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# SUPERCYCLICITY OF JOINT ISOMETRIES

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Abstract. Let $H$ be a separable complex Hilbert space. A commuting tuple $T = (T_1, \ldots, T_n)$ of bounded linear operators on $H$ is called a spherical isometry if $\sum_{i=1}^{n} T_i^* T_i = I$. The tuple $T$ is called a toral isometry if each $T_i$ is an isometry. In this paper, we show that for each $n \geq 1$ there is a supercyclic $n$-tuple of spherical isometries on $\mathbb{C}^n$ and there is no spherical or toral isometric tuple of operators on an infinite-dimensional Hilbert space.

## 1. Introduction

An $n$-tuple of operators is a finite sequence of length $n$ of commuting bounded linear operators $T_1, T_2, \ldots, T_n$ acting on a Hilbert space $H$. For an $n$-tuple $T = (T_1, T_2, \ldots, T_n)$, if there exists an element $x \in H$ such that $\operatorname{orb}(T, x) = \{Sx : S \in \mathcal{F}_T\}$ where $\mathcal{F}_T = \{T_1^{k_1} T_2^{k_2} \cdots T_n^{k_n} : k_i \geq 0,\ i = 1, 2, \ldots, n\}$, is dense in $H$ then $x$ is called a hypercyclic vector for $T$, and $T$ is said to be a hypercyclic $n$-tuple of operators. A vector $x \in H$ is called a supercyclic vector for $T$ if the set $\{\lambda Sx : S \in \mathcal{F}_T,\ \lambda \in \mathbb{C}\}$ is dense in $H$, and $T$ is said to be a supercyclic $n$-tuple of operators. These definitions generalize the notions of hypercyclicity and supercyclicity of a single operator to a tuple of operators. Hypercyclicity and supercyclicity of tuples of operators have been investigated in ([3], [4], [6], [7]). On the other hand, spherical isometries are a considerable part of tuples of operators. The authors in [5] proved that isometries on Hilbert spaces with dimension more than one are not supercyclic. Recently, this fact has been proved for $m$-isometric operators which are a generalization of isometric operators in some sense [2]. In this paper we see that spherical isometries are not supercyclic on infinite-dimensional Hilbert spaces. Let $A$ be a matrix we denote by $A^T$ and $\det A$ the transpose and the determinant of $A$ respectively.

In Section 2, we show that there is no supercyclic $n$-tuple of diagonalizable matrices on $\mathbb{C}^{n+1}$. The main result of this section is that there is a supercyclic

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$p$-tuple of spherical isometries on $\mathbb{C}^p$, $p \geq 1$. Also, it is proved that there is no spherical or toral isometric tuple of operators on an infinite-dimensional Hilbert space.

## 2. Supercyclicity of spherical isometries

In [4] it is shown that for each $n \geq 1$, there exists a hypercyclic $(n+1)$-tuple of diagonal matrices on $\mathbb{C}^n$, and there is no hypercyclic $n$-tuple of diagonalizable matrices on $\mathbb{C}^n$. We give the following result which is a generalization of [4, Theorem 3.6]. The technique employed in the proof is due to N. Feldman.

**Theorem 1.** *There is no supercyclic $n$-tuple of diagonalizable matrices on $\mathbb{C}^{n+1}$.*

*Proof.* The proof is on the same lines as for the case $n = 2$. We assume that there exists a supercyclic 2-tuple $(A, B)$ of diagonalizable matrices on $\mathbb{C}^3$. We can assume that the matrices $A$ and $B$ are diagonal thanks to simultaneously diagonalizability. Let

$$A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}, \ B = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}$$

and $\nu = \left[\begin{smallmatrix} \alpha \\ \beta \\ \gamma \end{smallmatrix}\right]$ be a supercyclic vector for the 2-tuple $(A, B)$. Therefore, if

$$E = \left\{ \begin{bmatrix} \lambda a_1^n b_1^k \alpha \\ \lambda a_2^n b_2^k \beta \\ \lambda a_3^n b_3^k \gamma \end{bmatrix} : \lambda \in \mathbb{C}, \ n, k \geq 0 \right\} = \{\lambda A^n B^k \nu : \lambda \in \mathbb{C}, \ n, k \geq 0\},$$

then $\overline{E} = \mathbb{C}^3$ which in turn implies that $\alpha$, $\beta$, $\gamma$, $a_i$ and $b_i$ for $i = 1, 2, 3$ are nonzero. Since the 2-tuple $(\frac{1}{a_1}A, \frac{1}{b_1}B)$ is also supercyclic, we may assume that $a_1 = b_1 = 1$. On the other hand, by applying the invertible matrix

$$\begin{bmatrix} \alpha^{-1} & 0 & 0 \\ 0 & \beta^{-1} & 0 \\ 0 & 0 & \gamma^{-1} \end{bmatrix}$$

to the set $E$ we conclude that $\overline{F} = \mathbb{C}^3$ where

$$F = \left\{ \begin{bmatrix} \lambda \\ \lambda a_2^n b_2^k \\ \lambda a_3^n b_3^k \end{bmatrix} : \lambda \in \mathbb{C}, \ n, k \geq 0 \right\}.$$

