Bull. Korean Math. Soc. **52** (2015), No. 5, pp. 1467–1480
http://dx.doi.org/10.4134/BKMS.2015.52.5.1467

GENERALIZED LUCAS NUMBERS OF THE FORM $5kx^2$ AND $7kx^2$

Olcay Karaatli and Refik Keskin

Abstract. Generalized Fibonacci and Lucas sequences (U_n) and (V_n) are defined by the recurrence relations $U_{n+1} = PU_n+QU_{n-1}$ and $V_{n+1} = PV_n + QV_{n-1}$, $n \geq 1$, with initial conditions $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = P$. This paper deals with Fibonacci and Lucas numbers of the form $U_n(P,Q)$ and $V_n(P,Q)$ with the special consideration that $P \geq 3$ is odd and $Q = -1$. Under these consideration, we solve the equations $V_n = 5kx^2$, $V_n = 7kx^2$, $V_n = 5kx^2 \pm 1$, and $V_n = 7kx^2 \pm 1$ when $k \mid P$ with $k > 1$. Moreover, we solve the equations $V_n = 5x^2 \pm 1$ and $V_n = 7x^2 \pm 1$.

1. Introduction

Let P and Q be nonzero integers such that $P^2 + 4Q \neq 0$. Generalized Fibonacci sequence $(U_n(P,Q))$ and Lucas sequence $(V_n(P,Q))$ are defined as follows:

$$U_0(P,Q) = 0, U_1(P,Q) = 1, U_{n+1}(P,Q) = PU_n(P,Q) + QU_{n-1}(P,Q) \ (n \geq 1)$$

and

$$V_0(P,Q) = 2, V_1(P,Q) = P, V_{n+1}(P,Q) = PV_n(P,Q) + QV_{n-1}(P,Q) \ (n \geq 1).$$

The numbers $U_n = U_n(P,Q)$ and $V_n = V_n(P,Q)$ are called the n-th generalized Fibonacci and Lucas numbers, respectively. Furthermore, generalized Fibonacci and Lucas numbers for negative subscripts are defined as

$$U_{-n} = U_{-n}(P,Q) = -U_n(P,Q)/(-Q)^n$$

and

$$V_{-n} = V_{-n}(P,Q) = V_n(P,Q)/(-Q)^n$$

for $n \geq 1$. If $\alpha = \frac{P+\sqrt{P^2+4Q}}{2}$ and $\beta = \frac{P-\sqrt{P^2+4Q}}{2}$ are the roots of the equation $x^2 - Px - Q = 0$, then we have the following well known expressions named

Received June 5, 2014; Revised January 22, 2015.
2010 *Mathematics Subject Classification.* 11B37, 11B39, 11B50.
Key words and phrases. generalized Fibonacci numbers, generalized Lucas numbers, congruences.

Binet's formulas

$$U_n = U_n(P,Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = V_n(P,Q) = \alpha^n + \beta^n$$

for all $n \in \mathbb{Z}$. Since $U_n = U_n(-P,Q) = (-1)^n U_n(P,Q)$ and $V_n = V_n(-P,Q) = (-1)^n V_n(P,Q)$, it will be assumed that $P \geq 1$. Moreover, we assume that $P^2 + 4Q > 0$. Special cases of the sequences (U_n) and (V_n) are known. If $P = Q = 1$, then $(U_n(1,1))$ is the familiar Fibonacci sequence (F_n) and the sequence $(V_n(1,1))$ is the familiar Lucas sequence (L_n). If $P = 2$ and $Q = 1$, we have the well known Pell sequence (P_n) and Pell–Lucas sequence (Q_n). For more information about generalized Fibonacci and Lucas sequences, the reader can follow [3, 8, 9, 10].

Generalized Fibonacci and Lucas numbers of the form kx^2 have been investigated by many authors and progress in determining the square or k times a square terms of U_n and V_n has been made in certain special cases. Interested readers can consult [5] or [14] for a brief history of this subject. In [11], the authors, applying only congruence properties of sequences, determined all indices n such that $U_n = x^2$, $U_n = 2x^2$, $V_n = x^2$, and $V_n = 2x^2$ for all odd relatively prime values of P and Q. Furthermore, the same authors [12] solved $V_n = kx^2$ under some assumptions on k. In [1], when P is odd, Cohn solved $V_n = Px^2$ and $V_n = 2Px^2$ with $Q = \pm 1$. In [14], the authors determined, assuming $Q = 1$, all indices n such that $V_n(P,1) = kx^2$ when $k \mid P$ and P is odd, where k is a square-free positive divisor of P. The values of n have been found for which $V_n(P,-1)$ is of the form kx^2, $2kx^2$, $kx^2 \pm 1$, and $2kx^2 \pm 1$ with $k \mid P$ and $k > 1$ [4]. Moreover, the values of n have been found for which $V_n(P,-1)$ is of the form $2x^2 \pm 1$, $3x^2 - 1$, and $6x^2 \pm 1$ [4] and the author give all integer solutions of the preceding equations. Our results in this paper add to the above list the values of n for which $V_n(P,-1)$ is of the form $5kx^2$, $5kx^2 \pm 1$, $7kx^2$, and $7kx^2 \pm 1$ when $k \mid P$ and $k > 1$. Furthermore, we determine all indices n such that $V_n(P,-1) = 5x^2 \pm 1$ and $V_n(P,-1) = 7x^2 \pm 1$ and then give all integer solutions of these equations. We need to state here that our method is elementary and used by Cohn, Ribenboim and McDaniel in [1] and [12], respectively.

