geq K_2 \geq \cdots$ of submodules of M such that K_i is a δ-supplement of N_i in M for each $i \geq 1$. By hypothesis, there exists a positive integer n such that $K_n = K_{n+j}$ for each $j \geq 1$. Since $K_i \cap N_i \leq \delta(M)$, we have $M/\delta(M) = N_i/\delta(M) \oplus (K_i + \delta(M))/\delta(M)$ for each $i \geq n$. It follows that $N_i = N_n$ for each $i \geq n$. Thus $M/\delta(M)$ is Noetherian, and hence finitely generated. Moreover, $M/\delta(M)$ is a semisimple module by [9, Lemma 2.12]. Then $M/\delta(M)$ is Artinian. Consequently, M is Artinian. □

Proposition 4.13. *Let M be a finitely generated δ-supplemented module. Then M is Artinian if and only if M satisfies DCC on δ-small submodules.*

Proof. By [9, Lemma 2.12] and [10, Proposition 2.6]. □

Corollary 4.14. *R is a right Artinian ring if and only if R is a δ-semiperfect ring which satisfies DCC on δ-small right ideals of R.*

Proof. By Corollary 4.11 and Proposition 4.13.

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