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SPECTRAL APPROXIMATIONS OF ATTRACTORS FOR CONVECTIVE CAHN-HILLIARD EQUATION IN TWO DIMENSIONS

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Abstract. In this paper, the long time behavior of the convective Cahn-Hilliard equation in two dimensions is considered, semidiscrete and completely discrete spectral approximations are constructed, error estimates of optimal order that hold uniformly on the unbounded time interval $0 \le t < \infty$ are obtained.

1. Introduction

Recently, more and more people are interested in the convective Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} + \gamma\Delta^2 u = \Delta\varphi(u) + \beta \cdot \nabla\psi(u),$$

which arises naturally as a continuous model for phase transition in binary systems, such as alloys, glass and polymer mixtures (see [2, 12]). Here, $u(x,t)$ denotes the concentration of one of two phases in a system which is undergoing phase separation. The convective term $\beta \cdot \nabla\psi(u)$ which is introduced to study how the phase transition is affected by the steady fluid flow.

In [15], Zaks et al. [15] studied the bifurcations of stationary periodic solutions of a convective Cahn-Hilliard equation; Eden and Kalantarov [3, 4] established some results on the existence of a compact attractor for the convective Cahn-Hilliard equation with periodic boundary conditions in one space dimension and three space dimensions; Based on the Schauder type estimates, Liu [8] not only established the global existence of classical solutions for convective Cahn-Hilliard equation, but also discussed the nonnegativity and the finite speed of propagation of perturbations of solutions for convective Cahn-Hilliard equation; Zhao et al. [17] considered the global attractor for the convective Cahn-Hilliard equation in two space dimensions with the functions $\varphi(s)$ and $\psi(s)$ are polynomials. In addition, Zhao and Liu [18, 19] also considered the

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optimal control problem for convective Cahn-Hilliard equation in 1D and 2D cases.

The study of long time behavior for dissipative nonlinear partial differential equations depended on the results of numerical experimentation to a great extent. For this reason, it is worth studying whether the numerical results are reliable and the calculation schemes are suitable. In [7], based on the finite element method, Hale et al. studied the approximate attractor of some types of nonlinear evolution equation; In [5], based on the finite element method, Elliott and Larsson considered the approximate attractor of Cahn-Hilliard equation; In [10], based on the a prior estimates, Lü and Lu obtained the existence and the convergence of global attractors for a fully discrete classical Galerkin spectral scheme of generalized Kdv-Burgers equation. For more results on the numerical approximation to long time behavior of nonlinear evolutions, we refer the reader to [9, 13, 16].

In this article, we consider the 2D convective Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} + \gamma\Delta^2 u - \Delta\varphi(u) - \beta\cdot\nabla\psi(u) = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2,\ t > 0, \tag{1.1}$$

where γ is a positive constant, β is a vector. Equation (1.1) is supplemented by the following boundary conditions

$$u(x_1 + 2\pi, x_2, t) = u(x_1, x_2 + 2\pi, t) = u(x_1, x_2, t), \quad x \in \mathbb{R}^2,\ t \geq 0, \tag{1.2}$$

and initial condition

$$u(x, 0) = u_0(x),\ x \in \mathbb{R}^2. \tag{1.3}$$

In this paper, we assume that the initial function has zero mean, i.e., $\int_\Omega u_0(x)dx = 0$, then it follows from (1.3) that $\int_\Omega u(x,t)dx = 0$ for $t > 0$. On the other hand, we use the following notation: $\Omega = [0, 2\pi]\times[0, 2\pi]$; $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$, $\|\cdot\|_m$ the norm of $L^m(\Omega)$, and $\|\cdot\| = \|\cdot\|_{L^2}$, $\|\cdot\|_\infty = \|\cdot\|_{L^\infty}$.

The outline of this paper is as follows. In the next section, the existence of discrete attractors $\mathcal{A}_N^\tau$ is obtained by the t-independent prior estimates of discrete solutions; In Section 3, the convergence of $\mathcal{A}_N^\tau$ is proved by the error estimates in $[0, +\infty)$ of the discrete solutions.

Finally in this section, we give the following lemmas which are necessary for further discussion.

Lemma 1.1 (Poincaré inequality [6]). *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain and $\|\cdot\|$ is the norm of $L^2(\Omega)$. Then we have the Poincaré inequality*:

- *$\forall v \in H_0^1(\Omega)$,*

$$\|v\| \leq \frac{|\Omega|}{\pi}\|Dv\|, \quad n = 1, \quad \|v\| \leq C(\Omega)\|Dv\|, \quad n \geq 2.$$

- *$\forall v \in H^1(\Omega)$,*

$$\|v\|^2 \leq \frac{|\Omega|^2}{2}\|Dv\|^2 + \frac{1}{|\Omega|}\left(\int_\Omega v dx\right)^2, \quad n = 1,$$

$$\|v\|^2 \le C(\Omega)\left[\|Dv\|^2 + \left(\int_\Omega v(x)dx\right)^2\right], \quad n \ge 2.$$

Lemma 1.2 (Sobolev interpolation inequality [11])**.** *Suppose that $u \in L^q(\Omega)$, $D^m u \in L^r(\Omega)$, $\Omega \subset \mathbb{R}^n$, $1 \le r \le \infty$, $0 \le j \le m$. Then there exists a constant $c = c(j, m, \Omega, p, q, r)$ independent of u such that*

$$\|D^j u\|_{L^p} \le c\|D^m u\|^{\alpha}_{W^{m,r}} \|u\|^{1-\alpha}_{L^q},$$

where

$$\frac{1}{p} = \frac{j}{n} + \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\alpha)\frac{1}{q}, \quad \frac{j}{m} < \alpha < 1.$$

2. Semidiscrete Galerkin spectral approximation

For any given positive integer N, $j = (j_1, j_2)$, $j \cdot x = j_1 x_1 + j_2 x_2$, let $S_N = \text{span}\{e^{ij\cdot x} : |j| \le N\}$, where $|j| = \max\{|j_1|, |j_2|\}$. Denote by $P_N : L^2_p(\Omega) \to S_N$ the orthogonal projection operator. For operator P_N and functions in S_N, we have the following results (see [1]):

(B1) P_N commutes with derivation on $H^2_p(\Omega)$, i.e.,

$$P_N \Delta u = \Delta P_N u, \ \forall u \in H^2_p(\Omega).$$

(B2) For any real $0 \le \mu \le \sigma$, there is a constant c independent of u, N such that

$$\|u - P_N u\|_{H^\mu} \le c N^{\mu-\sigma} \|u\|_{H^\sigma}, \ \forall u \in H^\sigma_p(\Omega).$$

2.1. Existence of approximation global attractors $\mathcal{A}_N$

By using Galerkin method, for each $N \ge |j|$, we find

$$u_N(x,t) = \sum_{|j|\le N} \beta_j(t) v_j(x) \in S_N$$

such that $\beta_j(t)$ satisfies the following ODEs

$$\left\{ \begin{array}{l} \big(u_{Nt} + \gamma\Delta^2 u_N - \Delta\varphi(u_N) - \beta\cdot\nabla\psi(u_N), v_j\big) = 0, \\ (u_N(\cdot,0), v_j) = (u_{N0}(\cdot), v_j), \ |j| \le N. \end{array} \right. \tag{2.1}$$

According to ODE theory, there exists a unique local solution to problem (2.1). What we should do is to show a prior estimates. In addition, we assume that $\|u_0\|_{H^2} \le R$, where R is a positive constant.

Lemma 2.1. *Suppose that γ is sufficiently large, $\varphi, \psi \in C^1$, $\varphi'(s) \ge -c_0$, $u_0(x) \in L^2(\Omega)$, then the solution $u_N(x,t)$ of problem (2.1) satisfies*

$$\|u_N(x,t)\|^2 \le \|u_0\|^2 e^{-c_1 t}, \quad t \ge 0,$$

$$\overline{\lim_{t\to\infty}} \|u_N(x,t)\|^2 \le \rho_0^2,$$

where c_0, c_1 and ρ_0^2 are positive constants independent of N, t.