2. Preliminaries

In this section, we present some theorems, lemmas, and identities, which will be needed during the proof of the main theorems. Instead of $U_n(P,-1)$ and $V_n(P,-1)$, we sometimes write U_n and V_n. Throughout the paper $\left(\frac{*}{*}\right)$ denotes the Jacobi symbol.

Now we can give the following lemma without proof since its proof can be done by induction.

Lemma 1. *Let n be a positive integer. Then*

$$V_n \equiv \begin{cases} \pm 2 \pmod{P^2} & \text{if } n \text{ is even,} \\ \pm nP \pmod{P^2} & \text{if } n \text{ is odd.} \end{cases}$$

We omit the proof of the following lemma due to Keskin and Demirtürk [2].

Lemma 2. *All positive integer solutions of the equation $x^2 - 5y^2 = 1$ are given by $(x, y) = (L_{3z}/2, F_{3z}/2)$ with z even natural number.*

The following theorem is well known (see [6] or [7]).

Theorem 1. *All positive integer solutions of the equation $x^2 - (P^2 - 4)y^2 = 4$ are given by $(x, y) = (V_n, U_n)$ with $n \geq 1$.*

The proofs of the following two theorems can be found in [13].

Theorem 2. *Let $n \in \mathbb{N} \cup \{0\}$, $m, r \in \mathbb{Z}$ and m be a nonzero integer. Then*

$$U_{2mn+r} \equiv U_r \pmod{U_m} \tag{2.1}$$

and

$$V_{2mn+r} \equiv V_r \pmod{U_m}. \tag{2.2}$$

Theorem 3. *Let $n \in \mathbb{N} \cup \{0\}$ and $m, r \in \mathbb{Z}$. Then*

$$U_{2mn+r} \equiv (-1)^n U_r \pmod{V_m} \tag{2.3}$$

and

$$V_{2mn+r} \equiv (-1)^n V_r \pmod{V_m}. \tag{2.4}$$

The following identities are well known.

$$U_{2n} = U_n V_n, \tag{2.5}$$

$$V_n^2 - (P^2 - 4)U_n^2 = 4, \tag{2.6}$$

$$V_{2n} = V_n^2 - 2. \tag{2.7}$$

From Lemma 1 and identity (2.6), we have

$$5 \mid V_n \text{ if and only if } 5 \mid P \text{ and } n \text{ is odd} \tag{2.8}$$

and

(2.9)

$7 \mid V_n$ if and only if $7 \mid P$ and n is odd or $P^2 \equiv 2 \pmod 7$ and $n \equiv 2 \pmod 4$.

If $r \geq 1$, then by (2.7),

$$V_{2^r} \equiv \pm 2 \pmod P. \tag{2.10}$$

If $r \geq 2$, then by (2.7),

$$V_{2^r} \equiv 2 \pmod P. \tag{2.11}$$

It is obvious that

$$5 \mid U_5 \text{ if } P^2 \equiv -1 \pmod 5, \tag{2.12}$$

$$7 \mid U_4 \ \text{ if } P^2 \equiv 2 \pmod 7. \tag{2.13}$$

From now on, unless otherwise stated, we assume that $P \geq 3$ is odd and $Q = -1$.

By using (2.2) together with the fact $8 \mid U_3$, we get

$$V_{6q+r} \equiv V_r \pmod 8. \tag{2.14}$$

Moreover, by using induction, it can be seen that

$$V_{2^r} \equiv 7 \pmod 8$$

and thus

$$\left(\frac{2}{V_{2^r}}\right) = 1 \tag{2.15}$$

and

$$\left(\frac{-1}{V_{2^r}}\right) = -1 \tag{2.16}$$

for $r \geq 1$.

If $r \geq 1$, then

$$\left(\frac{5}{V_{2^r}}\right) = \begin{cases} -1 & \text{if} \quad 5 \mid P, \\ 1 & \text{if} \quad P^2 \equiv 1 \pmod 5, \\ -1 & \text{if} \quad P^2 \equiv -1 \pmod 5. \end{cases} \tag{2.17}$$

Moreover,

$$\left(\frac{7}{V_{2^r}}\right) = \pm 1 \text{ if } 7 \mid P \text{ and } r \geq 1, \tag{2.18}$$

$$\left(\frac{7}{V_{2^r}}\right) = -1 \text{ if } 7 \mid P \text{ and } r \geq 2, \tag{2.19}$$

and

$$\left(\frac{7}{V_{2^r}}\right) = 1 \text{ if } P^2 \equiv 1 \pmod 7 \text{ and } r \geq 1. \tag{2.20}$$

If $r \geq 2$, then $V_{2^r} \equiv -1 \left(\bmod \frac{P^2-3}{2}\right)$ and thus

$$\left(\frac{(P^2-3)/2}{V_{2^r}}\right) = \left(\frac{P^2-3}{V_{2^r}}\right) = 1. \tag{2.21}$$

If $3 \nmid P$, then $V_{2^r} \equiv -1 \pmod 3$ for $r \geq 1$. Therefore we have

$$\left(\frac{3}{V_{2^r}}\right) = 1. \tag{2.22}$$

If $3 \mid P$, then $V_{2^r} \equiv -1 \pmod 3$ for $r \geq 2$ and therefore

$$\left(\frac{3}{V_{2^r}}\right) = 1. \tag{2.23}$$

Besides, we have

$$\left(\frac{P-1}{V_{2^r}}\right)=\left(\frac{P+1}{V_{2^r}}\right)=1. \tag{2.24}$$

3. Main theorems

We assume from this point on that n is a positive integer.