Now by applying the function $\log |z|$ to each coordinate of the set $F$ we have $\overline{G} = \mathbb{R}^3$ where

$$G = \left\{ \begin{bmatrix} \log |\lambda| \\ n \log |a_2| + k \log |b_2| + \log |\lambda| \\ n \log |a_3| + k \log |b_3| + \log |\lambda| \end{bmatrix} : \ n, k \geq 0, \ \lambda \in \mathbb{C} \right\}.$$

Let

$$S = \left\{ \begin{bmatrix} n \\ k \\ \log|\lambda| \end{bmatrix} : n, k \geq 0,\ \lambda \in \mathbb{C} \right\}$$

and

$$T = \begin{bmatrix} 0 & 0 & 1 \\ \log|a_2| & \log|b_2| & 1 \\ \log|a_3| & \log|b_3| & 1 \end{bmatrix}$$

then $T$ is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that $T(S)$ is dense in $\mathbb{R}^3$ thanks to $\overline{G} = \mathbb{R}^3$, hence $T$ is onto which implies that $T$ is invertible. Therefore, $S = T^{-1}(T(S))$ must be dense in $\mathbb{R}^3$ which is impossible. □

To prove the main result of this section, we need the following lemmas.

**Lemma 1.** *For $n > 1$ let $x_1, x_2, \dots, x_{n-1}$ and $y_1, y_2, \dots, y_{n-1}$ be complex numbers. If*

$$M_n = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ x_1 & y_1 & 1 & 1 & \cdots & 1 \\ x_2 & 1 & y_2 & 1 & \cdots & 1 \\ \vdots & & & & & \\ & & & & \ddots & \\ x_{n-1} & 1 & 1 & 1 & \cdots & y_{n-1} \end{bmatrix}_{n\times n}$$

*is invertible, then the solution of the equation $M_n Z = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$, called $Z_n = \begin{bmatrix} z_n^1 & z_n^2 & \cdots & z_n^n \end{bmatrix}^T$, satisfies the following recursive formula*

$$z_n^i = \begin{cases} \frac{(y_{n-1}-1)z_{n-1}^i}{(y_{n-1}-1)+(1-x_{n-1})z_{n-1}^1}, & 1 \leq i \leq n-1 \\ \frac{(1-x_{n-1})z_{n-1}^1}{(y_{n-1}-1)+(1-x_{n-1})z_{n-1}^1}, & i = n. \end{cases}$$

*Proof.* By Crammer's rule we have $z_n^i = \frac{\det M_n^i}{\det M_n}$, where $M_n^i$ is the matrix obtained from $M_n$ by replacing its $i^{th}$ column by $\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$. Expanding $\det M_n$ and $\det M_n^i$ along their $n^{th}$ rows, we can obtain the following recursive formulae:

$$\det M_n = (y_{n-1} - x_{n-1}z_{n-1}^1 - \sum_{k=2}^{n-1} z_{n-1}^k) \det M_{n-1},$$

$$\det M_n^i = \begin{cases} (y_{n-1} - 1) \det M_{n-1}^i, & 1 \leq i \leq n-1 \\ (1 - x_{n-1}z_{n-1}^1 - \sum_{k=2}^{n-1} z_{n-1}^k) \det M_{n-1}, & i = n. \end{cases}$$

Now an easy computation will give the conclusion; just note that $\sum_{i=1}^{n-1} z_{n-1}^i = 1$. □

**Lemma 2.** *Suppose that $p \geq 3$ is an integer, $t_1, \dots, t_{p-1}$ are real numbers and $f(p) = 2 - p + \sum_{i=1}^{p-1} t_i$. Then the matrix*

$$M_p = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ t_1 & b & 1 & \cdots & 1 \\ t_2 & 1 & b & \cdots & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ t_{p-1} & 1 & 1 & \cdots & b \end{bmatrix}_{p \times p}$$

*is invertible if and only if $b \notin \{1, f(p)\}$.*

*Proof.* Putting $m_p = \det M_p$ and writing the expansion of $m_p$ via the first column of $M_p$, we have $m_p = b_{p-1} - c_{p-1} \sum_{i=1}^{p-1} t_i$ where

$$b_{p-1} = \det \begin{bmatrix} b & 1 & 1 & \cdots & 1 \\ 1 & b & 1 & \cdots & 1 \\ 1 & 1 & b & \cdots & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & b \end{bmatrix}_{(p-1)\times(p-1)}$$

and

$$c_{p-1} = \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & b & 1 & \cdots & 1 \\ 1 & 1 & b & \cdots & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & b \end{bmatrix}_{(p-1)\times(p-1)} .$$

