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ON RELATIVE CLASS NUMBER AND CONTINUED FRACTIONS

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Abstract. The relative class number $H_d(f)$ of a real quadratic field $K = \mathbb{Q}(\sqrt{m})$ of discriminant d is the ratio of class numbers of $\mathcal{O}_f$ and $\mathcal{O}_K$, where $\mathcal{O}_K$ denotes the ring of integers of K and $\mathcal{O}_f$ is the order of conductor f given by $\mathbb{Z}+f\mathcal{O}_K$. In a recent paper of A. Furness and E. A. Parker the relative class number of $\mathbb{Q}(\sqrt{m})$ has been investigated using continued fraction in the special case when $\sqrt{m}$ has a diagonal form. Here, we extend their result and show that there exists a conductor f of relative class number 1 when the continued fraction of $\sqrt{m}$ is non-diagonal of period 4 or 5. We also show that there exist infinitely many real quadratic fields with any power of 2 as relative class number if there are infinitely many Mersenne primes.

1. Introduction

A real quadratic field K is an extension $\mathbb{Q}(\sqrt{m}) = \{a + b\sqrt{m} \mid a, b \in \mathbb{Q}\}$ for some square-free natural number m. The discriminant d of K is m if $m \equiv 1$ mod 4, otherwise $d = 4m$. In the former case, the ring $\mathcal{O}_K$ of integers of K is $\{a + b\frac{1+\sqrt{m}}{2} \mid a, b \in \mathbb{Z}\}$, and in the latter case, $\mathcal{O}_K = \{a + b\sqrt{m} \mid a, b \in \mathbb{Z}\}$. By Dirichlet's unit theorem, the units of $\mathcal{O}_K$ can be written as $\pm\xi_m^i$ $(i \in \mathbb{Z})$ where ξ_m is called the fundamental unit. The relative class number of K for a conductor f is the ratio $H_d(f)$ of the class numbers of the order $\mathcal{O}_f = \mathbb{Z}+f\mathcal{O}_K$ and $\mathcal{O}_K$. It was Dirichlet ([4]) who first obtained several interesting results concerning relative class number of real quadratic fields. Gauss's conjecture on existence of infinitely many real quadratic fields of class number 1 motivated Dirichlet to pose the question whether there are infinitely many real quadratic fields of relative class number 1 (see [2], [4]). The question was successfully addressed in [8], and a characterization in terms of the norm of the fundamental unit was given in [1]. In [6], A. Furness and E. A. Parker studied relative class number of $\mathbb{Q}(\sqrt{m})$ using continued fraction in the special case when $\sqrt{m}$ has a diagonal form [see (3.6)]. The main goal of this paper is to extend their approach and show that there exists a conductor f of relative class number

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1 when the continued fraction of $\sqrt{m}$ is non-diagonal of period 4 or 5. The following result of Dirichlet concerning the relative class number will be very useful to us (see [2]).

Theorem 1.1 (Dirichlet)**.** *Let $\theta(f)$ be the smallest positive integer such that $\xi_m^{\theta(f)}$ belong to $\mathcal{O}_f$ and $\psi(f) = f\prod_{q|f}\left(1-\left(\frac{d}{q}\right)\frac{1}{q}\right)$, where $\left(\frac{d}{q}\right)$ denotes the "Kronecker residue symbol" of d modulo a prime q. Then the relative class number for conductor f is given by*

$$H_d(f) = \frac{\psi(f)}{\theta(f)}. \tag{1.1}$$

The Kronecker residue symbol $\left(\frac{d}{q}\right)$ is the same as the Legendre symbol when q is an odd prime. For $q = 2$ and d odd, $\left(\frac{d}{q}\right)$ is 1 if $d \equiv \pm 1 \pmod 8$ and -1 if $d \equiv \pm 3 \pmod 8$. The relative class number is always an integer (see [2]), hence $\theta(f)$ always divides $\psi(f)$. We write the fundamental unit of $\mathcal{O}_K$ as

$$\xi_m = \alpha_0 + \beta_0\sqrt{m}, \qquad 2\alpha_0,\ 2\beta_0 \in \mathbb{Z}.$$

Note that $\xi_m^3 \in \mathbb{Z}[\sqrt{m}]$ and when $m \not\equiv 5 \pmod 8$, α_0 and β_0 are integers (see [7]). For the rest of the paper, we use the following notation:

$$\tilde{\beta_0}=\beta_0,\ \tilde{\alpha_0}=\alpha_0 \ \text{ if } \xi_m \in \mathbb{Z}[\sqrt{m}],\ \ \tilde{\beta_0}=2\beta_0,\ \tilde{\alpha_0}=2\alpha_0 \ \text{ if } \xi_m \notin \mathbb{Z}[\sqrt{m}]. \tag{1.2}$$