Proof. Setting $v_j = u_N(x,t)$ in (2.1), we derive that

$$\frac{1}{2}\frac{d}{dt}\|u_N\|^2 + \gamma\|\Delta u_N\|^2 = (\varphi(u_N), \Delta u_N) + (\beta \cdot \nabla\psi(u_N), u_N).$$

Then

$$\frac{1}{2}\frac{d}{dt}\|u_N\|^2 + \gamma\|\Delta u_N\|^2 + (\varphi'(u_N)\nabla u_N, \nabla u_N) = (\beta \cdot \nabla\psi(u_N), u_N). \tag{2.2}$$

Note that $\varphi'(s) \geq -c_0$. Hence

$$(\varphi'(u_N)\nabla u_N, \nabla u_N) \geq -c_0\|\nabla u_N\|^2. \tag{2.3}$$

On the other hand, a simple calculation shows that

$$(\beta \cdot \nabla\psi(u_N), u_N) = \beta \cdot \int_\Omega \psi'(u_N) u_N \nabla u_N dx = 0. \tag{2.4}$$

By Poincare's inequality, we obtain

$$\|u_N\|^2 \leq C(\Omega)\left\{\|\nabla u_N\|^2 + \left(\int_\Omega u_N dx\right)^2\right\} \leq C_1\|\nabla u_N\|^2. \tag{2.5}$$

We also have

$$C_1\|\nabla u_N\|^2 = -C_1\int_\Omega u_N \Delta u_N dx \leq \frac{1}{2}\|u_N\|^2 + \frac{C_1^2}{2}\|\Delta u_N\|^2. \tag{2.6}$$

Adding (2.5) and (2.6) together gives

$$\|u_N\|^2 \leq C_1^2\|\Delta u_N\|^2. \tag{2.7}$$

It then follows from (2.2)-(2.4) and (2.7) that

$$\frac{d}{dt}\|u_N\|^2 + \left(\frac{\gamma}{C_1^2} - \frac{c_0^2}{\gamma}\right)\|u_N\|^2 \leq 0, \tag{2.8}$$

where γ is large enough, which satisfies $\frac{\gamma}{C_1^2} - \frac{c_0^2}{\gamma} > 0$. Then, by uniform Gronwall's inequality, we obtain

$$\|u_N(x,t)\|^2 \leq \|u_0\|^2 e^{-\left(\frac{\gamma}{C_1^2} - \frac{c_0^2}{\gamma}\right)t}, \tag{2.9}$$

which implies that

$$\overline{\lim_{t\to\infty}}\|u_N(x,t)\|^2 \leq \rho_0^2. \tag{2.10}$$

Therefore, Lemma 2.1 is proved. □

Lemma 2.2. *In addition to the conditions of Lemma 2.1, we suppose that* $u_0(x) \in H_p^1(\Omega)$, $\varphi \in C^2$ *and* $\varphi(s)$, $\psi(s)$ *satisfy*

$$\varphi^{(i)}(r) \leq c|r|^{k-i} + c', \quad \psi'(r) \leq cr^2\sqrt{\varphi'(r)} + c',$$

where $k \leq 3$ *is a positive constant and* $i = 0, 1, 2$. *Then the solution* $u_N(x,t)$ *of problem* (2.1) *satisfies*

$$\|\nabla u_N(x,t)\|^2 \leq \|\nabla u_0\|^2 e^{-c_3 t} + c_4, \quad t \geq 0,$$

$$\overline{\lim_{t\to\infty}}\|\nabla u_N(x,t)\|^2 \le \rho_1^2,$$

where c_3, c_4 and ρ_1^2 are positive constants independent of N, t.

Proof. Setting $v_j = \Delta u_N(x,t)$ in (2.1), we derive that

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_N\|^2+\gamma\|\nabla\Delta u_N\|^2+(\Delta\varphi(u_N),\Delta u_N)+(\beta\cdot\nabla\psi(u_N),\Delta u_N)=0. \tag{2.11}$$

We also have

$$(\Delta\varphi(u_N),\Delta u_N)=\int_\Omega\varphi'(u_N)|\Delta u_N|^2dx+\int_\Omega\varphi''(u_N)|\nabla u_N|^2\Delta u_N dx. \tag{2.12}$$

Combining (2.11) and (2.12) together gives

(2.13)
$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u_N\|^2+\gamma\|\nabla\Delta u_N\|^2+\int_\Omega\varphi'(u_N)|\Delta u_N|^2dx\\
=&-\int_\Omega\varphi''(u_N)|\nabla u_N|^2\Delta u_N dx-\beta\cdot\int_\Omega\psi'(u_N)\nabla u_N\Delta u_N dx\\
\le& C\left(\int_\Omega|u_N\Delta u_N||\nabla u_N|^2dx+\int_\Omega|u_N^2\sqrt{\varphi'(u_N)}\nabla u_N\Delta u_N|dx+\|\nabla u_N\|^2\right)\\
\le&\frac{C}{2}\int_\Omega|\nabla u_N|^4dx+\frac{C}{2}\int_\Omega|u_N\Delta u_N|^2dx+\int_\Omega\varphi'(u_N)|\Delta u_N|^2dx\\
&+\frac{C^2|\beta|^2}{4}\int_\Omega u_N^4|\nabla u_N|^2dx+C\|\nabla u_N\|^2.
\end{aligned}$$

Using Sobolev's interpolation inequality, we obtain

$$\|u_N\|_4\le C\|\nabla\Delta u_N\|^{\frac{1}{6}}\|u_N\|^{\frac{5}{6}},\quad \|\nabla u_N\|_4\le C\|\nabla\Delta u_N\|^{\frac{1}{2}}\|u_N\|^{\frac{1}{2}},$$
$$\|u_N\|_8\le C\|\nabla\Delta u_N\|^{\frac{1}{4}}\|u_N\|^{\frac{3}{4}},\quad \|\Delta u_N\|_4\le C\|\nabla\Delta u_N\|^{\frac{5}{6}}\|u_N\|^{\frac{1}{6}}.$$

By Hölder's inequality and the above inequalities, we get

$$\|u_N\|_4^2\|\Delta u_N\|_4^2\le\frac{\gamma}{4C}\|\nabla\Delta u_N\|^2+C_3, \tag{2.14}$$

$$\|u_N\|_8^4\|\nabla u_N\|_4^2\le\frac{\gamma}{2C^2|\beta|^2}\|\nabla\Delta u_N\|^2+C_4, \tag{2.15}$$

and

$$\|\nabla u_N\|_4^4\le\frac{\gamma}{4C}\|\nabla\Delta u_N\|^2+C_5. \tag{2.16}$$

It then follows from (2.13)-(2.16) that

$$\frac{d}{dt}\|\nabla u_N\|^2+\frac{5\gamma}{4}\|\nabla\Delta u_N\|^2\le 2C\|\nabla u_N\|^2+2(C_3+C_4+C_5). \tag{2.17}$$

Using Sobolev's interpolation inequality again, we immediately obtain

$$\|\nabla u_N\|\le C\|\nabla\Delta u_N\|^{\frac{1}{3}}\|u_N\|^{\frac{2}{3}}.$$

Hence

$$2C\|\nabla u_N\|^2 \le \frac{\gamma}{4}\|\nabla\Delta u_N\|^2 + C_6. \tag{2.18}$$

Using (2.17) and (2.18), we deduce that

$$\frac{d}{dt}\|\nabla u_N\|^2 + \gamma\|\nabla u_N\|^2 \le C_7, \tag{2.19}$$

where $C_7 = 2(C_3 + C_4 + C_5) + C_6$. By uniform Gronwall's inequality, we obtain

$$\|\nabla u_N\|^2 \le \|\nabla u_0\|^2 e^{-\gamma t} + \frac{C_7}{\gamma}, \tag{2.20}$$

which implies that

$$\overline{\lim_{t\to\infty}}\|\nabla u_N(x,t)\|^2 \le \rho_1^2, \tag{2.21}$$

where $\rho_1^2 = \frac{C_7}{\gamma}$. Therefore, Lemma 2.2 is proved. □

Corollary 2.3. *In addition to the conditions of Lemma 2.2, there exists a unique solution $u_N(x,t)$ for problem (2.1) in $(0,+\infty)$, which satisfies*

$$\|u_N(x,t)\|_p \le C(R), \quad 0 < p < +\infty,\ t \ge 0,$$

where $C(R)$ is a positive constant dependent only on R.

Lemma 2.4. *In addition to the conditions of Lemma 2.2, we suppose that $u_0 \in H_p^2(\Omega)$, then the solution $u_N(x,t)$ of problem (2.1) satisfies*

$$\|\Delta u_N(x,t)\|^2 \le \|\Delta u_0\|^2 e^{-c_5 t} + c_6, \quad t \ge 0,$$

$$\overline{\lim_{t\to\infty}}\|\Delta u_N(x,t)\|^2 \le \rho_2^2,$$

where c_5, c_6 and ρ_2^2 are positive constants independent of N, t.

