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MULTIPLE SOLUTIONS TO DISCRETE BOUNDARY VALUE PROBLEMS FOR THE p-LAPLACIAN WITH POTENTIAL TERMS ON FINITE GRAPHS

Soon-Yeong Chung and Jea-Hyun Park

Abstract. In this paper, we prove the existence of at least three nontrivial solutions to nonlinear discrete boundary value problems

$$\begin{cases} -\Delta_{p,\omega}u(x) + V(x)|u(x)|^{q-2}u(x) = f(x,u(x)), & x \in S, \\ u(x) = 0, & x \in \partial S, \end{cases}$$

involving the discrete p-Laplacian on simple, finite and connected graphs $\overline{S}(S \cup \partial S, E)$ with weight ω, where $1 < q < p < \infty$. The approach is based on a suitable combine of variational and truncations methods.

1. Introduction

Recently, nonlinear difference equations have been studied in various fields, such as computer science, biological neural networks, dynamical systems, image processing and many others. In these fields, many authors have widely developed various methods and techniques to show the existence of multiple solutions to discrete boundary value problem

$$(1) \qquad \begin{cases} -\Delta(\phi_p(\Delta u(k-1))) = a(k)f(k,u(k)), & k \in [1,T], \\ u(0) = u(T+1) = 0, \end{cases}$$

where $p > 1$, T is a fixed positive integer, $[1,T]$ is the discrete interval (namely, $\{1,\ldots,T\}$), a is a positive function on $[1,T]$, $\Delta u(k) := u(k+1) - u(k)$ is the forward difference operator, $\phi_p(t) := |t|^{p-2}t$ and $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

To guarantee the existence of solutions to (1), researchers have proposed various asymptotic behaviors of f at zero or infinity. In [2], Agarwal, Perera and O'Regan proposed asymptotic behaviors of $f : (0,\infty) \to \mathbb{R}$ at zero and infinity. They proved the existence of multiples solutions to (1) with the asymptotic

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behaviors of f. In [10], He proved the existence of multiple solutions to (1) with the mixed boundary data with some asymptotic behaviors of $f : \mathbb{R}^+ \to \mathbb{R}^+$ at zero and infinity where $\mathbb{R}^+$ denotes the non-negative reals. In [5], Candito and Giovannelli worked with $\int_0^t f(\cdot, s)ds$ instead of f. For other results, see [1, 3, 4, 7, 11] and the references given therein.

In this paper, we consider a *graph* $\overline{S} = \overline{S}(S \cup \partial S, E)$ with weight ω and the *discrete p-Laplacian* $\Delta_{p,\omega}$ as generalizations the discrete interval $[1, T]$ and the operator $\Delta(\phi_p(\Delta(\cdot)))$, respectively. The definitions are in Section 2.

The reason why we consider the discrete p-Laplacian on weighted graphs is that it is one of the best methods to represent linear or nonlinear phenomena with network structures (for examples, see [6, 8, 9, 17, 18, 19]).

The main purpose of this paper is to show the existence of multiple solutions to the discrete boundary value problem on a weighted graph

$$\begin{cases} -\Delta_{p,\omega} u(x) + V(x)|u(x)|^{q-2}u(x) = f(x, u(x)), & x \in S, \\ u(x) = 0, & x \in \partial S, \end{cases} \tag{2}$$

where $1 < q < p < \infty$, $\overline{S} = (S \cup \partial S, E)$ is a simple, connected and finite graph with weight ω, V is a function on S with $V(x) > 0$ for all $x \in S$, and $f \in C(S \times \mathbb{R}, \mathbb{R})$, which means f is a function defined on $S \times \mathbb{R}$ satisfying for each $x \in S$, $f(x, t)$ is continuous with respect to t.

To show that there exist at least three non-trivial solutions to (2), we consider asymptotic behaviors of f as follows:

(i) a function f satisfies that for some $\alpha_0 \in \mathbb{R}$,

$$\lambda_{1,0} < \liminf_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} \le \limsup_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} < \alpha_0, \quad x \in S,$$

and

$$\limsup_{t\to -\infty} \frac{f(x,t)}{|t|^{p-2}t} < \lambda_{1,0} < \liminf_{t\to \infty} \frac{f(x,t)}{|t|^{p-2}t}, \quad x \in S.$$

In addition, we assume that

(ii) a function $V : S \to (0, \infty)$ satisfies $\sum_{x\in S} V(x) < \frac{q}{p}(\mu_0 - \lambda_{1,0})$,

where $\mu_0 := \min_{x\in S} \left[\liminf_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t}\right]$ and $\lambda_{1,0}$ is the smallest eigenvalue of the discrete p-Laplacian which is defined in Section 2. These conditions guarantee that (2) has at least three non-trivial solutions.

More precisely, we consider two functionals E^+ and E^- (for definitions, see Section 2). We show that these functionals E^+ and E^- have at least one and two non-trivial critical points, respectively. We finally show that these critical points are solutions to (2) by observing signs of the critical points.

We note that the case of $p = q$ had dealt with in [13]. In the paper, the existence of at least two solutions to (2) has been proved under some asymptotic behaviors of f at zero and at infinity.

2. Preliminaries

In this section, we start with the graph theoretic notions frequently used throughout this paper.

By a graph $\overline{S} = \overline{S}(S \cup \partial S, E)$, we mean a two finite and disjoint vertex set S and ∂S called *interior* and *boundary*, respectively, with a set E of unordered pairs of distinct elements of $S \cup \partial S$ whose elements are called *edges*. As conventionally used, we denote by $x \in \overline{S}$ the facts that x is a vertex in $S \cup \partial S$.

A graph $\overline{S}$ is said to be *simple* if it has neither multiple edges nor loops, and $\overline{S}$ is said to be connected if for every pair of vertices x and y, there exists a sequence (termed a *path*) of vertices $x = x_0, x_1, \ldots, x_{n-1}, x_n = y$ such that x_{j-1} and x_j are connected by an edge (termed *adjacent*) for $j = 1, \ldots, n$.

A *weight* on a graph $\overline{S}$ is a function $\omega : \overline{S} \times \overline{S} \to [0, \infty)$ satisfying

(i) $\omega(x, y) = \omega(y, x) > 0$ if $\{x, y\} \in E$,

(ii) $\omega(x, y) = 0$ if and only if $\{x, y\} \notin E$.

In this paper, we only consider a simple and connected graph $\overline{S}$ with weight ω. We note that since a graph $\overline{S}$ is simple, it is trivial that $\omega(x, x) = 0$ for all $x \in \overline{S}$.