Theorem 4. *If $V_n = 5kx^2$ for some integer x, then $n = 1$.*

Proof. Assume that $V_n = 5kx^2$ for some integer x. Then by (2.8), it follows that $5 \mid P$ and n is odd. Let $n = 6q + r$ with $r \in \{1, 3, 5\}$. Then by (2.14), we obtain $V_n = V_{6q+r} \equiv V_r \pmod 8$. Hence, we have $V_n \equiv V_1, V_3, V_5 \pmod 8$. This implies that $V_n = 5kx^2 \equiv P, 6P \pmod 8$ and therefore $kx^2 \equiv 5P, 6P \pmod 8$. On the other hand, using the fact that $k \mid P$, we may write $P = kM$ with odd M. Thus, we get $kMx^2 \equiv 5PM, 6PM \pmod 8$, implying that $Px^2 \equiv 5PM, 6PM \pmod 8$. Since P is odd, then the preceding congruence gives $x^2 \equiv 5M, 6M \pmod 8$. This shows that $M \equiv 5 \pmod 8$ since M is odd. Now suppose that $n > 1$. Then $n = 4q \pm 1 = 2 \cdot 2^r a \pm 1$, $2 \nmid a$ and $r \geq 1$. Substituting this value of n into $V_n = 5kx^2$ and using (2.4) give

$$5kx^2 = V_n = V_{4q\pm1} = V_{2\cdot 2^r a\pm1} \equiv -V_1 \equiv -P \pmod{V_{2^r}}.$$

This shows that

$$5x^2 \equiv -M \pmod{V_{2^r}}$$

since $(k, V_{2^r}) = 1$. According to the above congruence, we have

$$\left(\frac{5}{V_{2^r}}\right)=\left(\frac{-1}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right).$$

By using the facts that $M \equiv 5 \pmod 8$, $M \mid P$ and identity (2.10), we get

$$\left(\frac{M}{V_{2^r}}\right)=\left(\frac{V_{2^r}}{M}\right)=\left(\frac{\pm2}{M}\right)=\left(\frac{2}{M}\right)=-1.$$

On the other hand, this is impossible since $\left(\frac{5}{V_{2^r}}\right) = -1$ by (2.17) and $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (2.16). Thus, $n = 1$, as claimed. □

By using Theorem 1, we can give the following corollary.

Corollary 1. *The equation $25P^2x^4 - (P^2 - 4)y^2 = 4$ has no integer solutions.*

Theorem 5. *Let $k \mid P$ with $k > 1$. Then there is no integer x such that $V_n = 5kx^2 + 1$.*

Proof. Assume that $V_n = 5kx^2 + 1$ for some integer x. Then by Lemma 1, we have n is even. Let $n = 2m$ for some $m > 0$. Thus, by (2.7), we get $V_n = V_{2m} = V_m^2 - 2 = 5kx^2 + 1$, implying that $V_m^2 \equiv 3 \pmod 5$, a contradiction. □

Corollary 2. *The equation $(5Px^2 + 1)^2 - (P^2 - 4)y^2 = 4$ has no integer solutions.*

Theorem 6. *Let $k \mid P$ with $k > 1$. Then there is no integer x such that $V_n = 5kx^2 - 1$.*

Proof. Assume that $V_n = 5kx^2 - 1$ for some integer x. Then by Lemma 1, n is even. We divide the proof into three cases.

Case 1 : Assume that $5 \mid P$. Then by Lemma 1, it follows that $V_n \equiv \pm 2 \pmod 5$, which contradicts the fact that $V_n \equiv 4 \pmod 5$.

Case 2 : Assume that $P^2 \equiv -1 \pmod 5$. Then we immediately have from (2.12) that $5 \mid U_5$. Let $n = 10q + r$ with $r \in \{0, 2, 4, 6, 8\}$. By (2.2), we get $V_n = V_{10q+r} \equiv V_r \pmod{U_5}$, implying that $V_n \equiv V_0, V_2, V_4, V_6, V_8 \pmod 5$. This shows that $V_n \equiv 2 \pmod 5$, which contradicts the fact that $V_n \equiv 4 \pmod 5$.