Proof. Setting $v_j = \Delta^2 u_N(x,t)$ in (2.1), we derive that

$$\frac{1}{2}\frac{d}{dt}\|\Delta u_N\|^2 + \gamma\|\Delta^2 u_N\|^2 = (\Delta\varphi(u_N), \Delta^2 u_N) + (\beta\cdot\nabla\psi(u_N), \Delta^2 u_N). \tag{2.22}$$

We also have

$$\begin{aligned}(\Delta\varphi(u_N), \Delta^2 u_N) &\\ = (\varphi'(u_N)\Delta u_N + \varphi''(u_N)|\nabla u_N|^2, \Delta^2 u_N) &\\ \le \frac{\gamma}{4}\|\Delta^2 u_N\|^2 + \frac{2}{\gamma}\|\varphi'(u_N)\Delta u_N\|^2 + \frac{2}{\gamma}\|\varphi''(u_N)|\nabla u_N|^2\|^2,&\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}(\beta\cdot\nabla\psi(u_N), \Delta^2 u_N) &= \beta\cdot(\psi(u_N)\nabla u_N, \Delta^2 u_N)\\ &\le \frac{\gamma}{4}\|\Delta^2 u_N\|^2 + \frac{|\beta|^2}{\gamma}\|\psi'(u_N)\nabla u_N\|^2.\end{aligned} \tag{2.24}$$

Adding (2.22)-(2.24) together gives

$$\begin{aligned}
(2.25)\qquad & \frac{d}{dt}\|\Delta u_N\|^2+\gamma\|\Delta^2 u_N\|^2 \\
&\le \frac{4}{\gamma}\int_\Omega |\varphi'(u_N)\Delta u_N|^2dx+\frac{4}{\gamma}\int_\Omega |\varphi''(u_N)|\nabla u_N|^2|^2dx \\
&\quad+\frac{2|\beta|^2}{\gamma}\int_\Omega u_N^4|\varphi'(u_N)\nabla u_N|^2dx \\
&\le C\left(\int_\Omega u_N^4|\Delta u_N|^2dx+\int_\Omega u_N^2|\nabla u_N|^4dx+\int_\Omega u_N^6|\nabla u_N|^2dx\right) \\
&\le C\left(\|\Delta u_N\|_4^2+\|\nabla u_N\|_8^4+\|\nabla u_N\|_4^2\right).
\end{aligned}$$

By Sobolev's interpolation inequality, we conclude

$$(2.26)\qquad \|\Delta u_N\|_4^2\le (C\|\Delta^2 u_N\|^{\frac{5}{8}}\|u_N\|^{\frac{3}{8}})^2\le \frac{\gamma}{6}\|\Delta^2 u_N\|^2+C_8,$$

$$(2.27)\qquad \|\nabla u_N\|_4^2\le (C\|\Delta^2 u_N\|^{\frac{3}{8}}\|u_N\|^{\frac{5}{8}})^2\le \frac{\gamma}{6}\|\Delta^2 u_N\|^2+C_9,$$

$$(2.28)\qquad \|\nabla u_N\|_8^4\le (C\|\Delta^2 u_N\|^{\frac{7}{16}}\|u_N\|^{\frac{11}{16}})^4\le \frac{\gamma}{6}\|\Delta^2 u_N\|^2+C_{10}.$$

It then follows from (2.25)-(2.28) that

$$\frac{d}{dt}\|\Delta u_N\|^2+\frac{\gamma}{2}\|\Delta^2 u_N\|^2\le C_8+C_9+C_{10}.$$

By a Calderón-Zygmund type estimate, we have

$$(2.29)\qquad \frac{d}{dt}\|\Delta u_N\|^2+C_{11}(\|\Delta u_N\|^2+\|\nabla\Delta u_N\|^2)\le C_{12}.$$

Using uniform Gronwall's inequality, we derive that

$$(2.30)\qquad \|\Delta u_N\|^2\le \|\Delta u_0\|^2e^{-C_{11}t}+\frac{C_{12}}{C_{11}},$$

which implies that

$$(2.31)\qquad \overline{\lim_{t\to\infty}}\|\Delta u_N(x,t)\|^2\le \rho_2^2,$$

where $\rho_2^2=\frac{C_{12}}{C_{11}}$. Then, Lemma 2.4 is proved. □

Corollary 2.5. *In addition to the conditions of Lemma 2.4, there exists a unique solution $u_N(x,t)$ for problem* (2.1) *in* $(0,+\infty)$*, which satisfies*

$$\|u_N(x,t)\|_\infty\le C(R),\quad t\ge 0,$$

where $C(R)$ is a positive constant dependent only on R.

Lemma 2.6. *In addition to the conditions of Lemma 2.4, we suppose that $\varphi\in C^3$, $\psi\in C^2$, then the solution $u_N(x,t)$ of problem* (2.1) *satisfies*

$$\|\nabla\Delta u_N(x,t)\|\le \frac{E_0}{t}+E_1,\quad t>0,$$

where E_0 is a positive constant dependent on R and t, independent of N.

Proof. Setting $v_j = t^2\Delta^3 u_N(x,t)$ in (2.1), we derive that

$$\big(u_{Nt} + \gamma\Delta^2 u_N - \Delta\varphi(u_N) - \beta\cdot\psi(u_N), t^2\Delta^3 u_N\big) = 0. \tag{2.32}$$

Note that

$$(u_{Nt}, t^2\Delta^3 u_N) = -\frac{1}{2}\frac{d}{dt}\|t\nabla\Delta u_N\|^2 + \|t^{\frac{1}{2}}\nabla\Delta u_N\|^2, \tag{2.33}$$

and

$$(\gamma\Delta^2 u_N, t^2\Delta^3 u_N) = -\gamma\|\nabla\Delta^2 u_N\|^2. \tag{2.34}$$

In addition, we have

$$\begin{aligned}
&|(\beta\cdot\psi(u_N), t^2\Delta^3 u_N)| \\
&= |\int_\Omega t^2\nabla(\psi'(u_N)\nabla u_N)\nabla\Delta^2 u_N dx| \\
&\le \int_\Omega t^2|\psi'(u_N)\Delta u_N\nabla\Delta^2 u_N|dx + \int_\Omega t^2|\psi''(u_N)|\nabla u_N|^2\nabla\Delta^2 u_N|dx \\
&\le C\left(\int_\Omega t^2|\Delta u_N\nabla\Delta^2 u_N|dx + \int_\Omega t^2||\nabla u_N|^2\nabla\Delta^2 u_N|dx\right) \\
&\le \frac{\gamma}{2}\|t\nabla\Delta^2 u_N\|^2 + C(\|\Delta u_N\|^2 + \|\nabla u_N\|_4^4) \\
&\le \frac{\gamma}{2}\|t\nabla\Delta^2 u_N\|^2 + C_{13}
\end{aligned} \tag{2.35}$$

and

$$\begin{aligned}
&|(\Delta\varphi(u_N), t^2\Delta^3 u_N)| \\
&= |(\nabla\Delta\varphi(u_N), t^2\nabla\Delta^2 u_N)| \\
&\le \int_\Omega t^2|\varphi'(u_N)\nabla\Delta u_N\nabla\Delta^2 u_N|dx + 3\int_\Omega t^2|\varphi''(u_N)\nabla u_N\Delta u_N\nabla\Delta^2 u_N|dx \\
&\quad + \int_\Omega t^2|\varphi'''(u_N)|\nabla u_N|^3\nabla\Delta^2 u_N|dx \\
&\le C\left(\int_\Omega t^2|\nabla\Delta u_N\nabla\Delta^2 u_N|dx + 3\int_\Omega t^2|\nabla u_N\Delta u_N\nabla\Delta^2 u_N|dx\right. \\
&\quad \left. + \int_\Omega t^2||\nabla u_N|^3\nabla\Delta^2 u_N|dx\right) \\
&\le \frac{\gamma}{2}\|t\nabla\Delta^2 u_N\|^2 + C(\|\nabla\Delta u_N\|^2 + \|\nabla u_N\Delta u_N\|^2 + \|\nabla u_N\|_6^6) \\
&\le \frac{\gamma}{2}\|t\nabla\Delta^2 u_N\|^2 + C(\|\nabla\Delta u_N\|^2 + \|\nabla u_N\|_\infty^2\|\Delta u_N\|^2 + \|\nabla u_N\|_6^6) \\
&\le \frac{\gamma}{2}\|t\nabla\Delta^2 u_N\|^2 + C_{14}\|\nabla\Delta u_N\|^2 + C_{15}.
\end{aligned} \tag{2.36}$$

We have used Sobolev embedding $H^3(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ in (2.36), which is correct in 2D case. Combing (2.32)-(2.36) together gives

$$\frac{d}{dt}\|t\nabla\Delta u_N\|^2 \leq 2C_{14}\|\nabla\Delta u_N\|^2 + 2(C_{13}+C_{15}). \tag{2.37}$$

It then follows from (2.29) and (2.37) that

$$\|\nabla\Delta u_N(x,t)\| \leq \frac{C_{16}}{t} + C_{17}, \quad t > 0, \tag{2.38}$$

where C_{16} and C_{17} are two positive constant dependent on R and t, independent of N. □

Now, from Lemma 2.1, Lemma 2.2, Lemma 2.4, Lemma 2.6 and the compact argument, we have:

Theorem 2.7. *Suppose that γ is sufficiently large, $u_0 \in H_p^2(\Omega)$, $\varphi \in C^2$ and $\psi \in C^1$ also satisfy*

$$\varphi'(r) > 0, \quad \varphi^{(i)}(r) \leq c|r|^{k-i} + c', \quad \psi'(r) \leq cr^2\sqrt{\varphi'(r)} + c',$$

where $k \leq 3$ is a positive constant and $i = 0, 1, 2$. Then there exists a unique global solution $u(x,t)$ for problem (1.1)-(1.3)*, such that*

$$u(x,t) \in L^\infty(\mathbb{R}^+; H_p^2(\Omega)) \bigcap L^2(\mathbb{R}^+; H_p^4(\Omega)).$$

Furthermore, if $\varphi \in C^3$, $\psi \in C^2$, then

$$\|\nabla\Delta u_N(x,t)\| \leq \frac{E_0}{t} + E_1, \quad t > 0,$$

where E_0 and E_1 are two positive constant dependent on R and t.