Throughout this paper, a function on a graph is understood as a function defined on the set of vertices of the graph. For a nonempty subset T of vertices in $\overline{S}$, the integration of a function $u : T \to \mathbb{R}$ is defined by

$$\int_T u := \sum_{x \in T} u(x).$$

For $p > 1$, the *p-directional derivative* of a function $u : \overline{S} \to \mathbb{R}$ in the direction y is defined by

$$D_{p,\omega,y} u(x) := |u(y) - u(x)|^{p-2}(u(y) - u(x))\sqrt{\omega(x, y)}$$

for $x \in \overline{S}$. The *p-gradient* $\nabla_{p,\omega}$ of a function $u : \overline{S} \to \mathbb{R}$ is defined to be

$$\nabla_{p,\omega} u(x) := (D_{p,\omega,y} u(x))_{y \in \overline{S}}$$

for $x \in \overline{S}$. In the case of $p = 2$, we write simply ∇_ω instead of $\nabla_{2,\omega}$.

The *discrete p-Laplacian* $\Delta_{p,\omega}$ of a function $u : \overline{S} \to \mathbb{R}$ is defined by

$$\Delta_{p,\omega} u(x) := \sum_{y \in \overline{S}} |u(y) - u(x)|^{p-2}(u(y) - u(x))\omega(x, y), \ x \in S.$$

We note that for any pair of functions $u : \overline{S} \to \mathbb{R}$ and $v : \overline{S} \to \mathbb{R}$, we have

$$2 \int_{\overline{S}} v(-\Delta_{p,\omega} u) = \int_{\overline{S}} \nabla_\omega v \cdot \nabla_{p,\omega} u, \tag{3}$$

where $\mathbb{A} \cdot \mathbb{B} := \sum_{i=1}^{n} a_i b_i$ for $\mathbb{A} = (a_1, \ldots, a_n)$ and $\mathbb{B} = (b_1, \ldots, b_n)$. Even though the formula (3) is proved in [12], we here give the sketch of proof. It follows from the definition of the discrete p-Laplacian that

$$\begin{aligned} 2\int_{\overline{S}} v(-\Delta_{p,\omega}u) &= -2\sum_{x,y\in\overline{S}} |u(y)-u(x)|^{p-2}(u(y)-u(x))v(x)\omega(x,y) \\ &= -\sum_{x,y\in\overline{S}} |u(y)-u(x)|^{p-2}(u(y)-u(x))v(x)\omega(x,y) \\ &\quad -\sum_{x,y\in\overline{S}} |u(x)-u(y)|^{p-2}(u(x)-u(y))v(y)\omega(y,x). \end{aligned}$$

Since $\omega(x,y)=\omega(y,x)$ for all $x,y\in\overline{S}$, we have that

$$\begin{aligned} &-\sum_{x,y\in\overline{S}} |u(y)-u(x)|^{p-2}(u(y)-u(x))v(x)\omega(x,y) \\ &-\sum_{x,y\in\overline{S}} |u(x)-u(y)|^{p-2}(u(x)-u(y))v(y)\omega(y,x) \\ =&\sum_{x,y\in\overline{S}} |u(y)-u(x)|^{p-2}(u(y)-u(x))(v(y)-v(x))\omega(x,y). \end{aligned}$$

Therefore, by the definition of p-gradient, we have the formula (3).

In this paper, we define a set $\mathcal{A}_0$ as follows:

$$\mathcal{A}_0 := \{u:\overline{S}\to\mathbb{R} \mid u(z)=0, z\in\partial S\},$$

and the norm $\|\cdot\|_p$ on $\mathcal{A}_0$ is defined by

$$\|u\|_p := \left(\sum_{x\in\overline{S}} |u(x)|^p\right)^{\frac{1}{p}}$$

for $u\in\mathcal{A}_0$.

For $V:S\to\mathbb{R}$, the operator $-\Delta_{p,\omega}+V$ has the smallest eigenvalue $\lambda_{1,V}$ which is variationally defined as

$$\lambda_{1,V} := \inf_{\substack{\phi\not\equiv 0\\ \phi\in\mathcal{A}_0}} \frac{\frac{1}{2}\int_{\overline{S}}\nabla_\omega\phi\cdot\nabla_{p,\omega}\phi+\int_S V|\phi|^p}{\int_S|\phi|^p}.$$

Then there exists $\phi\in\mathcal{A}_0$ such that

$$\lambda_{1,V} = \frac{\frac{1}{2}\int_{\overline{S}}\nabla_\omega\phi\cdot\nabla_{p,\omega}\phi+\int_S V|\phi|^p}{\int_S|\phi|^p},$$

and the function ϕ is called an eigenfunction corresponding to $\lambda_{1,V}$.

We note that the multiplicity of $\lambda_{1,V}$ is one and there exists an eigenfunction ϕ corresponding to $\lambda_{1,V}$ such that $\|\phi\|_p=1$ and $\phi(x)>0$ for all $x\in S$. In particular, if $V\equiv 0$, then the smallest eigenvalue is positive (for more detail, see [14]). As conventionally used, $\lambda_{1,0}$ denotes an eigenvalue in the case of $V\equiv 0$. Also, by ϕ_1, we denote an eigenfunction corresponding to $\lambda_{1,0}$ satisfying $\phi_1(x)>0$ for all x in S and $\int_{\overline{S}}|\phi_1|^p=1$.

For a function $f \in C(S \times \mathbb{R}, \mathbb{R})$, let consider two functionals defined by

$$E^+[u] := \frac{1}{2p}\int_{\overline{S}} \nabla_{p,\omega} u \cdot \nabla_\omega u + \frac{1}{q}\int_S V|u^+(x)|^q - \int_S F(x, u^+(x)), \quad u \in \mathcal{A}_0$$

and

$$E^-[u] := \frac{1}{2p}\int_{\overline{S}} \nabla_{p,\omega} u \cdot \nabla_\omega u + \frac{1}{q}\int_S V|u^-(x)|^q - \int_S F(x, u^-(x)), \quad u \in \mathcal{A}_0,$$

where $u^+(x) := \max\{u(x), 0\}$, $u^-(x) := \min\{u(x), 0\}$, and $F(x, u(x)) := \int_0^{u(x)} f(x,s)ds$. Then if the function f satisfies

$$f(x, 0) = 0 \text{ for all } x \in S,$$

then the functionals E^+ and E^- are differentiable. Moreover, the partial derivatives with respect to $u(x)$ are given by

$$\frac{d}{du(x)}E^+[u] = -\Delta_{p,\omega}u(x) + V(x)|u^+(x)|^{q-2}u^+(x) - f(x, u^+(x)),$$

and

$$\frac{d}{du(x)}E^-[u] = -\Delta_{p,\omega}u(x) + V(x)|u^-(x)|^{q-2}u^-(x) - f(x, u^-(x)).$$

As conventionally used, we denote the partial derivatives with respect to $u(x)$ by $E^{+\prime}[u](x)$ and $E^{-\prime}[u](x)$.