Case 3 : Assume that $P^2 \equiv 1 \pmod 5$. Since n is even, $n = 2m$ for some $m > 0$. Hence, we get

$$5kx^2 - 1 = V_n = V_{2m} = V_m^2 - 2$$

by (2.7). If m is odd, then $P \mid V_m$ by Lemma 1 and so we get $k \mid 1$, a contradiction. Thus, m is even. Let $m = 2u$ for some $u > 0$. Then $n = 4u$ and therefore

$$5kx^2 - 1 = V_{4u} \equiv V_0 \equiv 2 \pmod{U_2}$$

by (2.2), which implies that

$$5kx^2 \equiv 3 \pmod P.$$

Since $k \mid P$, it follows that $k \mid 3$ and therefore $k = 3$. So, we conclude that $3 \mid P$. In this case, we have $n = 4u$ and thus by (2.2), we have

$$15x^2 - 1 = V_n = V_{4u} \equiv V_0 \pmod{U_2},$$

implying that

$$15x^2 \equiv 3 \pmod P.$$

Since $3 \mid P$, it is seen that $5x^2 \equiv 1 \pmod{P/3}$. This shows that $\left(\frac{5}{P/3}\right) = 1$. We are in the case that $P^2 \equiv 1 \pmod 5$. So, $P \equiv 1, 4 \pmod 5$. A simple computation shows that $P/3 \equiv \pm 2 \pmod 5$. Hence, we have

$$1 = \left(\frac{5}{P/3}\right) = \left(\frac{P/3}{5}\right) = \left(\frac{\pm 2}{5}\right) = -1,$$

a contradiction. □

Corollary 3. *The equation $(5Px^2 - 1)^2 - (P^2 - 4)y^2 = 4$ has no integer solutions.*

Theorem 7. *If $V_n = 5x^2 + 1$, then $n = 1$ and $V_1 = 5x^2 + 1$ where x is even.*

Proof. Assume that $V_n = 5x^2 + 1$ for some integer x. If n is even, then $n = 2m$ and therefore $V_n = V_{2m} = V_m^2 - 2$ by (2.7). This implies that $V_m^2 \equiv 3 \pmod 5$, a contradiction. Thus, n is odd.

Case 1 : Assume that $5 \mid P$. Since n is odd, it follows from Lemma 1 that $5 \mid V_n$, implying that $5 \mid 1$, a contradiction.

Case 2 : Assume that $P^2 \equiv 1 \pmod 5$. If $n > 1$, then $n = 4q \pm 1 = 2 \cdot 2^r a \pm 1$, $2 \nmid a$ and $r \geq 1$. Thus,

$$5x^2 = V_n - 1 \equiv -V_1 - 1 \equiv -(P+1) \pmod{V_{2^r}}$$

by (2.4). By using (2.16), (2.17), and (2.24), it is seen that

$$1 = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{5}{V_{2^r}}\right)\left(\frac{P+1}{V_{2^r}}\right) = -1,$$

a contradiction. So, $n = 1$. This implies that $P = 5x^2 + 1$ with x even.

Case 3 : Assume that $P^2 \equiv -1 \pmod 5$. If $P \equiv 1 \pmod 4$, since n is odd, we can write $n = 4q \pm 1$. Thus, $5x^2 + 1 = V_n \equiv V_1 \equiv P \pmod{U_2}$ by (2.2). This implies that

$$5x^2 \equiv -1 \pmod P. \tag{3.1}$$

Since $P^2 \equiv -1 \pmod 5$, it follows that $P \equiv \pm 2 \pmod 5$. Hence, we have

$$\left(\frac{5}{P}\right) = \left(\frac{P}{5}\right) = \left(\frac{\pm 2}{5}\right) = -1.$$

On the other hand, using the fact that $P \equiv 1 \pmod 4$, we get

$$\left(\frac{-1}{P}\right) = (-1)^{\frac{P-1}{2}} = 1.$$

And so, (3.1) is impossible. Now if $P \equiv 3 \pmod 4$, let $n = 6q + r$ with $r \in \{1, 3, 5\}$. Hence, by (2.14), we have

$$5x^2 + 1 = V_n = V_{6q+r} \equiv V_r \equiv V_1, V_3, V_5 \pmod 8,$$

implying that

$$5x^2 + 1 \equiv P, 6P \pmod 8.$$

If x is even, then $P, 6P \equiv 1 \pmod 4$, which is impossible since $P \equiv 3 \pmod 4$. If x is odd, then $P, 6P \equiv 6 \pmod 8$, which is impossible since $P \equiv 3, 7 \pmod 8$. This completes the proof. □

Corollary 4. *The equation* $(5x^2 + 1)^2 - (P^2 - 4)y^2 = 4$ *has integer solutions only when* $P = 5x^2 + 1$ *with* x *even.*

Theorem 8. *If* $V_n = 5x^2 - 1$, *then* $n = 1$ *and* $V_1 = 5x^2 - 1$ *with* x *is even, or* $n = 2$ *and* $P = L_{3z}/2$ *and* $x = F_{3z}/2$ *where* z *is even.*

Proof. Assume that $V_n = 5x^2 - 1$ for some integer x. Dividing the proof into three cases, we have

Case 1 : Assume that $5 \mid P$. Lemma 1 implies that $V_n \equiv 0, \pm 2 \pmod 5$, which contradicts $V_n = 5x^2 - 1$.