In the conditions of Theorem 2.7, the solution operator of problem (1.1)-(1.3) generate an operator semigroup $S(t)$. Similarly, for the solution $u(x,t)$ of problem (1.1)-(1.3), we also have the a prior estimates as Lemma 2.1, Lemma 2.2, Lemma 2.4 and Lemma 2.6. Thus, the operator $S(t)$ is a continuous operator from $H_p^2(\Omega)$ to itself. On the other hand, set $B_0 = \{u \in H_p^2(\Omega) : \|u\|^2_{H^2(\Omega)} \leq 2(\rho_0^2 + \rho_1^2 + \rho_2^2)\}$. It is easy to see that B_0 is an absorbing set, and $S(t)$ is uniform compact for t large enough. Therefore, the semigroup of operator $S(t)$ has a compact global attractor $\mathcal{A} \subset H_p^2(\Omega)$. It is similar completely, the solution operator of problem (2.1) also generate an operator semigroup $S_N(t)$ on S_N, which possesses an attractor $\mathcal{A}_N$.

2.2. Convergence of the global attractors $\mathcal{A}_N$

Define the semidistance of two sets A and B in $H^2(\Omega)$ as follows

$$d(A,B) = \sup_{x\in A}\inf_{y\in B}\|x-y\|_{H^2(\Omega)}.$$

We say that $\mathcal{A}_N$ converge to $\mathcal{A}$, if

$$d(\mathcal{A}_N, \mathcal{A}) \to 0 \quad \text{as } N \to +\infty.$$

Set

$$u-u_N=u-P_Nu+P_Nu-u_N=\eta+\theta. \tag{2.39}$$

Based on (1.1)-(1.3) and (2.1), θ satisfies

(2.40)
$$\begin{cases}(\theta_t,v)+\gamma(\Delta\theta,\Delta v)=(\varphi(u)-\varphi(u_N),\Delta v)-\beta\cdot(\psi(u)-\psi(u_N),\nabla v),\ \forall v\in S_N\\ \theta(x,0)=0.\end{cases}$$

Theorem 2.8. *In addition to the conditions of Theorem 2.7, we suppose that* $u(x,t)$ *is the solution of problem* (1.1)-(1.3) *and* $u_N(x,t)$ *is the solution of semi-discrete approximation* (2.1)*. Then*

$$\|u(x,t)-u_N(x,t)\|_{H^2}\le C\left(\frac{E_0}{t}+E_1+C_{17}\right)N^{-1},$$

where C_{17} *is a positive constant dependent on* $\|u_0\|_{H^2}$*.*

Proof. Setting $v=\theta$ in (2.40), we deduce that

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2+\gamma\|\Delta\theta\|^2=(\varphi(u)-\varphi(u_N),\Delta\theta)-(\beta\cdot[\psi(u)-\psi(u_N)],\nabla\theta). \tag{2.41}$$

Note that

$$\begin{aligned}(\varphi(u)-\varphi(u_N),\Delta\theta)&=(\varphi'(\xi u+(1-\xi)u_N)(u-u_N),\Delta\theta)\\ &\le\|\varphi'(\xi u+(1-\xi)u_N)\|_\infty\|u-u_N\|\|\Delta\theta\|\\ &\le C(\|\eta\|+\|\theta\|)\|\Delta\theta\|\\ &\le\frac{\gamma}{2}\|\Delta\theta\|^2+C_{18}(\|\eta\|^2+\|\theta\|^2),\end{aligned} \tag{2.42}$$

and

(2.43)
$$\begin{aligned}-(\beta\cdot[\psi(u)-\psi(u_N)],\nabla\theta)=&\ -(\beta\cdot[\psi'(\zeta u+(1-\zeta)u_N)(u-u_N)],\nabla\theta)\\ &\le|\beta|\|\psi'(\zeta u+(1-\zeta)u_N)\|_\infty\|u-u_N\|\|\nabla\theta\|\\ &\le C(\|\eta\|+\|\theta\|)\|\nabla\theta\|\\ &\le\frac{\gamma}{2}\|\Delta\theta\|^2+C_{19}(\|\eta\|^2+\|\theta\|^2).\end{aligned}$$

Adding (2.41)-(2.43) together gives

$$\frac{d}{dt}\|\theta\|^2\le 2(C_{18}+C_{19})(\|\eta\|^2+\|\theta\|^2).$$

Using Gronwall's inequality, we get

$$\|\theta(\cdot,t)\|^2\le\|\theta(\cdot,0)\|^2e^{2(C_{18}+C_{19})t}+\|\eta\|^2\le CN^{-6}\|\nabla\Delta u\|^2\le C_{17}N^{-6}.$$

By inverse inequality (see [1]), we derive that

$$\begin{aligned}\|\nabla\theta(\cdot,t)\|&\le CN^1\|\theta(\cdot,t)\|\le CC_{17}N^{-2},\\ \|\Delta\theta(\cdot,t)\|&\le CN^2\|\theta(\cdot,t)\|\le CC_{17}N^{-1}.\end{aligned}$$

In the end, by Theorem 2.7, we obtain

$$\begin{aligned}\|u(\cdot,t)-u_N(\cdot,t)\|_{H^2(\Omega)} &\le \|\eta(\cdot,t)\|_{H^2(\Omega)}+\|\theta(\cdot,t)\|_{H^2(\Omega)}\\ &\le CN^{-1}\|\nabla\Delta(\cdot,t)\|+CC_{17}N^{-1}\\ &\le C(\frac{E_0}{t}+E_1+C_{17})N^{-1}.\end{aligned}$$

□

Therefore, for any compact interval $J \in (0,+\infty)$,

$$\sup_{u_0\in H_p^2(\Omega)\bigcap S_N}\ \sup_{t\in J} d(S_N(t)u_0, S(t)u_0)\to 0 \quad \text{as } N\to+\infty.$$

Through Theorem I.1.2 of [14], we get the following result.

Theorem 2.9. *In addition to the conditions of Theorem 2.7,*

$$d(\mathcal{A}_N,\mathcal{A})\to 0 \quad \text{as } N\to+\infty.$$

3. Fully discrete Galerkin spectral approximation

3.1. Existence of approximation global attractors $\mathcal{A}_N^\tau$

Let τ be the mesh size in the variable t, $t_k = k\tau$, $u^k = u(x,t_k)$, $\bar\partial_t u^k = \frac{1}{\tau}(u^k-u^{k-1})$. The fully discrete Galerkin spectral scheme for solving problem (1.1)-(1.3) is to find $u_N^k\in S_N$ such that

$$\left\{\begin{array}{l}(\bar\partial_t u_N^k, v)+\gamma(\Delta u_N^k,\Delta v)=(\varphi(u_N^k),\Delta v)-(\beta\cdot\psi(u_N^k),\nabla v),\\ u_N^0=P_Nu_0.\end{array}\right. \tag{3.1}$$

Lemma 3.1. *In addition to the conditions of Lemma 2.1, the solution u_N^k of problem* (2.1) *satisfies*

$$\|u_N^n\|^2\le\frac{1}{(1+c_1\tau)^n}\|u_0\|^2\le\|u_0\|^2\triangleq\Upsilon_0^2,\quad\forall n\ge1,$$

$$\overline{\lim_{n\to\infty}}\|u_N^n\|^2\le(\varrho_0')^2.$$

$$\tau^2\sum_1^n\|\bar\partial_t u_N^k\|^2\le C_1'(1+t_n),\quad\forall n\ge1,$$

where the constant $C_1'=C_1'(\|u_0\|)$ is independent of N, n and τ.