We note that it is clear that if a function $u \in \mathcal{A}_0$ is either a critical point of E^+ and $u(x) \geq 0$ for all $x \in S$ or a critical point of E^- and $u(x) \leq 0$ for all $x \in S$, then the function u is a solution to (2).

3. Multiple solutions

In this section, we show the existence of non-trivial critical points using the mountain pass theorem. So we first discuss conditions for f to guarantee that E^+ and E^- satisfy the *Palais-Smale condition* (simply, (PS) condition):

(PS) Suppose that Ω is a real Banach space. A functional $E \in \mathcal{C}^1(\Omega; \mathbb{R})$ satisfies the Palais-Smale condition if for any sequence $(u_n) \subset \Omega$ satisfying

(a) $E[u_n]$ is bounded and

(b) $E'[u_n] \to 0$ as $n \to \infty$,

the sequence (u_n) has a convergent subsequence. A sequence satisfying (a) and (b) is called a (PS) sequence for E.

Theorem 3.1. *If a function $f \in C(S \times \mathbb{R}, \mathbb{R})$ satisfies that $f(x, 0) = 0$ for all x in S and*

$$\lambda_{1,0} < \liminf_{t\to\infty} \frac{f(x,t)}{|t|^{p-2}t}, \quad x \in S,$$

then the functional E^+ satisfies the (PS) *condition.*

Proof. We assume that $\{u_n\}$ is the (PS) sequence. Since $E^{+\prime}[u_n](x) \to 0$ as $n \to \infty$ for all $x \in S$, there exists $\epsilon_n \in \mathcal{A}_0$, $n = 1, \ldots$ such that

$$-\Delta_{p,\omega} u_n(x) = -V(x)|u_n^+(x)|^{q-2}u_n^+(x) + f(x, u_n^+(x)) + \epsilon_n(x) \tag{4}$$

for all $x \in S$. Thus $\epsilon_n(x) \to 0$ for all x. Taking as a non-trivial test function u_n^- in (4), since $\int_{\overline{S}} \nabla_{p,\omega} u_n \cdot \nabla_\omega u_n^- \geq \int_{\overline{S}} \nabla_{p,\omega} u_n^- \cdot \nabla_\omega u_n^-$, we have

$$\int_S \epsilon_n u_n^- = \frac{1}{2}\int_{\overline{S}} \nabla_{p,\omega} u_n \cdot \nabla_\omega u_n^- \geq \frac{1}{2}\int_{\overline{S}} \nabla_{p,\omega} u_n^- \cdot \nabla_\omega u_n^-.$$

It follows from the definition of $\lambda_{1,0}$ that

$$\lambda_{1,0} \leq \frac{\frac{1}{2}\int_{\overline{S}} \nabla_\omega \phi \cdot \nabla_{p,\omega}\phi}{\int_S |\phi|^p}$$

for all $\phi \in \mathcal{A}_0$ with $\phi \not\equiv 0$. Hence it implies that

$$\int_S \epsilon_n u_n^- \geq \frac{1}{2}\int_{\overline{S}} \nabla_{p,\omega} u_n^- \cdot \nabla_\omega u_n^- \geq \lambda_{1,0}\|u_n^-\|_p^p \tag{5}$$

for all $n \in \mathbb{N}$. If there exists a subsequence $\{u_{n_j}\}$ such that $\|u_{n_j}^-\|_p \to \infty$, then

$$\int_S \epsilon_{n_j} u_{n_j}^- - \lambda_{1,0}\|u_{n_j}^-\|_p^p \to -\infty.$$

It contradicts (5). Hence $\lim_{n_j\to\infty} \|u_{n_j}^-\|_p < \infty$ for all subsequence $\{u_{n_j}\}$. We now suppose that $\|u_{n_j}\|_p \to \infty$ for some subsequence $\{u_{n_j}\}$. For each $j \in \mathbb{N}$, we define a function $w_{n_j} \in \mathbb{A}_0$ as follows:

$$w_{n_j}(x) := \frac{u_{n_j}(x)}{\|u_{n_j}\|_p}, \quad x \in S.$$

Then there exists a function $w_0 \in \mathcal{A}_0$ such that $w_{n_j}(x) \to w_0$ and $\|w_0\|_p = 1$. Since $\lim_{n_j\to\infty} \|u_{n_j}^-\|_p < \infty$ and $\|u_{n_j}\|_p \to \infty$, the function w_0 satisfies that $w_0 \geq 0$ and $w_0 \not\equiv 0$. Since $q < p$,

$$\lambda_{1,0} < \liminf_{t\to\infty} \frac{f(x,t)}{|t|^{p-2}t} \leq \liminf_{t\to\infty} \frac{-V(x)|t|^{q-2}t + f(x,t)}{|t|^{p-2}t}$$

which implies that there exists $\mu_1 > \lambda_{1,0}$ such that for $\epsilon \in \big(0, (\mu_1 - \lambda_{1,0})/2\big)$,

$$(\mu_1 - \epsilon)|t|^{p-2}t - C_1 \leq -V(x)|t|^{q-2}t + f(x,t), \quad t \in (\min_{j,x}\{u_{n_j}^-(x)\}, \infty)$$

for some constant $C_1 > 0$. Hence we have

$$\begin{aligned}(\mu_1 - \epsilon)|w_0(x)|^{p-2}w_0(x) &\leq \lim_{j\to\infty} \frac{-V(x)|u_{n_j}(x)|^{q-2}u_{n_j}(x) + f(x, u_{n_j}(x))}{\|u_{n_j}\|_p^{p-1}} \\ &= \lim_{j\to\infty} \frac{-\Delta_{p,\omega} u_{n_j}(x) + \epsilon_{n_j}(x)}{\|u_{n_j}\|_p^{p-1}} \\ &= -\Delta_{p,\omega} w_0(x)\end{aligned}$$

for all $x \in S$. Since $w_0 \geq 0$ and $w_0 \not\equiv 0$, by Theorem 5.1 in [14], the function w_0 is strictly positive on S. Without loss of generality, we assume that an eigenfunction ϕ_1 corresponding to $\lambda_{1,0}$ satisfies $0 < \phi_1 < w_0$ on S. Then we have that for any $\epsilon \in \big(0, (\mu_1 - \lambda_{1,0})/2\big)$,

$$-\Delta_{p,\omega} w_0(x) > (\lambda_{1,0} + \epsilon) w_0^{p-1}(x) > \lambda_{1,0}\phi_1(x), \quad x \in S.$$