Case 2 : Assume that $P^2 \equiv 1 \pmod 5$. If $n > 1$ is odd, then $n = 4q \pm 1 = 2 \cdot 2^r a \pm 1$, $2 \nmid a$ and $r \geq 1$. Thus,

$$5x^2 - 1 = V_n \equiv -V_1 \equiv -P \pmod{V_{2^r}},$$

implying that

$$5x^2 \equiv -(P-1) \pmod{V_{2^r}}.$$

By using (2.16), (2.17), and (2.24), it is seen that

$$1 = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{5}{V_{2^r}}\right)\left(\frac{P-1}{V_{2^r}}\right) = -1,$$

which is impossible. So $n = 1$ and $P = 5x^2 - 1$ with x even. Now if n is even, then $n = 2m$ for some $m > 0$. If $m > 1$ is odd, then $n = 2(4q \pm 1) = 2 \cdot 2^r a \pm 2$, $2 \nmid a$ and $r \geq 2$. Thus,

$$5x^2 - 1 = V_n \equiv -V_2 \equiv -(P^2 - 2) \pmod{V_{2^r}},$$

implying that

$$5x^2 \equiv -(P^2 - 3) \pmod{V_{2^r}}.$$

By using (2.16), (2.17), and (2.21), it is seen that

$$1 = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{5}{V_{2^r}}\right)\left(\frac{P^2-3}{V_{2^r}}\right) = -1,$$

a contradiction. Hence, we have $m = 1$ and therefore $n = 2$. Substituting this value of n into $V_n = 5x^2 - 1$ gives $V_2 = P^2 - 2 = 5x^2 - 1$, i.e., $P^2 - 5x^2 = 1$. By Lemma 2, we get $P = L_{3z}/2$ and $x = F_{3z}/2$ with z even natural number. If m is even, then $m = 2u$ for some $u > 0$ and therefore $n = 4u = 2 \cdot 2^r a$, $2 \nmid a$ and $r \geq 1$. Thus, by (2.4)

$$5x^2 - 1 = V_n = V_{4u} \equiv -V_0 \equiv -2 \pmod{V_{2^r}},$$

implying that

$$5x^2 \equiv -1 \pmod{V_{2^r}}.$$

But this is impossible since $\left(\frac{5}{V_{2^r}}\right) = 1$ by (2.17) and $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (2.16).

Case 3 : Assume that $P^2 \equiv -1 \pmod 5$. So, $P \equiv \pm 2 \pmod 5$. If n is odd, then we can write $n = 4q \pm 1$. Thus, $5x^2 - 1 = V_n \equiv V_1 \pmod{U_2}$ by (2.2). This implies that $5x^2 \equiv 1 \pmod P$. But this is impossible since

$$1 = \left(\frac{5}{P}\right) = \left(\frac{P}{5}\right) = \left(\frac{\pm 2}{5}\right) = -1.$$

If $n \equiv 2 \pmod 4$, where $n \geq 6$, then $n = 2(4q) \pm 2$. This shows that $5x^2 - 1 = V_n \equiv V_2 \pmod{V_2}$ by (2.4), implying that $5x^2 \equiv 1 \pmod{P^2 - 2}$. But this is impossible since

$$1 = \left(\frac{5}{P^2-2}\right) = \left(\frac{P^2-2}{5}\right) = \left(\frac{-3}{5}\right) = -1.$$

And so $n = 2$. Substituting this value of n into $V_n = 5x^2 - 1$ gives $P^2 - 2 = 5x^2 - 1$, implying that $P^2 \equiv 1 \pmod 5$, which is impossible since $P^2 \equiv -1 \pmod 5$. Now if $n \equiv 0 \pmod 4$, then $n = 4u$ for some u. By (2.7), we have $5x^2 - 1 = V_n = V_{4u} = V_{2u}^2 - 2$, which implies that $V_{2u}^2 - 1 = 5x^2$. That is,

$(V_{2u}-1)(V_{2u}+1)=5x^2$. Clearly, $d=(V_{2u}-1,V_{2u}+1)=1$ or 2. If $d=1$, then either

$$V_{2u}-1=a^2,\ V_{2u}+1=5b^2 \tag{3.2}$$

or

$$V_{2u}-1=5a^2,\ V_{2u}+1=b^2 \tag{3.3}$$

for some integers a and b. It can be easily seen that (3.2) and (3.3) are impossible. If $d=2$, then either

$$V_{2u}-1=2a^2,\ V_{2u}+1=10b^2 \tag{3.4}$$

or

$$V_{2u}-1=10a^2,\ V_{2u}+1=2b^2 \tag{3.5}$$

for some integers a and b. Obviously, by (2.7), we get (3.5) is impossible since $V_u^2\equiv 3 \pmod 5$ in this case. Suppose (3.4) is satisfied. Then by (2.7), we have

$$V_u^2-3=2a^2. \tag{3.6}$$

If $3\nmid a$, then $a^2\equiv 1 \pmod 3$ and therefore we get $V_u^2\equiv 2 \pmod 3$, which is impossible. Hence, we have $3\mid a$. This implies that $3\mid V_u$. For the case when $3\mid a$ and $3\mid V_u$, we easily see from (3.6) that $9\mid 3$, a contradiction. This completes the proof. □

Corollary 5. *The equation* $(5x^2-1)^2-(P^2-4)y^2=4$ *has integer solutions only when* $P=5x^2-1$ *with* x *even or* $P=L_{3z}/2$ *with* z *even.*

Theorem 9. *If* $V_n=7kx^2$ *with* $k\mid P$ *and* $k>1$, *then* $n=1$.