Proof. Setting $v=u_N^k$ in (3.1), we derive that

$$(\bar\partial_t u_N^k,u_N^k)+\gamma\|\Delta u_N^k\|^2=(\varphi(u_N^k),\Delta u_N^k)-(\beta\cdot\psi(u_N^k),\nabla u_N).$$

Note that

$$(\bar\partial_t u_N^k,u_N^k)=\frac12\bar\partial_t\|u_N^k\|^2+\frac{\tau}{2}\|\bar\partial_t u_N^k\|^2,$$

$$\|u_N^k\|^2\le C_1\|\nabla u_N\|^2\le C_1^2\|\Delta u_N^k\|^2,$$

$$(\varphi(u_N^k),\Delta u_N^k)=-(\varphi'(u_N^k),|\nabla u_N^k|^2)\le c_0\|\nabla u_N^k\|^2,$$

and

$$(\beta\cdot\psi(u_N^k),\nabla u_N)=0.$$

Summing up, we get

$$\bar{\partial}_t \|u_N^k\|^2 + \tau\|\bar{\partial}_t u_N^k\|^2 + \frac{\gamma}{2C_1^2}\|u_N^k\|^2 + \left(\frac{\gamma}{2C_1} - 2c_0\right)\|\nabla u_N^k\|^2 \leq 0, \tag{3.2}$$

where γ satisfies $\frac{\gamma}{2C_1} - 2c_0 > 0$. Hence

$$(1 + \frac{\gamma}{2C_1^2}\tau)\|u_N^k\|^2 \leq \|u_N^{k-1}\|^2. \tag{3.3}$$

Multiplying (3.3) by $(1 + \frac{\gamma}{2C_1^2}\tau)^{k-1}$ and summing them for k from 1 to n, we have

$$\|u_N^n\|^2 \leq (1 + \frac{\gamma}{2C_1^2}\tau)^{-k}\|u_0\|^2 \leq \|u_0\|^2 \triangleq \Upsilon_0^2,$$

which implies that

$$\overline{\lim_{n\to\infty}}\|u_N^n\|^2 \leq (\varrho_0')^2.$$

Taking the sum of (3.2) for k from $k_0 + 1$ to n, we complete the proof of the lemma. □

Corollary 3.2. *For any given $\varrho_0 > \varrho_0'$ and $R_0 > 0$, if $\|u_0\| \leq R_0$, then*

$$\|u_N^n\|^2 \leq \varrho_0^2, \quad \forall n \geq n_0 = \left(\ln \frac{R_0^2}{\varrho_0^2 - (\varrho_0')^2}\right) / \ln(1 + c_1\tau).$$

Lemma 3.3. *In addition to the conditions of Lemma 2.2, the solution u_N^k of problem (2.1) satisfies*

$$\|\nabla u_N^n\|^2 \leq \frac{1}{(1 + c_3\tau)^n}\|\nabla u_0\|^2 + c_4 \leq \|\nabla u_0\|^2 + c_4 \triangleq \Upsilon_1^2, \quad n \geq 1,$$

$$\|\nabla u_N^k\|^2 \leq \varrho_1^2, \quad \forall n \geq n_0 + N_0 \triangleq n_1,$$

$$\tau^2 \sum_{k=1}^{n} \|\bar{\partial}_t \nabla u_N^k\|^2 \leq C_2'(1 + t_n), \quad \forall n \geq 1,$$

where n_0 is given by Corollary 3.2, N_0 is an arbitrary positive integer, r is an arbitrary positive number such that $N_0\tau = r$, the constant ϱ_1 is independent of N, n, τ and $\|u_0\|_{H^1}$, the two constants $C_2' = C_2'(\|u_0\|_{H^1})$ and $\Upsilon_1 = \Upsilon_1(\|u_0\|_{H^1})$ are independent of N, n and τ.

Proof. Setting $v = \Delta u_N^k$ in (3.1), we derive that

$$\begin{aligned} &\frac{1}{2}\bar{\partial}_t\|\nabla u_N^k\|^2 + \frac{\tau}{2}\|\bar{\partial}_t \nabla u_N^k\|^2 + \gamma\|\nabla\Delta u_N^k\|^2 + (\varphi'(u_N^k), |\Delta u_N^k|^2) \\ = &-(\varphi''(u_N^k)|\nabla u_N^k|^2, \Delta u_N^k) - (\beta \cdot \psi'(u_N^k)\nabla u_N^k, \Delta u_N^k). \end{aligned}$$

Similar to the proof of Lemma 2.2, use Sobolev's interpolation inequality. Simple calculations show that

$$\bar{\partial}_t\|\nabla u_N^k\|^2 + \tau\|\bar{\partial}_t \nabla u_N^k\|^2 + \gamma\|\nabla\Delta u_N^k\|^2 \leq C_7.$$

By Poincaré's inequality, we get

$$\|\nabla u_N^k\|^2 \leq C_1\|\Delta u_N^k\|^2 \leq C_1^2\|\nabla\Delta u_N^k\|^2.$$

Hence

$$\bar{\partial}_t\|\nabla u_N^k\|^2+\tau\|\bar{\partial}_t\nabla u_N^k\|^2+\frac{\gamma}{2C_1^2}\|\nabla u_N^k\|^2+\frac{\gamma}{2C_1}\|\Delta u_N^k\|^2\le C_7, \tag{3.4}$$

that is

$$(1+\frac{\gamma}{2C_1^2}\tau)\|\nabla u_N^k\|^2\le\|\nabla u_N^{k-1}\|^2+C_7\tau. \tag{3.5}$$

Multiplying (3.5) by $(1+\frac{\gamma}{2C_1^2}\tau)^{k-1}$ and summing then for k from 1 to n, we have

$$\begin{aligned}\|\nabla u_N^k\|^2&\le\frac{1}{(1+\frac{\gamma}{2C_1^2}\tau)^n}\|\nabla u_0\|^2+\frac{2C_1^2C_7}{\gamma}\\&\le\|\nabla u_0\|^2+\frac{2C_1^2C_7}{\gamma}\triangleq\Upsilon_1^2.\end{aligned} \tag{3.6}$$

It then follows from (3.2) that

$$\left(\frac{\gamma}{2C_1}-2c_0\right)\tau\sum_{k_0+1}^{k_0+N_0}\|\nabla u_N^k\|^2\le\|u_N^{k_0}\|^2. \tag{3.7}$$

By applying the discrete uniform Gronwall's inequality, we deduce that

$$\begin{aligned}\|\nabla u_N^k\|^2&\le\left(\frac{\tau}{r}\sum_{k_0+1}^{k_0+N_0}\|\nabla u_N^k\|^2+\tau\sum_{k_0+1}^{k_0+N_0}C_7\right)e^{-\tau\sum_{k_0+1}^{k_0+N_0}\frac{\gamma}{2C_1^2}}\\&\le\varrho_1^2,\quad\forall n\ge n_1=n_0+N_0.\end{aligned}$$

Taking the sum of (3.4), we complete the proof of Lemma 3.3. □

Corollary 3.4. *Under the hypotheses of Lemma 3.3, we have*

$$\|u_N^n\|_q\le C(\varrho_0,\varrho_1),\quad\forall n\ge n_1,\ 0<q<\infty,$$

$$\|u_N^n\|_q\le C(\Upsilon_0,\Upsilon_1),\quad\forall n\ge 1,\ 0<q<\infty.$$

Lemma 3.5. *In addition to the conditions of Lemma 2.4, we suppose that* $u_0\in H_p^2(\Omega)$ *satisfying* $\|\Delta u_0\|^2\le R^2$. *Then we have*

$$\|\Delta u_N^n(x,t)\|^2\le\varrho_2^2,\quad\forall n\ge n_2=n_1+N_0,$$

$$\|\Delta u_N^n\|^2\le\Upsilon_2^2,\quad\forall n\ge 1,$$

$$\tau^2\sum_{k=1}^{n}\|\bar{\partial}_t\Delta u_N^k\|^2+\tau\sum_{k=1}^{n}\|\nabla\Delta u_N^k\|^2\le C_3'(1+t_n),\quad\forall n\ge 1,$$

where n_1 *is given by Lemma 3.3,* N_0 *is an arbitrary positive integer,* r *is an arbitrary positive number such that* $N_0\tau=r$, *the constant* ϱ_2 *is independent of* N, n, τ, *and* $\|u_0\|_{H^2}$, *the two constants* $\Upsilon_2=\Upsilon_2(\|u_0\|_{H^2})$ *and* $C_3'=C_2'(\|u_0\|_{H^2})$ *are independent of* N, n *and* τ.