Hence the functions w_0 and ϕ_1 are super-solution and sub-solution, respectively, to the following equation:

$$\begin{cases} -\Delta_{p,\omega} u(x) = (\lambda_{1,0} + \epsilon)|u(x)|^{p-2}u(x), & x \in S, \\ u(x) = 0, & x \in \partial S. \end{cases} \tag{6}$$

By Theorem D in Appendix, for any $\epsilon \in (0, \frac{\mu_2 - \lambda_{1,0}}{2})$, there exists a solution u to the equation (6) which implies that the principle eigenvalue $\lambda_{1,0}$ is not isolated. It is a contradiction by Theorem C in Appendix. Hence $\|u_n\|_p$ is bounded. □

Theorem 3.2. *If a function $f \in C(S \times \mathbb{R}, \mathbb{R})$ satisfies that $f(x, 0) = 0$ for all x in S and*

$$\limsup_{t \to -\infty} \frac{f(x,t)}{|t|^{p-2}t} < \lambda_{1,0}, \quad x \in S,$$

then the functional E^- satisfies the (PS) *condition.*

Proof. We assume that $\{u_n\}$ is the (PS) sequence. Since $E^{-\prime}[u_n](x) \to 0$ as $n \to \infty$ for all $x \in S$, there exists $\epsilon_n \in \mathcal{A}_0$, $n = 1, \ldots$ such that

$$-\Delta_{p,\omega} u_n(x) = -V(x)|u_n^-(x)|^{q-2}u_n^-(x) + f(x, u_n^-(x)) + \epsilon_n(x) \tag{7}$$

for all $x \in S$. Thus $\epsilon_n(x) \to 0$ for all x. Taking as a test function u_n^- in (7), since $\int_{\overline{S}} \nabla_{p,\omega} u_n \cdot \nabla_\omega u_n^- \geq \int_{\overline{S}} \nabla_{p,\omega} u_n^- \cdot \nabla_\omega u_n^-$, we have

$$\frac{1}{2}\int_{\overline{S}} \nabla_{p,\omega} u_n^- \cdot \nabla_\omega u_n^- \leq -\int_S V|u_n|^q + \int_S f(x, u_n^-)u_n^- + \int_S \epsilon_n u_n^-.$$

Since the function f satisfies

$$\limsup_{t \to -\infty} \frac{f(x,t)}{|t|^{p-2}t} < \lambda_{1,0}, \quad x \in S,$$

there exists $\mu_2 < \lambda_{1,0}$ such that

$$f(x,t) \geq \mu_2|t|^{p-2}t - C, \quad (x,t) \in S \times (-\infty, 0)$$

for some constant $C > 0$. Hence we have

$$\frac{1}{2}\int_{\overline{S}} \nabla_{p,\omega} u_n^- \cdot \nabla_\omega u_n^- \leq -\int_S V|u_n^-|^q + \int_S \mu_2|u_n^-|^p + \int_S (\epsilon_n - C)u_n^-.$$

By the definition of $\lambda_{1,0}$, we have

$$0 \leq \int_S (\mu_2 - \lambda_{1,0})|u_n^-|^p - \int_S V|u_n^-|^q + \int_S (\epsilon_n - C)u_n^- \tag{8}$$

for all $n \in \mathbb{N}$. Since $1 < q < p$, if there exists a subsequence $\{u_{n_j}\}$ such that $\|u_{n_j}^-\|_p \to \infty$, then

$$\int_S (\mu_2 - \lambda_{1,0})|u_{n_j}^-|^p - \int_S V|u_{n_j}^-|^q + \int_S (\epsilon_n - C)u_{n_j}^- \to -\infty \text{ as } j \to \infty.$$

It contradicts (8). Hence $\lim_{n_j\to\infty} \|u_{n_j}^-\|_p < \infty$ for all subsequence $\{u_{n_j}\}$. We now suppose that $\|u_{n_j}\|_p \to \infty$ for some subsequence $\{u_{n_j}\}$. For each $j \in \mathbb{N}$, we define a function $w_{n_j} \in \mathbb{A}_0$ as follows:

$$w_{n_j}(x) := \frac{u_{n_j}(x)}{\|u_{n_j}\|_p}, \quad x \in S.$$

Then there exists a function $w_0 \in \mathcal{A}_0$ such that $w_{n_j}(x) \to w_0$ and $\|w_0\|_p = 1$. Moreover, $w_0 \geq 0$ and $w_0 \not\equiv 0$ from the fact that $\lim_{n_j\to\infty} \|u_{n_j}^-\|_p < \infty$ and $\|u_{n_j}\|_p \to \infty$. From (7), we have

$$\begin{aligned}\frac{-\Delta_{p,\omega}u_{n_j}(x)}{\|u_{n-j}\|_p^{p-1}} &= \frac{-V(x)|u_{n_j}^-(x)|^{p-2}u_{n_j}^-(x) + f(x, u_{n_j}^-(x)) + \epsilon_{n_j}(x)}{\|u_{n-j}\|_p^{p-1}} \\ &= 0\end{aligned}$$

for all $j \in \mathbb{N}$. Hence the function w_0 satisfies that

$$\begin{cases} -\Delta_{p,\omega}w_0(x) = 0, & x \in S \\ w_0(x) = 0, & x \in \partial S. \end{cases}$$

By Theorem 5.1 in [14], the above equation has a unique solution $w_0 \equiv 0$ which contradicts the fact $w_0 \not\equiv 0$. Hence $\|u_n\|_p$ is bounded. □

Remark 3.3. By the similar arguments in Theorem 3.1 and Theorem 3.2, we can see that the functional E defined by

$$E[u] := \frac{1}{2p}\int_{\overline{S}} \nabla_{p,\omega}u \cdot \nabla_\omega u + \frac{1}{q}\int_S V|u|^q - \int_S F(x,u), \quad u \in \mathcal{A}_0,$$

satisfies the (PS) condition if a function $f \in C(S\times\mathbb{R}, \mathbb{R})$ satisfies that $f(x, 0) = 0$ for all $x \in S$ and

$$\limsup_{t\to-\infty} \frac{f(x,t)}{|t|^{p-2}t} < \lambda_{1,0} < \liminf_{t\to\infty} \frac{f(x,t)}{|t|^{p-2}t}, \quad x \in S.$$

In [2], it had been proved that in the case of $V \equiv 0$, E satisfies the (PS) condition.