Proof. Suppose that $V_n=7kx^2$ for some integer x. Then by Lemma 1, we see that n is odd. And since $7\mid V_n$ and n is odd, it follows from (2.9) that $7\mid P$. Then by (2.14), we have $V_n\equiv V_1,V_3,V_5\equiv P,6P \pmod 8$. This implies that $7kx^2\equiv P,6P \pmod 8$, i.e., $kx^2\equiv 2P,7P \pmod 8$. Since $k\mid P$, we may write $P=kM$ with odd M. Thus, we get $kMx^2\equiv 2PM,7PM \pmod 8$. And so $x^2\equiv 2M,7M \pmod 8$. This shows that $M\equiv 7 \pmod 8$ since M is odd. Now suppose $n>1$. Then $n=4q\pm 1=2\cdot 2^r a\pm 1$, $2\nmid a$ and $r\geq 1$. Substituting this value of n into $V_n=7kx^2$ and using (2.4) give

$$7kx^2=V_n\equiv -V_1\equiv -P \pmod{V_{2^r}}.$$

This shows that

$$7x^2\equiv -M \pmod{V_{2^r}}$$

since $(k,V_{2^r})=1$. Thus, we have

$$1=\left(\frac{7}{V_{2^r}}\right)\left(\frac{-1}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right).$$

If $r\geq 2$, then by (2.16), it is seen that $\left(\frac{-1}{V_{2^r}}\right)=-1$. On the other hand, $\left(\frac{7}{V_{2^r}}\right)=-1$ by (2.19). Besides, using $M|P$ and identity (2.11), we get $\left(\frac{M}{V_{2^r}}\right)=$

$(-1)\left(\frac{2}{M}\right) = -1$. This means that $1 = -1$, a contradiction. Thus, $r = 1$. But in this case, since $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (2.16), $\left(\frac{7}{V_{2^r}}\right) = 1$ by (2.18), and $\left(\frac{M}{V_{2^r}}\right) = 1$ by (2.10), we again get a contradiction. So, we conclude that $n = 1$. □

Corollary 6. *The equation $49P^2x^4 - (P^2-4)y^2 = 4$ has no integer solutions.*

Theorem 10. *Let $k \mid P$ with $k > 1$. Then the equation $V_n = 7kx^2 + 1$ has no integer solutions.*

Proof. Suppose that $V_n = 7kx^2 + 1$ for some integer x. Then by Lemma 1, we have n is even and therefore $n = 2m$ for some $m > 0$. By (2.7), we get $V_n = V_{2m} = V_m^2 - 2 = 7kx^2 + 1$, implying that $V_m^2 \equiv 3 \pmod 7$, a contradiction. □

Corollary 7. *The equation $(7Px^2+1)^2 - (P^2-4)y^2 = 4$ has no integer solutions.*

Theorem 11. *Let $k \mid P$ with $k > 1$. Then the equation $V_n = 7kx^2 - 1$ has no integer solutions.*

Proof. Suppose that $V_n = 7kx^2 - 1$ for some integer x. Then by Lemma 1, it is seen that n is even. Dividing the proof into four cases, we have,

Case 1 : Assume that $7 \mid P$. Then by Lemma 1, it follows that $V_n \equiv \pm 2 \pmod 7$, which contradicts the fact that $V_n \equiv 6 \pmod 7$.

Case 2 : Assume that $P^2 \equiv 2 \pmod 7$. Then it is easily seen from (2.13) that $7 \mid U_4$. Since n is even, $n = 8q + r$ with $r \in \{0, 2, 4, 6\}$, so by (2.2), $V_n \equiv V_0, V_2, V_4, V_6 \pmod{U_4}$, implying that $V_n \equiv 0, 2 \pmod 7$, which is impossible.

Case 3 : Assume that $P^2 \equiv 4 \pmod 7$. Then $P \equiv 2, 5 \pmod 7$. On the other hand, it can be easily seen that $V_n \equiv 2, 5 \pmod 7$ in this case. But this contradicts the fact that $V_n \equiv 6 \pmod 7$.

Case 4 : Assume that $P^2 \equiv 1 \pmod 7$. Since $n = 2m$, we get $V_n = V_{2m} = V_m^2 - 2 = 7kx^2 - 1$, i.e., $V_m^2 = 7kx^2 + 1$. If m is odd, then $P \mid V_m$ by Lemma 1 and therefore we obtain $k \mid 1$, a contradiction. Thus, m is even. Let $m = 2u$ for some $u > 0$. Then $n = 4u$ and therefore by (2.2), we have

$$7kx^2 - 1 = V_{4u} \equiv V_0 \equiv 2 \pmod{U_2},$$

which implies that

$$7kx^2 \equiv 3 \pmod P.$$

Since $k \mid P$, it follows that $k \mid 3$ and this means that $k = 3$. Hence, we find that $3 \mid P$. In this case, we have $n = 4u = 2 \cdot 2^r a$, $2 \nmid a$ and $r \geq 1$. And thus,

$$21x^2 - 1 = V_n \equiv -V_0 \equiv -2 \pmod{V_{2^r}}.$$

This shows that

$$\left(\frac{3}{V_{2^r}}\right)\left(\frac{7}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right).$$

If $r \geq 2$, then $\left(\frac{3}{V_{2^r}}\right) = 1$ by (2.23) and $\left(\frac{7}{V_{2^r}}\right) = 1$ by (2.20). But since $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (2.16), we get a contradiction. So, $r = 1$. This means that