Proof. Setting $v = \Delta^2 u_N^k$ in (3.1), we derive that

$$\begin{aligned}(3.8)\qquad &\frac{1}{2}\bar{\partial}_t\|\Delta u_N^k\|^2 + \frac{\tau}{2}\|\bar{\partial}_t\Delta u_N^k\|^2 + \gamma\|\Delta^2 u_N^k\|^2 \\ &= (\Delta\varphi(u_N^k), \Delta^2 u_N^k) + (\beta\cdot\nabla\psi(u_N^k), \Delta^2 u_N^k).\end{aligned}$$

When $\mathbf{k} \geq \mathbf{n_1}$, similar to the proof of Lemma 2.2, use Sobolev's interpolation inequality. Simple calculations show that

$$\begin{aligned}(3.9)\qquad (\beta\cdot\nabla\psi(u_N^k), \Delta^2 u_N^k) &\leq \frac{\gamma}{8}\|\Delta^2 u_N^k\|^2 + \frac{|\beta|^2}{\gamma}\|\psi'(u_N^k)\nabla u_N^k\|^2 \\ &\leq \frac{\gamma}{8}\|\Delta^2 u_N^k\|^2 + C\|u_N^k\|_{12}^6\|\nabla u_N^k\|_4^2 \\ &\leq \frac{\gamma}{4}\|\Delta^2 u_N^k\|^2 + C(\varrho_0, \varrho_1)\end{aligned}$$

and

(3.10)

$$\begin{aligned}&(\Delta\varphi(u_N^k), \Delta^2 u_N^k) \\ &\leq \frac{\gamma}{8}\|\Delta^2 u_N^k\|^2 + \frac{2}{\gamma}\|\varphi'(u_N^k)\Delta u_N^k\|^2 + \frac{2}{\gamma}\|\varphi''(u_N^k)|\nabla u_N^k|^2\|^2 \\ &\leq \frac{\gamma}{8}\|\Delta^2 u_N^k\|^2 + C(\|u_N^k\|_8^4\|\Delta u_N^k\|_4^2 + \|u_N^k\|_4^2\|\nabla u_N^k\|_8^4 + \|\nabla u_N^k\|_{12}^6\|\nabla u_N^k\|_4^2) \\ &\leq \frac{\gamma}{8}\|\Delta^2 u_N^k\|^2 + C(\|\Delta u_N^k\|_4^2 + \|\nabla u_N^k\|_8^4 + \|\nabla u_N^k\|_4^2) \\ &\leq \frac{\gamma}{4}\|\Delta^2 u_N^k\|^2 + C(\varrho_0, \varrho_1).\end{aligned}$$

It then follows from (3.8)-(3.10) that

$$\bar{\partial}_t\|\Delta u_N^k\|^2 + \tau\|\bar{\partial}_t\Delta u_N^k\|^2 + \gamma\|\Delta^2 u_N\|^2 \leq C(\varrho_0, \varrho_1).$$

Hence

$$(3.11)\qquad \bar{\partial}_t\|\Delta u_N^k\|^2 + \tau\|\bar{\partial}_t\Delta u_N^k\|^2 + C_{11}(\|\Delta u_N^k\|^2 + \|\nabla\Delta u_N^k\|^2) \leq C(\varrho_0, \varrho_1).$$

By (3.4), we obtain

$$(3.12)\qquad \left(\frac{\gamma}{2C_1} - 2c_0\right)\tau\sum_{k_0+1}^{k_0+N_0}\|\Delta u_N^k\|^2 \leq \|\nabla u_N^{k_0}\|^2 + C_7 r.$$

By applying the discrete uniform Gronwall's inequality, we deduce that

$$\begin{aligned}(3.13)\qquad \|\Delta u_N^k\|^2 &\leq \left(\frac{\tau}{r}\sum_{k_0+1}^{k_0+N_0}\|\Delta u_N^k\|^2 + \tau\sum_{k_0+1}^{k_0+N_0}C(\varrho_0+\varrho_1)\right)e^{-\tau\sum_{k_0+1}^{k_0+N_0}C_{11}} \\ &\leq \varrho_2^2, \quad \forall n \geq n_2 = n_1 + N_0.\end{aligned}$$

For $\mathbf{k} \geq \mathbf{1}$, as the proof of inequality (3.11), we have

$$(3.14)\qquad \bar{\partial}_t\|\Delta u_N^k\|^2 + \tau\|\bar{\partial}_t\Delta u_N^k\|^2 + C_{11}(\|\Delta u_N^k\|^2 + \|\nabla\Delta u_N^k\|^2) \leq C(\Upsilon_0, \Upsilon_1).$$

Taking the sum of (3.14) for k from 1 to n, we obtain
(3.15)

$$\|\Delta u_N^k\|^2 + \tau^2 \sum_{k=1}^{n} \|\bar{\partial}_t \Delta u_N^k\|^2 + C_{11}\tau \sum_{k=1}^{n} \|\nabla \Delta u_N^k\|^2 \le C_3'(1+t_n), \quad \forall n \ge 1.$$

Combining (3.13) and (3.15), the proof of this lemma is completed. □

Corollary 3.6. *Under the hypotheses of Lemma 3.3, we have*

$$\|u_N^n\|_\infty \le C(\varrho_0, \varrho_1), \quad \forall n \ge n_2,$$
$$\|u_N^n\|_\infty \le C(\Upsilon_0, \Upsilon_1), \quad \forall n \ge 1.$$

Theorem 3.7. *Suppose that γ is sufficiently large, $u_0 \in H_p^2(\Omega)$, $\varphi \in C^2$ and $\psi \in C^1$ are also satisfy*

$$\varphi'(r) > 0, \quad \varphi^{(i)}(r) \le c|r|^{k-i} + c', \quad \psi'(r) \le cr^2\sqrt{\varphi'(r)} + c',$$

where $k \le 3$ is a positive constant and $i = 0, 1, 2$. Then the semigroup of operator $\{S_N^\tau(n)\}_{n\ge 0}$ generated by problem (3.1) has a compact global attractor $\mathcal{A}_N^\tau \subset H_p^2(\Omega)\bigcap S_N$.

Proof. Set $H = H_p^2(\Omega)\bigcap S_N$, S_N^τ is a semigroup operator, i.e., the solution operator generated by problem (3.1).

(I) By using the results of Lemma 3.1, Lemma 3.3, Lemma 3.5, and assuming that $u_N^0 \in \mathcal{B} = \{u_N^0 | \|u_N^0\|_{H^2} \le R_0\} \subset H_p^2(\Omega)\bigcap S_N$, we have

$$\|S_N^\tau(n)u_N^0\|_{H^2} = \|u_N^n\|_{H^2} \le (\varrho_0^2 + \varrho_1^2 + \varrho_2^2)^{\frac{1}{2}}, \quad \forall n \ge n_1(R).$$

Therefore

$$\mathcal{B}_0 = \{u_N^n \in H_p^2(\Omega)\bigcap S_N | \|u_N^n\|_{H^2} \le (\varrho_0^2 + \varrho_1^2 + \varrho_2^2)^{\frac{1}{2}}\}$$

is a bounded absorbing set of the semigroup of operator $\{S_N^\tau(n)\}_{n\ge 0}$.

(II) From Lemma 3.1, Lemma 3.3, Lemma 3.5 and their corollaries, we have

$$\|S_N^\tau(n)u_N^0\|_{H^2} \le (\Upsilon_0^2 + \Upsilon_1^2 + \Upsilon_2^2)^{\frac{1}{2}}, \quad \forall n \ge 0.$$

This means that $\{S_N^\tau(n)\}$ is uniformly bounded in $H_p^2(\Omega)\bigcap S_N$. Since a closed bounded set is a compact set in the finite dimensional space $H_p^2(\Omega)\bigcap S_N$, the operator $S_N^\tau(n)$ is uniformly compact for any $n \ge 0$.

On the other hand, it is easy to see that the continuity of operator $S_N^\tau(n)$ is from its boundness. Hence, the proof is completed. □

3.2. Convergence of the global attractors $\mathcal{A}_N^\tau$

Let $G_N : L_p^2(\Omega) \to S_N$ be the integral projection operator, i.e., for any given $u \in L^2(\Omega)$, we have

(3.16) $$(\nabla(G_N u), \nabla v) = (u, v), \quad \forall v \in S_N.$$

Then for any u, $v \in L^2(\Omega)$, we have $(G_N u, v) = (u, G_N v)$.

Lemma 3.8. *For the integral projection operator G_N, the following results hold*:

(1) $\|\Delta(G_N u)\| = \|P_N u\|, \quad \forall u \in L_p^2(\Omega)$;
(2) $\|G_N(\nabla u)\| = \|\nabla(G_N u)\|, \quad \forall u \in H_p^1(\Omega)$;
(3) $\|G_N(\Delta u)\| = \|\nabla[G_N(\nabla u)]\| = \|\Delta(G_N u)\|, \quad \forall u \in H_p^2(\Omega)$;
(4) $\|G_N^2(\Delta u)\| = \|\nabla[G_N^2(\nabla u)]\| = \|\Delta(G_N^2 u)\|, \quad \forall u \in H_p^2(\Omega)$.