Multiplying the first row of the matrix in $c_{p-1}$ by $-1$ and adding it to all other rows, we get $c_{p-1} = (b-1)^{p-2}$. On the other hand, evaluating $b_{p-1}$ by using the first column of the relevant matrix, we find the recursive formula $b_{p-1} = b.b_{p-2} - (p-2)c_{p-2} = b.b_{p-2} - (p-2)(b-1)^{p-3}$. Now, an easy use of the mathematical induction implies that $b_{p-1} = (b-1)^{p-2}(b+p-2)$ and hence we have

$$m_p = (b-1)^{p-2}(b - f(p)).$$

Thus, the proof is completed. □

In the sequel, some comments are in order. Let $f : [0,1] \to [0,1]$ be defined by $f(x) = \text{frac}(10x)$ where $\text{frac}(x)$ denotes the fractional part of real number $x$. For each fixed $n \in \mathbb{N}$, let $F_{2n} : [0,1]^{2n} \longrightarrow [0,1]^{2n}$ be defined by $F_{2n}(x_1, x_2, \dots, x_{2n}) = (f(x_1), f(x_2), \dots, f(x_{2n}))$. By Proposition 3.1 of [4], there exists $(x_1, x_2, \dots, x_{2n}) \in [0,1]^{2n}$ which has a dense orbit under $F_{2n}$. Let

$$\triangle^n = \{\mathbf{x} \in [0,1]^{2n} : \overline{orb(F_{2n}, \mathbf{x})} = [0,1]^{2n}\}$$

and

$$\triangle^n_{\frac{1}{2}} = \left\{ \begin{bmatrix} \exp(\alpha q_1 x_1 + 2\pi i x_2) \\ \exp(\alpha q_2 x_3 + 2\pi i x_4) \\ \vdots \\ \exp(\alpha q_n x_{2n-1} + 2\pi i x_{2n}) \end{bmatrix} : (x_1, x_2, \dots, x_{2n}) \in \triangle^n,\ q \in \mathbb{N}, \alpha = \ln 2 \right\}.$$

By Corollary 3.5 of [4], for every $(a_1, a_2, \dots, a_n) \in \triangle^n_{\frac{1}{2}}$, $|a_i| > 1$ $(i = 1, 2, \dots, n)$. Moreover, if $A_n$ and $B_{nk}$ $(1 \le k \le n)$ are diagonal matrices with, respectively, the diagonals $(a_1, a_2, \dots, a_n)$ and $(b_{1k}, b_{2k}, \dots, b_{nk})$ where $b_{kk} = \frac{1}{2}$ and $b_{ik} = 1$ $(i \neq k)$, then the $(n+1)$-tuple $(A_n, B_{n1}, \dots, B_{nn})$ is hypercyclic on $\mathbb{C}^n$ with the hypercyclic vector $\nu = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$. Hence the set

$$E = \left\{ \begin{bmatrix} 2^{-k_1} {a_1}^m \\ 2^{-k_2} {a_2}^m \\ \vdots \\ 2^{-k_n} {a_n}^m \end{bmatrix} : m \ge 0,\ k_i \ge 0,\ i = 1, 2, \dots, n \right\}$$

is dense in $\mathbb{C}^n$.

**Theorem 2.** *There is a supercyclic spherical isometric $p$-tuple on $\mathbb{C}^p$, $p \ge 1$.*

*Proof.* The case $p = 1$ is obvious. Now, let $p = n + 1 \ge 2$ and consider the $p$-tuple $(A'_n, B'_{n1}, \dots, B'_{nn})$ of diagonal matrices on $\mathbb{C}^p$, where $A'_n$ and $B'_{nk}$ $(1 \le k \le n)$ have, respectively, the diagonals $(1, a_1, a_2, \dots, a_n)$ and $(1, b_{1k}, \dots, b_{nk})$. Also, let $\nu' = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$. Then the density of $E$ in $\mathbb{C}^n$ implies that

$$\{\lambda (A'_n)^{k_1} (B'_{n1})^{k_2} \cdots (B'_{nn})^{k_p} \nu' : \lambda \in \mathbb{C},\ k_i \ge 0,\ i = 1, 2, \dots, n+1\}$$