$n = 4u$ with u odd. By (2.7), we have $21x^2 - 1 = V_n = V_{4u} = V_{2u}^2 - 2$, which implies that $V_{2u}^2 - 1 = 21x^2$. That is, $(V_{2u} - 1)(V_{2u} + 1) = 21x^2$. Clearly, $d = (V_{2u} - 1, V_{2u} + 1) = 1$ or 2. If $d = 1$, then either

$$V_{2u} - 1 = a^2,\ V_{2u} + 1 = 21b^2,$$
$$V_{2u} - 1 = 3a^2,\ V_{2u} + 1 = 7b^2,$$
$$V_{2u} - 1 = 7a^2,\ V_{2u} + 1 = 3b^2,$$

or

$$V_{2u} - 1 = 21a^2,\ V_{2u} + 1 = b^2.$$

A simple computation shows that all the above equalities are impossible. Now if $d = 2$, then

(3.7) $$V_{2u} - 1 = 2a^2,\ V_{2u} + 1 = 42b^2,$$

(3.8) $$V_{2u} - 1 = 6a^2,\ V_{2u} + 1 = 14b^2,$$

(3.9) $$V_{2u} - 1 = 14a^2,\ V_{2u} + 1 = 6b^2,$$

or

(3.10) $$V_{2u} - 1 = 42a^2,\ V_{2u} + 1 = 2b^2.$$

If we combine the two equations in (3.7), we get $a^2 \equiv 6 \pmod 7$, a contradiction. By using (2.7), we see that (3.9) and (3.10) are impossible since $V_u^2 \equiv 3 \pmod 7$ in both cases. Now assume that (3.8) is satisfied. Let $u > 1$, and so $2u = 2 \cdot 2^r a \pm 2$, $2 \nmid a$ and $r \geq 2$. Thus, we obtain by (2.4) that

$$14b^2 - 1 = V_{2u} = V_{2\cdot 2^r a \pm 2} \equiv -V_2 \equiv -(P^2 - 2) \pmod{V_{2^r}},$$

implying that

$$7b^2 \equiv -(P^2 - 3)/2 \pmod{V_{2^r}}.$$

But this is impossible since $\left(\frac{7}{V_{2^r}}\right) = 1$ by (2.20), $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (2.16), and $\left(\frac{(P^2-3)/2}{V_{2^r}}\right) = 1$ by (2.21). As a consequence, we get $u = 1$ and therefore $n = 4$. Substituting $n = 4$ into $V_n = 21x^2 - 1$ gives $V_4 = (P^2 - 2)^2 - 2 = 21x^2 - 1$, i.e., $(P^2 - 2)^2 - 21x^2 = 1$. Since all positive integer solutions of the equation $u^2 - 21v^2 = 1$ are given by $(u, v) = (V_s(110, -1)/2, 12U_s(110, -1))$ with $s \geq 1$, we get $P^2 - 2 = V_s(110, -1)/2$ for some $s \geq 0$. Thus, $V_s(110, -1) = 2P^2 - 4$. It can be shown that s is odd. Taking $s = 4q \pm 1$ and using this into $V_s(110, -1)/2$ give

$$2P^2 - 4 = V_s \equiv V_1 \pmod{V_1},$$

implying that

$$2P^2 \equiv 4 \pmod 5.$$

Hence, we readily obtain that $P^2 \equiv 2 \pmod 5$, which is impossible. This completes the proof. □

Corollary 8. *The equation* $(7Px^2 - 1)^2 - (P^2 - 4)y^2 = 4$ *has no integer solutions.*

Theorem 12. *If $V_n = 7x^2 + 1$, then $n = 1$ and $V_1 = 7x^2 + 1$ where x is even.*

Proof. Suppose that $V_n = 7x^2 + 1$ for some integer x. If n is even, then $n = 2m$ and therefore $V_n = V_{2m} = V_m^2 - 2$ by (2.7). This implies that $V_m^2 \equiv 3 \pmod 7$, which is impossible. Thus, n is odd. Dividing the remainder of the proof into four cases, we have

Case 1 : Assume that $7 \mid P$. Since n is odd, it follows from Lemma 1 that $7 \mid V_n$, implying that $7 \mid 1$, a contradiction.

Case 2 : Assume that $P^2 \equiv 1 \pmod 7$. If $n > 1$, then $n = 4q \pm 1 = 2 \cdot 2^r a \pm 1$, $2 \nmid a$ and $r \geq 1$. Thus,

$$7x^2 = V_n - 1 \equiv -V_1 - 1 \equiv -(P+1) \pmod{V_{2^r}}$$

by (2.4). By using (2.20), (2.16), and (2.24), it is seen that

$$1 = \left(\frac{7}{V_{2^r}}\right)\left(\frac{-1}{V_{2^r}}\right)\left(\frac{P+1}{V_{2^r}}\right) = -1,$$

a contradiction. So, $n = 1$. This implies that $P = 7x^2 + 1$ with x even.

Case 3 : Assume that $P^2 \equiv 2 \pmod 7$. Hence, $P \equiv 3, 4 \pmod 7$. Moreover, it can be easily seen that $7 \mid V_2$. Since $n = 4q \pm 1$, it follows from (2.4) that $V_n = V_{4q\pm1} \equiv \pm V_1 \pmod{V_2}$, implying that $V_n \equiv 3, 4 \pmod 7$, which is impossible.