Similar to Lemma 3.5, the following result can be proved easily.

Lemma 3.9. *Under the hypotheses of Lemma 3.5, we have the estimates for the smooth solution $u(x,t)$ of problem* (1.1)-(1.3) :

$$\begin{aligned}
&\int_0^t \|\nabla u_t\|^2 ds \le C(1+t),\\
&t\|u_t\|^2 + \int_0^t s\|\Delta u_t\|^2 ds \le C(1+t^2),\\
&t^2\|\nabla u_t\|^2 + \int_0^t s^2\|\nabla\Delta u_t\|^2 ds \le C(1+t^3),\\
&t^3\|\Delta u_t\|^2 + \int_0^t s^3(\|u_{tt}\|^2 + \|\Delta^2 u_t\|^2) ds \le C(1+t^4),\\
&t^4\|\nabla\Delta u_t\|^2 + \int_0^t s^4\|\nabla u_{tt}\|^2 ds \le C(1+t^5),
\end{aligned}$$

where the constant C is independent of t.

Theorem 3.10. *Suppose that the conditions of Theorem 3.7 hold. Then*

$$d(\mathcal{A}_N^\tau, \mathcal{A}) \to 0 \quad \textit{as } \tau \to 0,\ N \to +\infty.$$

Proof. Let $\|u_0\|_{H^2} \le R_0$. On account of Theorem 2.7, this theorem will be proved by taking the error estimates of the solution u_N^n of discrete problem (3.1). Now, we accomplish them through two steps.

Step 1. Take the error estimates of the solution v^n of the linear scheme as follows:

$$\begin{aligned}
&(\bar\partial_t v_N^k + \gamma\Delta^2 v_N^k - \gamma\Delta v_N^k - \Delta\varphi(u_N^k) - \beta\cdot\nabla\psi(u_N^k), \zeta) = (-\gamma\Delta u_N^k, \zeta),\\
&v_N^0 = P_N u_0, \quad \forall\zeta \in S_N.
\end{aligned} \tag{3.17}$$

Set $u^k - v_N^k = u^k = P_N u^k + P_N u^k - v_N^k = \rho^k + \theta^k$. Hence, θ^k satisfies

$$\begin{aligned}
&(\bar\partial_t\theta^k + \gamma\Delta^2\theta^k - \gamma\Delta\theta^k - (\bar\partial_t u^k - u_t^k), \zeta) = 0, \quad \forall\zeta \in S_N,\\
&\theta^0 = 0.
\end{aligned} \tag{3.18}$$

Letting $\zeta = \theta^k$ in (3.18), we derive that

$$\frac{1}{2}\bar\partial_t\|\theta^k\|^2 + \frac{\tau}{2}\|\bar\partial_t\theta^k\|^2 + \gamma(\|\Delta\theta^k\|^2 + \|\nabla\theta^k\|^2) = (\bar\partial_t u^k - u_t^k, \theta^k).$$

From the definition of G_N, we find that

$$(\bar{\partial}_t u^k - u_t^k, \theta^k) = (\nabla[G_N(\bar{\partial}_t u^k - u_t^k)], \nabla\theta^k) \le \frac{\gamma}{2}\|\nabla\theta^k\|^2 + \frac{2}{\gamma}\|\nabla[\bar{\partial}_t u^k - u_t^k)]\|.$$

Noticing that

$$\begin{aligned}\|\nabla[\bar{\partial}_t u^k - u_t^k)]\|^2 &= \frac{1}{\tau^2}\|\int_{t_{k-1}}^{t_k}(s-t_{k-1})\nabla G_N u_{tt} ds\|^2 \\ &\le \frac{1}{\tau^2}\int_{t_{k-1}}^{t_k}\frac{(s-t_{k-1}^2}{s^2}ds\int_{t_{k-1}}^{t_k}s^2\|\nabla G_N u_{tt}\|^2 ds \\ &\le \frac{\tau}{t_k^2}\int_{t_{k-1}}^{t_k}s^2\|\nabla G_N u_{tt}\|^2 ds.\end{aligned}$$

Therefore, we deduce that

$$\bar{\partial}_t\|\theta^k\|^2+\tau\|\bar{\partial}_t\theta^k\|^2+2\gamma\|\Delta\theta^k\|^2+\gamma\|\nabla\theta^k\|^2 \le \frac{4\tau}{\gamma t_k^2}\int_{t_{k-1}}^{t_k}s^2\|\nabla G_N u_{tt}\|^2 ds. \tag{3.19}$$

Multiplying (3.19) by t_k^2, taking the sum for k from 1 to n, using $\|\nabla G_N u_{tt}\| \le C\|\nabla\Delta u_t\|$, we get

$$\begin{aligned} &t_n^2\|\theta^n\|^2 + \tau^2\sum_{k=1}^n t_k^2\|\bar{\partial}_t\theta^k\|^2 + \gamma\tau\sum_{k=1}^n t_k^2(2\|\Delta\theta^k\|^2 + \|\nabla\theta^k\|^2) \\ &\le 3\tau\sum_{k=1}^n t_k\|\theta^k\|^2 + \frac{4\tau^2}{\gamma}\sum_{k=1}^n\int_{t_{k-1}}^{t_k}s^2\|\nabla G_N u_{tt}\|^2 ds \\ &\le 3\tau\sum_{k=1}^n t_k\|\theta^k\|^2 + C\tau^2\int_0^{t_n}s^2\|\nabla\Delta u_t\|^2 ds \\ &\le 3\tau\sum_{k=1}^n t_k\|\theta^k\|^2 + C\tau^2(1+t_n^3).\end{aligned} \tag{3.20}$$

Now, we estimate $\tau\sum_{k=1}^n t_k\|\theta^k\|^2$ in (3.20). Set $\zeta = G_N\theta^k$ in (3.18), we have

$$(\bar{\partial}_t\theta^k + \gamma\Delta^2\theta^k - \gamma\Delta\theta^k - (\bar{\partial}_t u^k - u_t^k), G_N\theta^k) = 0.$$

Noticing that

$$\begin{aligned}&(\bar{\partial}_t\theta^k, G_N\theta^k) = \frac{1}{2}\bar{\partial}_t\|\nabla G_N\theta^k\|^2, \\ &(\gamma\Delta^2\theta^k - \gamma\Delta\theta^k, G_N\theta^k) = \gamma\|\nabla\theta^k\|^2 + \gamma\|\theta^k\|^2, \\ &(\bar{\partial}_t u^k - u_t^k, G_N\theta^k) \le \frac{\gamma}{2}\|\theta^k\|^2 + \frac{1}{2\gamma}\|G_N(\bar{\partial}_t u^k - u_t^k)\|^2.\end{aligned}$$

Hence

$$\bar{\partial}_t\|\nabla G_N\theta^k\|^2 + \gamma(2\|\nabla\theta^k\|^2 + \|\theta^k\|^2) \le \frac{1}{\gamma}\|G_N(\bar{\partial}_t u^k - u_t^k)\|^2. \tag{3.21}$$

Multiplying (3.21) by τt_k, taking the sum for k from 1 to n, using $\|G_N u_{tt}\| \le C\|\Delta u_t\|$, we get

$$
\begin{aligned}
(3.22)\qquad & t_n\|\nabla G_N\theta^n\|^2+\gamma\tau\sum_{k=1}^{n}t_k(2\|\nabla\theta^k\|^2+\|\theta^k\|^2)\\
&\le \frac{\tau}{\gamma}\sum_{k=1}^{n}t_k\|G_N(\bar\partial_t u^k-u_t^k)\|^2+\tau\sum_{k=1}^{n}\|\nabla G_N\theta^k\|^2\\
&\le \frac{\tau^2}{\gamma}\int_0^{t_n}s\|G_N u_{tt}\|^2ds+\tau\sum_{k=1}^{n}\|\nabla G_N\theta^k\|^2\\
&\le C\tau^2(1+t_n^2)+\tau\sum_{k=1}^{n}\|\nabla G_N\theta^k\|^2.
\end{aligned}
$$

To estimate $\tau\sum_{k=1}^{n}\|\nabla G_N\theta^k\|^2$ in (3.22), set $\zeta=G_N^2\theta^k$ in (3.18). Therefore

$$(\bar\partial_t\theta^k+\gamma\Delta^2\theta^k-\gamma\Delta\theta^k-(\bar\partial_t u^k-u_t^k),G_N^2\theta^k)=0.$$