In the next lemma, we observe behaviors of $E^\pm$ at the origin $\mathbb{O}$. If the behavior of the function f at the origin $\mathbb{O}$ satisfies

$$\limsup_{t\to0} \frac{f(x,t)}{|t|^{p-2}t} < \alpha_0, \quad x \in S$$

for some $\alpha_0 \in \mathbb{R}$, then there exists $r_0 > 0$ such that $E^\pm[u] > 0$ for all $u \in \mathcal{A}_0$ with $\|u\|_p = r_0$, namely, the origin $\mathbb{O}$ is a local minimizer of $E^\pm$. Hence we note that the origin is a solution of (2) and there is no solution near the origin.

Lemma 3.4. *For a given function $V : S \to (0, \infty)$, if a function $f \in C(S \times \mathbb{R}, \mathbb{R})$ satisfies that there exists $\alpha_0 \in \mathbb{R}$ such that*

$$\limsup_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} < \alpha_0, \quad x \in S, \tag{9}$$

then the origin $\mathbb{O}$ is a local minimizer of E^+ and E^-.

Proof. It follows from (9) that there exist $M < \alpha_0$ and $\delta > 0$ such that

$$F(\cdot, t) \le \frac{M}{p}|t|^p, \quad |t| < \delta.$$

Hence for each $u \in \mathcal{A}_0$ satisfying $\|u\|_p = 1$ and $r \in (0, \delta)$, we have

$$E^+[ru] \ge \frac{\lambda_{1,0}}{p} r^p + \left[\frac{\min_{x\in S}\{V(x)\}}{q} - \frac{M}{p} r^{p-q}\right] r^q \int_S |u^+|^p.$$

Hence $E^+[ru] > 0$ for $r \in (0, \left(\frac{p \min V}{qM}\right)^{\frac{1}{p-q}}]$. Thus the origin $\mathbb{O}$ is a local minimizer of E^+. Similarly, $\mathbb{O}$ is also a local minimizer of E^-. □

To use the mountain pass theorem, we now discuss conditions of f and V to guarantee that there exists $t_1 > 0$ such that $E^+[t_1\phi_1] < 0$ and $E^-[-t_1\phi_1] < 0$.

Lemma 3.5. *Suppose that $f \in C(S \times \mathbb{R}, \mathbb{R})$ satisfies that*

$$\lambda_{1,0} < \liminf_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t}, \quad x \in S. \tag{10}$$

If a function $V : S \to \mathbb{R}$ satisfies

$$\int_S |V| < \frac{q}{p}(\mu_0 - \lambda_{1,0}), \tag{11}$$

where $\mu_0 := \min_{x\in S}\left[\liminf_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t}\right]$, then there exists $t_1 > 0$ such that $E^+[t_1\phi_1] < 0$ and $E^-[-t_1\phi_1] < 0$.

Proof. It follows from the definition of μ_0 that for sufficiently small $\epsilon > 0$ and for each $T > 0$, there exists $r_T > p$ such that

$$\begin{cases} f(x,t) > (\mu_0 - \epsilon)|t|^{p-2}t - \frac{r}{r-p}|t|^{r-2}t, & 0 < t < T,\ r > r_T, \\ f(x,t) < (\mu_0 - \epsilon)|t|^{p-2}t - \frac{r}{r-p}|t|^{r-2}t, & -T < t < 0,\ r > r_T \end{cases}$$

which implies that

$$F(x,t) > \frac{\mu_0 - \epsilon}{p}|t|^p - \frac{1}{r-p}|t|^r, \quad 0 < |t| < T,\ r > r_T.$$

Hence for a positive eigenfunction ϕ_1 corresponding to $\lambda_{1,0}$ with $\|\phi_1\|_p = 1$, the functional E^+ satisfies

$$E^+[t\phi_1] < \frac{|t|^p}{p}\left[\frac{1}{2}\int_{\overline{S}} \nabla_{p,\omega}\phi_1 \cdot \nabla_\omega \phi_1\right] + \frac{|t|^q}{q}\int_S V\phi_1^q$$

$$
\begin{aligned}
&\quad - \frac{\mu_0 - \epsilon}{p} \int_S \phi_1^p + \frac{1}{r-p}|t|^r \int_S \phi_1^r \\
&< \frac{\lambda_{1,0} - \mu_0 + \epsilon}{p}|t|^p + \left[\left(\frac{1}{q} \int_S |V| \right) |t|^{q-p} + \frac{1}{r-p}|t|^{r-p} \right] |t|^p. \qquad (12)
\end{aligned}
$$

Now we put $h(t) := \left(\frac{1}{q} \int_S |V| \right) |t|^{q-p} + \frac{1}{r-p}|t|^{r-p}$ for $t \in \mathbb{R} \setminus \{0\}$. Since $r > p$, the function h has a minimizer $t_1 \neq 0$ which implies

$$
0 = h'(t_1) = \left(\frac{q-p}{q} \int_S |V| \right) |t_1|^{q-p-2} t_1 + |t_1|^{r-p-2} t_1.
$$

Thus

$$
t_1 = \pm \left[\left(\frac{p-q}{q} \right) \int_S |V| \right]^{\frac{1}{r-q}}.
$$

We note that if T increases, then r_T also increases. Hence for sufficiently large T, we have

$$
0 < \left[\left(\frac{p-q}{q} \right) \int_S |V| \right]^{\frac{1}{r-q}} < T, \quad r > r_T.
$$

Thus from (12),

$$
E^+[t_1\phi_1] < \left[\frac{\lambda_{1,0} - \mu_0 + \epsilon}{p} + \left(\frac{1}{q} \int_S |V| \right) |t_1|^{q-p} + \frac{1}{r-p}|t_1|^{r-p} \right] |t_1|^p.
$$

Since V satisfies

$$
\int_S |V| < \frac{q}{p}(\mu_0 - \lambda_{1,0} - 2\epsilon),
$$

we obtain

$$
\begin{aligned}
E^+[t_1\phi_1] &< \frac{\lambda_{1,0} - \mu_0 + \epsilon}{p} + \left(\frac{1}{q} \int_S |V| \right) |t_1|^{q-p} + \frac{1}{r-p}|t_1|^{r-p} \\
&< \frac{\lambda_{1,0} - \mu_0 + \epsilon}{p} + \left(\frac{\mu_0 - \lambda_{1,0} - 2\epsilon}{p} \right) \left[\left(\frac{p-q}{q} \right) \int_S |V| \right]^{\frac{q-p}{r-q}} \\
&\qquad + \frac{1}{r-p} \left[\left(\frac{p-q}{q} \right) \int_S |V| \right]^{\frac{r-p}{r-q}} \\
&\to \frac{-\epsilon}{p} \text{ as } r \to \infty.
\end{aligned}
$$

Hence

$$
E^+[t_1\phi_1] < 0.
$$

By the similar argument in the above proof, we can prove that $E^-[-t_1\phi_1] < 0$. $\square$

Now we are in a position to prove the main result of this paper.