is dense in $G = \mathbb{C}^p - \{(0, \lambda_1, \dots, \lambda_n) : \lambda_i \in \mathbb{C},\ i = 1, 2, \dots, n\}$. But $G$ is also dense in $\mathbb{C}^p$, so we conclude that the vector $\nu'$ is supercyclic for the $p$-tuple $(A'_n, B'_{n1}, \dots, B'_{nn})$. We claim that there are scalars $\alpha_1, \alpha_2, \dots, \alpha_p$ such that the $p$-tuple $(\alpha_1 A'_n, \alpha_2 B'_{n1}, \dots, \alpha_p B'_{nn})$ is a supercyclic spherical isometry on $\mathbb{C}^p$. For $p = 2$ if $|a_1| > 1$ and we put

$$\alpha_1 = \sqrt{\frac{3}{4|a_1|^2 - 1}},\ \alpha_2 = \sqrt{\frac{4(|a_1|^2 - 1)}{4|a_1|^2 - 1}},\ A'_1 = \begin{bmatrix} 1 & 0 \\ 0 & a_1 \end{bmatrix},\ B'_{11} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix},$$

then $(\alpha_1 A'_1, \alpha_2 B'_{11})$ is a spherical isometric 2-tuple on $\mathbb{C}^2$. Let $p = n + 1 \ge 3$. The $p$-tuple $(\alpha_1 A'_n, \alpha_2 B'_{n1}, \dots, \alpha_p B'_{nn})$ is spherical isometry if and only if there is a solution $Z_p = \begin{bmatrix} z_p^1 & \cdots & z_p^p \end{bmatrix}^T$ for the equation $M_p Z = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$, where

$$M_p = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ |a_1|^2 & \frac{1}{4} & 1 & \cdots & 1 \\ |a_2|^2 & 1 & \frac{1}{4} & \cdots & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ |a_{p-1}|^2 & 1 & 1 & \cdots & \frac{1}{4} \end{bmatrix}$$

and $z_p^i > 0$ for $i = 1, 2, \ldots, p$. Note that by Lemma 2, $M_p$ is invertible for all $p \geq 3$. Suppose that $(a_1, a_2, \ldots, a_p) \in \triangle_{\frac{1}{2}}^p$ is the point corresponding to $(x_1, x_2, \ldots, x_{2p}) \in \triangle^p$. Since $(x_1, x_2, \ldots, x_{2p})$ has a dense orbit under $F_{2p}$, it is easy to see that $(x_1, x_2, \ldots, x_{2p-2})$ has a dense orbit under $F_{2p-2}$ and clearly $(a_1, a_2, \ldots, a_{p-1}) \in \triangle_{\frac{1}{2}}^{p-1}$ is the point corresponding to $(x_1, x_2, \ldots, x_{2p-2}) \in \triangle^{p-1}$. Therefore, by Lemma 1, the existence of $z_p^i > 0$, $i = 1, 2, \ldots, p$, such that $M_p Z_p = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$, depends on the existence of $z_{p-1}^i > 0$, $i = 1, 2, \ldots, p-1$, such that $M_{p-1} Z_{p-1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$ and obviously, this backward process can be continued to conclude finally that the existence of $z_p^i > 0$, for $i = 1, 2, \ldots, p$ depends on the existence of $z_2^i > 0$, $i = 1, 2$, such that $M_2 Z_2 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, which is proved. □

We finish this note by giving the following result which is a consequence of Example 2.11 and Corollary 4.2 of [4].

**Proposition 1.** *There is no supercyclic spherical or toral isometry on an infinite-dimensional Hilbert space.*

*Proof.* We prove the assertion for spherical isometries. The same argument gives the proof for toral isometries. Suppose that $H$ is an infinite-dimensional Hilbert space and $(A_1, A_2, \ldots, A_n)$ is a supercyclic spherical isometry on $H$. By a result of [1], the tuple $(A_1, A_2, \ldots, A_n)$ is subnormal; that is, there is a Hilbert space $K$ containing $H$ and a commuting tuple $(S_1, S_2, \ldots, S_n)$ of normal operators on $K$ such that $A_i = S_i \mid_H$ for $i = 1, 2, \ldots, n$. Let $\theta$ be an irrational multiple of $\pi$ and $a$, $b$ be two relatively prime integers greater than one. Moreover, suppose that $I$ is the identity operator on $H$. By Corollary 4.2 of [4] the set $\{\frac{a^{k_1} e^{ik_2\theta}}{b^{k_3}} : k_1, k_2, k_3 \geq 0\}$ is dense in $\mathbb{C}$; thus, the supercyclicity of $(A_1, A_2, \ldots, A_n)$ implies the hypercyclicity of $(n+3)$-tuple $(aI, \frac{1}{b}I, e^{i\theta}I, A_1, A_2, \ldots, A_n)$, but this is a subnormal tuple and, by Corollary 3.9 of [4], can not be hypercyclic. □

*Remark* 1. In fact the proof of Proposition 1 shows that on an infinite-dimensional Hilbert space. There is no supercyclic n-tuple of subnormal operators with commuting normal extensions.

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