Noticing that

$$
\begin{aligned}
&(\bar\partial_t\theta^k,G_N^2\theta^k)=\frac{1}{2}\bar\partial_t\|G_N\theta^k\|^2+\frac{\tau}{2}\|\nabla G_N\theta^k\|^2,\\
&(\gamma\Delta^2\theta^k-\gamma\Delta^k,G_N^2\theta^k)=\gamma\|\theta^k\|^2+\gamma\|\nabla G_N\theta^k\|^2,\\
&(\bar\partial_t u^k-u_t^k,G_N^2\theta^k)\le\frac{\gamma}{2}\|\nabla G_N\theta^k\|^2+\frac{1}{2\gamma}\|\nabla G_N^2(\bar\partial_t u^k-u_t^k)\|^2.
\end{aligned}
$$

Thus

$$(3.23)\qquad \bar\partial_t\|G_N\theta^k\|^2+\gamma(2\|\theta^k\|^2+\|\nabla G_N\theta^k\|^2)\le\frac{1}{\gamma}\|\nabla G_N^2(\bar\partial_t u^k-u_t^k)\|^2.$$

Taking the sum of (3.23) for k from 1 to n, applying $\|\nabla G_N^2 u_{tt}\|\le C\|\nabla u_t\|$, we obtain

$$
\begin{aligned}
(3.24)\qquad & \|G_N\theta^k\|^2+\gamma\tau\sum_{k=1}^{n}(2\|\theta^k\|^2+\|\nabla G_N\theta^k\|^2)\\
&\le\frac{\tau}{\gamma}\sum_{k=1}^{n}\|\nabla G_N^2(\bar\partial_t u^k-u_t^k)\|^2\le\frac{\tau^2}{\gamma}\int_0^{t_n}\|\nabla G_N^2(\bar\partial_t u^k-u_t^k)\|^2ds\\
&\le C\tau^2(1+t_n).
\end{aligned}
$$

Adding (3.20), (3.22) and (3.24) together gives

$$
\begin{aligned}
(3.25)\qquad & t_n^2\|\theta^k\|^2+\tau^2\sum_{k=1}^{n}t_k^2\|\bar\partial_t\theta^k\|^2+\gamma\tau\sum_{k=1}^{n}t_k^2(2\|\Delta\theta^k\|^2+\|\nabla\theta^k\|^2)\\
&\quad+\gamma\tau\sum_{k=1}^{n}t_k(2\|\nabla\theta^k\|^2+\|\theta^k\|^2)+\gamma\tau\sum_{k=1}^{n}(2\|\theta^k\|^2+\|\nabla G_N\theta^k\|^2)\\
&\le C\tau^2(1+t_n^3).
\end{aligned}
$$

Setting $\zeta = \Delta\theta^k$ in (3.18), we derive that

$$\bar{\partial}_t\|\nabla\theta^k\|^2 + 2\gamma\|\nabla\Delta\theta^k\|^2 + \gamma\|\Delta\theta^k\|^2 \le \frac{1}{\gamma}\|\bar{\partial}_t u^k - u_t^k\|^2.$$

Multiply above inequality by τt_k^3, take the sum for k from 1 to n. Simple calculations show

$$t_n^3\|\nabla\theta^n\|^2 \le C\tau^2(1+t_n^4). \tag{3.26}$$

Setting $\zeta = \Delta^2\theta^k$ in (3.18), we derive that

$$\bar{\partial}_t\|\Delta\theta^k\|^2 + 2\gamma\|\Delta^2\theta^k\|^2 + \gamma\|\nabla\Delta\theta^k\|^2 \le \frac{1}{\gamma}\|\nabla[\bar{\partial}_t u^k - u_t^k]\|^2.$$

Multiply above inequality by τt_k^4, take the sum for k from 1 to n. Simple calculations show

$$t_n^4\|\nabla\theta^n\|^2 \le C\tau^2(1+t_n^5). \tag{3.27}$$

Step 2. Take the error estimates of solution u_N^n of problem (3.1). Set $v_N^k - u_N^k = e^k$. Thus, e^k satisfies

(3.28)

$$\begin{aligned}
&(\bar{\partial}_t e^k + \gamma\Delta e^k + \gamma\Delta\theta^k - \Delta(\varphi(u^k) - \varphi(u_N^k)) - \beta\cdot\nabla(\psi(u^k) - \psi(u_N^k)), \zeta) = 0,\\
&\forall\zeta \in S_N,\ k = 1, 2, \dots,\\
&e^0 = 0.
\end{aligned}$$

Setting $\zeta = e^k$ in (3.28), we get

$$\begin{aligned}
&\frac{1}{2}\bar{\partial}_t\|e^k\|^2 + \frac{\tau}{2}\|\bar{\partial}_t e^k\|^2 + \gamma\|\Delta e^k\|^2\\
&= (\Delta(\varphi(u^k) - \varphi(u_N^k)) + \beta\cdot\nabla(\psi(u^k) - \psi(u_N^k)), e^k) - \gamma(\Delta\theta^k, e^k)\\
&\le \|\varphi'(\lambda_1 u^k + (1-\lambda_1)u_N^k)\|_\infty\|u^k - u_N^k\|\|\Delta e^k\|\\
&\quad + |\beta|\|\psi'(\lambda_2 u^k + (1-\lambda_2)u_N^k)\|_\infty\|u^k - u_N^k\|\|\nabla e^k\| + \gamma\|\theta^k\|\|\Delta e^k\|\\
&\le C\|u^k - u_N^k\|\|\Delta e^k\| + C\|u^k - u_N^k\|\|\nabla e^k\| + \gamma\|\theta^k\|\|\Delta e^k\|\\
&\le \frac{\gamma}{2}\|\Delta e^k\|^2 + C(\|\rho^k\|^2 + \|\theta^k\|^2 + \|e^k\|^2),
\end{aligned}$$

where $\lambda \in (0,1)$. Hence

$$\bar{\partial}_t\|e^k\|^2 + \tau\|\bar{\partial}_t e^k\|^2 + \gamma\|\Delta e^k\|^2 \le C(\|\rho^k\|^2 + \|\theta^k\|^2 + \|e^k\|^2).$$

Using discrete Gronwall's inequality, we obtain

$$\|e^n\|^2 + \gamma\tau\|\Delta e^k\|^2 \le C\tau\sum_{k=1}^{n}(\rho^k\|^2 + \|\theta^k\|^2) \le C(N^{-4} + \tau^2). \tag{3.29}$$

Setting $\zeta = \Delta e^k$ in (3.28), using Sobolev's interpolation inequality, we immediately obtain

$$\bar{\partial}_t\|\nabla e^k\|^2 + \gamma\|\nabla\Delta e^k\|^2 \le C(\|\nabla\rho^k\|^2 + \|\nabla\theta^k\|^2 + \|e^k\|^2).$$

Multiplying above inequality by τt_k^2, taking the sum for k from 1 to n, using (3.29) and (3.25), we get

$$t_n^2\|\nabla e^n\|^2 + \gamma\tau\sum_{k=1}^{n}\|\nabla\Delta e^k\|^2 \le C(N^{-2}+\tau^2+\tau\sum_{k=1}^{n}(\|\nabla e^k\|^2 + t_k^2\|\nabla\theta^k\|^2) \le C(N^{-2}+\tau^2). \tag{3.30}$$

Setting $\zeta = \Delta^2 e^k$ in (3.28), using Sobolev's interpolation inequality, we immediately obtain

$$\begin{aligned}\bar{\partial}_t\|\Delta e^k\|^2 + \gamma\|\Delta^2 e^k\|^2 &\le C(\|\Delta\rho^k\|^2 + \|\Delta\theta^k\|^2 + \|e^k\|^2)\\ &\le C(N^{-2}\|\nabla\Delta u\|^2 + \|\Delta\theta^k\|^2 + \|e^k\|^2).\end{aligned}$$

Multiplying above inequality by τt_k^2, taking the sum for k from 1 to n, using (3.29) and (3.25), we get

$$t_n^2\|\Delta e^n\|^2 + \gamma\tau\sum_{k=1}^{n}\|\Delta^2 e^k\|^2 \le C(N^{-2}+\tau^2+\tau\sum_{k=1}^{n}t_k^2\|\nabla\theta^k\|^2) \le C(N^{-2}+\tau^2). \tag{3.31}$$

By using the triangle inequality, we have

$$\|u^n - u_N^n\|_{H^2}^2 \le 2(\|\rho^n\|_{H^2}^2 + \|\theta^n\|_{H^2}^2 + \|e^n\|_{H^2}^2) \le C(N^{-2}+\tau^2), \quad \forall t\in(0,+\infty).$$

Then, we complete the proof of the theorem. □

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