Theorem 3.6. *Suppose that $f \in C(S \times \mathbb{R}, \mathbb{R})$ satisfies that*

$$\lambda_{1,0} < \liminf_{t \to 0} \frac{f(x,t)}{|t|^{p-2}t} \leq \limsup_{t \to 0} \frac{f(x,t)}{|t|^{p-2}t} < \alpha_0, \quad x \in S \tag{13}$$

for some $\alpha_0 > 0$ and

$$\limsup_{t \to -\infty} \frac{f(x,t)}{|t|^{p-2}t} < \lambda_{1,0} < \liminf_{t \to \infty} \frac{f(x,t)}{|t|^{p-2}t}, \quad x \in S. \tag{14}$$

If a function $V : S \to (0, \infty)$ satisfies $\int_S V < \frac{q}{p}(\mu_0 - \lambda_{1,0})$, where $\mu_0 := \min_{x \in S} \left[\liminf_{t \to 0} \frac{f(x,t)}{|t|^{p-2}t}\right]$, then the equation (2) has at least three non-trivial solutions.

Proof. By Lemma 3.4 and Lemma 3.5, the functional E^+ satisfies that
(i) there exists $r_0 > 0$ such that $E^+[u] > 0$ for all $u \in \mathcal{A}_0$ with $\|u\|_p = r_0$,
(ii) there exists $t_1 > 0$ such that $E^+[t_1 \phi_1] < 0$.
Hence by the mountain pass theorem,

$$c = \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} E^+[g(t)] > 0 \tag{15}$$

is a critical value of E^+ where

$$\Gamma := \{g \in C([0,1]; \mathbb{R}^n) \mid g(0) = \mathbb{O},\ g(1) = t_1 \phi_1\}.$$

Since $c > 0$ and $E^+[\mathbb{O}] = 0$, it is clear that all critical points corresponding to the critical value c is non-trivial.

We now show that there exists a non-negative critical point corresponding to the critical value c. It is easy to show that there is no critical point of E^+ on $\{v \in \mathcal{A}_0 \mid v \leq 0\}$ without the origin $\mathbb{O}$. Moreover, for every $u \in \mathcal{A}_0$ with $u^+ \not\equiv 0$ and $u^- \not\equiv 0$, it is clear that

$$E^+[u] > E^+[u^+]$$

which implies that

$$\max_{0 \leq t \leq 1} E^+[g(t)] > \max_{0 \leq t \leq 1} E^+[(g(t))^+]$$

for all $g \in \Gamma$ such that $(g(t))^+ \not\equiv 0$ for $t \in (0,1]$. Therefore all critical points corresponding to the critical value c is on the path g^+. Hence there exists a non-negative critical point u_1 corresponding to the critical value c. Since the assumption (13) implies $f(x,0) = 0$ for all $x \in S$, E^+ is differentiable. Therefore, u_1 is a non-trivial and non-negative solution to Eq. (2).

As applying the similar argument with the above proof to the functional E^-, we can see that there exists a non-trivial and non-positive solution u_2 of (2).

Now, we show the existence of the third solution to the equation (2). From the condition that

$$\limsup_{t \to -\infty} \frac{f(x,t)}{|t|^{p-2}t} < \lambda_1,$$

there exists $\mu_1 < \lambda_1$ such that

$$F(x,t) \le \frac{\mu_1}{p}|t|^p + C, \quad t < 0$$

for some $C \in \mathbb{R}$. Hence for $t > 0$ and $u \in \mathcal{A}_0$ with $\|u\|_p = 1$,

$$E^-[tu] > \frac{\lambda_1 - \mu_1}{p} t^p - C' t \int_S |u^-| \to \infty \text{ as } t \to \infty,$$

which implies that the functional E^- is coercive. Thus E^- has a global minimizer u_3. By Lemma 3.5, there exists $t_1 > 0$ such that $E^-[t_1\phi_1] < 0$. Hence the global minimizer u_3 is non-trivial. Moreover, since the functional E^- satisfies that

$$\begin{aligned} E^-[u] &= \frac{1}{2p}\int_{\overline{S}} \nabla_{p,\omega} u \cdot \nabla_\omega u + \frac{1}{q}\int_S V|u^-(x)|^q - \int_S F(x, u^-(x)) \\ &\ge \frac{1}{2p}\int_{\overline{S}} \nabla_{p,\omega} u^- \cdot \nabla_\omega u^- + \frac{1}{q}\int_S V|u^-(x)|^q - \int_S F(x, u^-(x)) \\ &= E^-[u^-] \end{aligned}$$

for all $u \in \mathcal{A}_0$, the global minimizer u_3 is non-positive.

We now show that $u_2 \not\equiv u_3$. It follows from $E^-[-t_1\phi_1] < 0$ that $E^-[u_3] < 0$. Since $E^-[u_2] > 0$, we have $u_2 \not\equiv u_3$. Therefore, u_3 is the third non-trivial and non-positive solution to the equation (2). $\square$

We note that in Theorem 3.6, it is optimal that the function V is strictly positive. Let us see the next example.

Example 3.7. Let assume that a weighted graph $\overline{S}$ whose interior and boundary are given by

$$S = \{x_1\}, \quad \text{and} \quad \partial S = \{x_0, x_2\},$$

and weights on edges are given by

$$\omega(x_0, x_1) = 1, \quad \omega(x_1, x_2) = 1, \quad \text{and} \quad \omega(x_0, x_2) = 0,$$

as Figure 1.

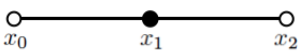

Figure 1. The graph $\overline{S}$.