Case 4 : Assume that $P^2 \equiv 4 \pmod 7$. Hence, $P \equiv 2, 5 \pmod 7$. In this case, it can be easily shown by induction that

$$V_n \equiv \begin{cases} 2 \pmod 7 & \text{if } n \text{ is even,} \\ P \pmod 7 & \text{if } n \text{ is odd.} \end{cases}$$

This implies that $V_n \equiv 2, 5 \pmod 7$, which is impossible. This completes the proof. □

Corollary 9. *The equation $(7x^2 + 1)^2 - (P^2 - 4)y^2 = 4$ has integer solutions only when $P = 7x^2 + 1$ with even x.*

Theorem 13. *If $V_n = 7x^2 - 1$, then $n = 1$ and $V_1 = 7x^2 - 1$ with x is even, or $n = 2$ and $P = V_k(16, -1)/2$ and $x = 3U_k(16, -1)$ where k is even.*

Proof. Suppose that $V_n = 7x^2 - 1$ for some $x > 0$.

Case 1 : Assume that $7 \mid P$. Then by Lemma 1, $V_n \equiv 0, \pm 2 \pmod 7$, and so $V_n \neq 7x^2 - 1$.

Case 2 : Assume that $P^2 \equiv 1 \pmod 7$. If n is odd, then we can write $n = 4q \pm 1 = 2 \cdot 2^r a \pm 1$, $2 \nmid a$ and $r \geq 1$. Hence, we get

$$7x^2 - 1 = V_n \equiv -V_1 \equiv -P \pmod{V_{2^r}},$$

implying that

$$7x^2 \equiv -(P-1) \pmod{V_{2^r}}.$$

By using (2.20), (2.16), and (2.24), we immediately have

$$1 = \left(\frac{7}{V_{2^r}}\right)\left(\frac{-1}{V_{2^r}}\right)\left(\frac{P-1}{V_{2^r}}\right) = -1,$$

which is impossible. If $n \equiv 0 \pmod 4$, then $n = 4u$ for some u. Hence, we get

$$7x^2 - 1 = V_n \equiv -V_0 \equiv -2 \pmod{V_{2^r}},$$

implying that

$$7x^2 \equiv -1 \pmod{V_{2^r}}.$$

But this is impossible since $\left(\frac{7}{V_{2^r}}\right) = 1$ by (2.20) and $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (2.16). If $n \equiv 2 \pmod 4$ with $n \geq 6$, then $n = 2(4q \pm 1) = 2 \cdot 2^r a \pm 2$, $2 \nmid a$ and $r \geq 2$. Hence, we have

$$7x^2 - 1 = V_n \equiv -V_2 \equiv -(P^2 - 2) \pmod{V_{2^r}},$$

implying that

$$7x^2 \equiv -(P^2 - 3) \pmod{V_{2^r}}.$$

By using (2.20), (2.16), and (2.21), we readily obtain

$$1 = \left(\frac{7}{V_{2^r}}\right)\left(\frac{-1}{V_{2^r}}\right)\left(\frac{P^2-3}{V_{2^r}}\right) = -1,$$

a contradiction. So $n = 2$, and $V_n = 7x^2 - 1$ gives $V_2 = P^2 - 2 = 7x^2 - 1$, i.e., $P^2 - 7x^2 = 1$. Since all positive integer solutions of the equation $u^2 - 7v^2 = 1$ are given by $(u, v) = (V_k(16, -1)/2, 3U_k(16, -1))$ with $k \geq 1$, it follows that $P = V_k(16, -1)/2$ for positive even k, since P is odd.

Case 3 : Assume that $P^2 \equiv 2 \pmod 7$. And so $7 \mid V_2$, and if we write $n = 4q + r$ with $r \in \{0, 1, 2, 3\}$, then

$$V_n = V_{2 \cdot 2q \pm r} \equiv \pm V_r \equiv \pm \{V_0, V_1, V_2, V_3\} \pmod{V_2},$$

i.e.,

$$V_n \equiv 0, 2, 3, 4, 5 \pmod 7,$$

which is impossible.

Case 4 : Assume that $P^2 \equiv 4 \pmod 7$. So, $P \equiv 2, 5 \pmod 7$. Using the fact that

$$V_n \equiv \begin{cases} 2 \pmod 7 \text{ if } n \text{ is even} \\ P \pmod 7 \text{ if } n \text{ is odd} \end{cases}$$

gives $V_n \equiv 2, 5 \pmod 7$, which is impossible. This completes the proof. □

Corollary 10. *The equation* $(7x^2 - 1)^2 - (P^2 - 4)y^2 = 4$ *has integer solutions only when* $P = 7x^2 - 1$ *with* x *even or* $P = V_k(16, -1)/2$ *with* k *even.*

Acknowledgement. The authors would like to thank the anonymous referee for careful reading and suggestions that improved the clarity of this manuscript.

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Olcay Karaatli
Sakarya University Faculty of Arts and Science
TR54187 Sakarya, Turkey
E-mail address: okaraatli@sakarya.edu.tr

Refik Keskin
Sakarya University
Faculty of Arts and Science
TR54187 Sakarya, Turkey
E-mail address: rkeskin@sakarya.edu.tr