We now consider a function $f : S \times \mathbb{R} \to \mathbb{R}$ satisfying that
(i) $f(x_1, t)$ is continuous with respect to t,
(ii) f is defined on $(-\infty, -2) \cup (-1, \infty)$ as

$$f(x_1, t) = \begin{cases} 3|t|t, & t > -1, \\ |t|t, & t < -2. \end{cases}$$

If $p=3$ and $q=2$, then the function f satisfies

$$\liminf_{t\to 0}\frac{f(x_1,t)}{|t|^{p-2}t}=\limsup_{t\to 0}\frac{f(x_1,t)}{|t|^{p-2}t}=3,$$

$$\limsup_{t\to -\infty}\frac{f(x_1,t)}{|t|^{p-2}t}=1,\quad \liminf_{t\to \infty}\frac{f(x_1,t)}{|t|^{p-2}t}=3,$$

and by the definitions of $\lambda_{1,0}$ and μ_0, we have

$$\lambda_{1,0}=\inf_{\substack{\phi\not\equiv 0\\ \phi\in\mathcal{A}_0}}\frac{\frac{1}{2}\int_{\overline{S}}\nabla_\omega\phi\cdot\nabla_{p,\omega}\phi}{\int_S|\phi|^p}=\inf_{\substack{\phi\not\equiv 0\\ \phi\in\mathcal{A}_0}}\frac{2\phi^p(x_1)}{\phi^p(x_1)}=2,$$

and

$$\mu_0=\min_{x\in S}\left[\liminf_{t\to 0}\frac{f(x,t)}{|t|^{p-2}t}\right]=\liminf_{t\to 0}\frac{f(x_1,t)}{|t|^{p-2}t}=3.$$

Hence the function f satisfies (13) and (14). Finally, taking $V(x_1)=\frac{1}{3}$, it holds that

$$\int_S V=V(x_1)=\frac{1}{3}<\frac{q}{p}(\mu_0-\lambda_{1,0})=\frac{2}{3}.$$

Therefore, by Theorem 3.6, there exist one non-negative critical point of E^+ and two non-positive critical points of E^-.

Now, let us find the three critical points by analyzing the functionals E^+ and E^-. Since $S=\{x_1\}$ and $f(x_1,t)=3|t|t$ for $t\in(-1,\infty)$, the functional E^+ is given by

$$\begin{aligned}E^+[u]&=\frac{2}{3}|u(x_1)|^3+\frac{1}{6}|u^+(x_1)|^2-\int_0^{u^+(x_1)}f(x_1,t)dt\\&=-\frac{1}{3}|u(x_1)|^3+\frac{1}{6}|u^+(x_1)|^2\end{aligned}$$

for $u(x_1)>0$. As the graph of E^+ in Figure 2, the functional E^+ has a positive critical point.

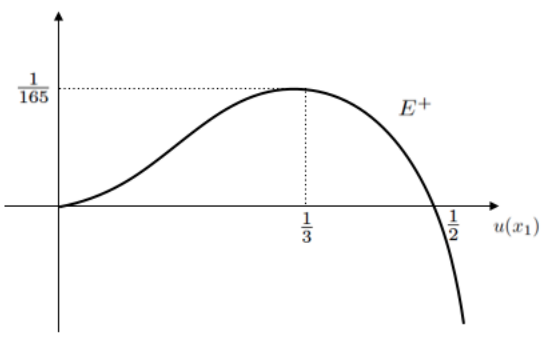

FIGURE 2. The graph of E^+ on $(0,\infty)$.

We now find two critical points of the functional E^-. By the definition of f, the functional E^- is given by

$$E^-[u] = -\frac{1}{3}|u(x_1)|^3 + \frac{1}{6}|u^-(x_1)|^2 \text{ for } -1 < u(x_1) < 0,$$

and

$$E^-[u] = \frac{1}{6}|u(x_1)|^3 + \frac{1}{6}|u^-(x_1)|^2 \text{ for } u(x_1) < -2.$$

The graph of E^- on $(-\infty, -2) \cup (-1, 0)$ is in Figure 3. Hence the functional

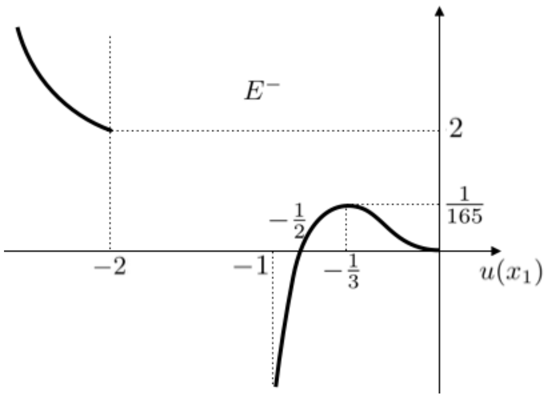

FIGURE 3. The graph of E^- on $(-\infty, 0)$.

E^- has a negative critical point on $(-1, 0)$. Moreover, even though we skip the graph of E^- on the interval $[-2, -1]$ in Figure 3, since f is continuous and $f(x_1, 0) = 0$, the functional E^- is continuous and differentiable. Hence it is clear that there exists a negative critical point on $(-2, -1)$.

We note that it is optimal that $V > 0$. In this example, if $V(x_1) = 0$, then the functionals E^+ and E^- are given by

$$E^+[u] = -\frac{1}{3}|u(x_1)|^3 \text{ on } (0, \infty),$$

and

$$E^-[u] = \begin{cases} -\frac{1}{3}|u(x_1)|^3 \text{ on } (-1, 0), \\ \frac{1}{6}|u(x_1)|^3 \text{ on } (-\infty, -2). \end{cases}$$

Hence in the case of $V(x_1) = 0$, there is no critical points of E^+ and E^- on $(0, \infty)$ and $(-1, 0)$, respectively.

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Appendix

In this section, we discuss that $\lambda_{1,V}$ is isolated for all $V : S \to \mathbb{R}$ and we also deal with the sub-super solution method.

Here, we give the discrete version of Picone's identity and the comparison principle which are proved in [16, Theorem 4.1] and [15, Theorem 1], respectively.

Lemma A ([16])**.** *Let u be a non-negative function and v be a positive function on a graph G. Then*

$$\nabla_\omega u \cdot \nabla_{p,\omega} u - \nabla_\omega \left(\frac{u^p}{v^{p-1}} \right) \cdot \nabla_{p,\omega} v \geq 0 \text{ on } G.$$

Moreover, the equality holds if and only if $u \equiv tv$ for some constant $t > 0$.

Lemma B ([15])**.** *For a function $V : S \to \mathbb{R}$ with $\lambda_{1,V} > 0$, suppose that $u_i : \bar{S} \to \mathbb{R}$, $i = 1, 2$ satisfy the inequalities*

$$(16) \qquad \begin{cases} -\Delta_{p,\omega} u_2(x) + V(x)|u_2(x)|^{p-2}u_2(x) \\ \geq -\Delta_{p,\omega} u_1(x) + V(x)|u_1(x)|^{p-2}u_1(x), & x \in S, \\ u_2(z) \geq u_1(z), & z \in \partial S. \end{cases}$$

If we assume in addition that

$$\begin{cases} -\Delta_{p,\omega} u_2(x) + V(x)|u_2(x)|^{p-2}u_2(x) \geq 0, & x \in S, \\ u_2(z) \geq 0, & z \in \partial S, \end{cases}$$

then $u_1 \leq u_2$ in S. Moreover, the equalities of (16) are hold if and only if $u_1 \equiv u_2$ in $\bar{S}$.

Using Lemma A and Lemma B, we prove the following two results.

Theorem C. *For $V : S \to \mathbb{R}$, $\lambda_{1,V}$ is isolated.*

Proof. Suppose that for $\epsilon > 0$, there exists $\psi_\epsilon \in \mathcal{A}_0$ with $\int_S |\psi_\epsilon|^p = 1$ such that

$$\begin{cases} -\Delta_{p,\omega} \psi_\epsilon(x) + V(x)|\psi_\epsilon(x)|^{p-2}\psi_\epsilon(x) \\ = (\lambda_{1,V} + \epsilon)|\psi_\epsilon(x)|^{p-2}\psi_\epsilon(x), & x \in S, \\ \psi_\epsilon(x) = 0, & x \in \partial S. \end{cases}$$

Since the multiplicity of $\lambda_{1,V}$ is one, there exists a subsequence ψ_{ϵ_n} such that $\psi_{\epsilon_n} \to \phi_0$ as $n \to \infty$ where ϕ_0 is an eigenfunction corresponding to $\lambda_{1,V}$ satisfying $\phi_0 > 0$ and $\int_S |\phi_0|^p = 1$. Hence for sufficiently small ϵ, $\psi_\epsilon > 0$ on S. Since ϕ_0 is an eigenfunction, we have that

$$-\epsilon \phi_0^{p-1}(x) = -\Delta_{p,\omega} \phi_0(x) + \Delta_{p,\omega} \psi_\epsilon(x) \left(\frac{\phi_0^{p-1}(x)}{\psi_\epsilon^{p-1}(x)} \right), \quad x \in S.$$

It implies that

$$0 > -\epsilon \int_{\bar{S}} \phi_0^p = \frac{1}{2} \int_{\bar{S}} \nabla_{p,\omega} \phi_0 \cdot \nabla_\omega \phi_0 - \nabla_{p,\omega} \psi_\epsilon \cdot \nabla_\omega \left(\frac{\phi_0^p}{\psi_\epsilon} \right)$$

which is contradicted by Lemma A. $\square$

We now give the method of sub-super-solutions for the discrete p-Laplacian.

Theorem D. *For a function $f \in C(S \times \mathbb{R}; \mathbb{R})$, suppose that $\underline{u}$ and $\overline{u}$ in $\mathcal{A}_0$ with $\underline{u} \leq \overline{u}$ are sub-solution and super-solution to the equation*

$$\begin{cases} -\Delta_{p,\omega} u(x) = f(x, u(x)), & x \in S \\ u(x) = 0, & x \in \partial S. \end{cases} \tag{17}$$

If the given function f satisfies that there exists $\lambda > 0$ such that $f(\cdot, t) + \lambda |t|^{p-2} t$ is nondecreasing in S, then there exists a solution u of Eq. (17) *such that $\underline{u} \leq u \leq \overline{u}$.*

Proof. We put $u_0 = \underline{u}$ and then given u_k, $k = 0, 1, \ldots$ inductively define u_{k+1} as follows:

$$\begin{cases} -\Delta_{p,\omega} u_{k+1}(x) + \lambda |u_{k+1}(x)|^{p-2} u_{k+1}(x) & \\ = f(x, u_k(x)) + \lambda |u_k(x)|^{p-2} u_k(x), & x \in S \\ u(x) = 0, & x \in \partial S \end{cases}$$

for sufficiently large $\lambda > 0$ satisfying that $f(\cdot, t) + \lambda |t|^{p-2} t$ is nondecreasing. We first show that $\underline{u}(x) \leq u_1(x) \leq u_2(x) \leq \cdots$ for all $x \in S$. Since u_0 is a sub-solution to (17), we have

$$-\Delta_{p,\omega} u_1 + \lambda |u_1|^{p-2} u_1 \geq -\Delta_{p,\omega} u_0 + \lambda |u_0|^{p-2} u_0$$

in S. By Lemma B, $u_1(x) \geq u_0(x)$ for all $x \in S$. Now, assume inductively

$$u_{k-1}(x) \leq u_k(x), \quad x \in S.$$

Since $f(\cdot, t) + \lambda |t|^{p-2} t$ is nondecreasing, we have

$$-\Delta_{p,\omega} u_{k+1} + \lambda |u_{k+1}|^{p-2} u_{k+1} \geq -\Delta_{p,\omega} u_k + \lambda |u_k|^{p-2} u_k$$

in S. By Lemma B, we have $u_{k+1}(x) \geq u_k(x)$ for all $x \in S$. Now, we show that $u_k(x) \leq \overline{u}(x)$ for all $x \in S$. For $k = 0$, by the assumption, it is clear. Assume for induction $u_k(x) \leq \overline{u}(x)$, $x \in S$. Then we have

$$-\Delta_{p,\omega} u_{k+1} + \lambda |u_{k+1}|^{p-2} u_{k+1} \leq -\Delta_{p,\omega} \overline{u} + \lambda |\overline{u}|^{p-2} \overline{u}$$

in S. By Lemma B, $u_{k+1}(x) \leq \overline{u}(x)$ for all $x \in S$. Thus we have $\underline{u} \leq u_1 \leq u_2 \leq \cdots \leq \overline{u}$ in $\overline{S}$ which implies that there exists $u \in \mathcal{A}_0$ such that $\lim_{n \to \infty} u_n = u$ in $\overline{S}$. Hence the function u is a solution to (17) and $\underline{u} \leq u \leq \overline{u}$ in S. □

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Soon-Yeong Chung
Department of Mathematics and Program of Integrated Biotechnology
Sogang University
Seoul 121-742, Korea
E-mail address: sychung@sogang.ac.kr

Jea-Hyun Park
Department of Mathematics
Kunsan National University
Kunsan 573-701, Korea
E-mail address: parkjhm@kunsan.ac.kr