2. Continued fraction approach

In [6], the authors showed that the existence of relative class number 1 can be related to the continued fraction of $\sqrt{m}$. Their results were for those m for which $\sqrt{m}$ has a continued fraction of diagonal form (see (3.6)). We are now going to extend that approach and prove that whenever $\sqrt{m}$ is represented by a continued fraction of period 4 or 5, there exists a conductor f with relative class number 1. When m is a square-free positive integer, it is well-known (e.g., see [9]) that the continued fraction of $\sqrt{m}$ is periodic of the form

$$n + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_r + \cfrac{1}{2n + \cfrac{1}{a_1 + \cdots}}}}}, \qquad \text{where } n = \lfloor\sqrt{m}\rfloor, \text{ and } a_i = a_{r+1-i}. \tag{2.1}$$

It is standard to denote it as

$$\sqrt{m} = \langle n, \overline{a_1, a_2, \ldots, a_r, 2n}\rangle. \tag{2.2}$$

The main result of this paper is the following theorem.

Theorem 2.1. *Let m be a square-free positive integer such that $\sqrt{m}$ is represented by a continued fraction of period 4 or 5. Then there exists a prime divisor p of m such that the relative class number for p is 1. In other words, $H_d(p) = 1$ where d is the discriminant of the number field $\mathbb{Q}(\sqrt{m})$.*

In order to prove the above theorem, we will use Dirichlet's formula (1.1) which involves the fundamental unit ξ_m of the real quadratic field $\mathbb{Q}(\sqrt{m})$. It is well known that ξ_m is closely related to the continued fraction for $\sqrt{m}$. Hence we will first look into properties of the continued fraction for $\sqrt{m}$.

3. Useful properties of the continued fraction of $\sqrt{m}$

It is convenient for us to write the continued fraction of $\sqrt{m}$ in (2.2) as

$$\sqrt{m} = \langle n, x\rangle = n + x^{-1}, \text{ where } x = \langle\overline{a_1, a_2, \ldots, a_r, 2n}\rangle. \tag{3.1}$$

The i-th convergent of the continued fraction of x is defined as

$$\frac{h_i}{k_i} = a_1 + \cfrac{1}{a_2 + \frac{1}{\cdots + \frac{1}{a_{i+1}}}} .$$

It is easy to verify that the following recurrence relations are satisfied by h_i and k_i:

$$\begin{aligned} &h_0 = a_1, \quad k_0 = 1, \qquad h_1 = 1 + a_1a_2, \quad k_1 = a_2, \\ &h_i = a_ih_{i-1} + h_{i-2}, \qquad k_i = a_ik_{i-1} + k_{i-2}, \\ &h_ik_{i-1} - h_{i-1}k_i = (-1)^{i-1}. \end{aligned} \tag{3.2}$$

Conventionally, the explicit expressions in terms of the a_is for the partial quotients h_i and k_i are denoted by

$$h_{i-1} = [a_1, a_2, a_3, \ldots, a_i], \quad k_{i-1} = [a_2, a_3, \ldots, a_i], \tag{3.3}$$

(e.g. see [3]),

$$\begin{aligned} &h_2 = [a_1, a_2, a_3] = a_1a_2a_3 + a_1 + a_3, \\ &h_3 = [a_1, a_2, a_3, a_4] = a_1a_2a_3a_4 + a_3a_4 + a_1a_4 + a_1a_2 + 1. \end{aligned} \tag{3.4}$$

The Euler's rule (see [3]) tells us that the expressions for partial quotients in terms of the a_is are unchanged when we take the a_is in the reverse order, i.e.,

$$[a_1, a_2, a_3, \ldots, a_k] = [a_k, a_{k-1}, \ldots, a_2, a_1]. \tag{3.5}$$

We first prove the following lemma, which completely generalizes an analogue in [6] for $\sqrt{m}$ with a continued fraction of diagonal form, i.e.,

$$\sqrt{m} = \langle n, \overline{a, \ldots, a, 2n}\rangle. \tag{3.6}$$

Lemma 3.1. *Let $\sqrt{m} = \langle n, \overline{a_1, a_2, \ldots, a_r, 2n}\rangle$ and let $\frac{h_i}{k_i}$ be the i-th convergent of the purely periodic continued fraction $x = \langle\overline{a_1, a_2, \ldots, a_r, 2n}\rangle$. Then we have*

$$m = n^2 + \frac{k_r}{h_{r-1}}.$$

In particular, h_{r-1} divides k_r.

Proof. Note that $\sqrt{m} = \langle n, x\rangle = n + x^{-1}$. Now, x can also be written as

$$x = \langle a_1, a_2, \dots, a_r, 2n, \overline{a_1, a_2, \dots, a_r, 2n}\rangle = \langle a_1, a_2, \dots, a_r, 2n, x\rangle.$$

Therefore, the $(r+1)$-th convergent of $\langle a_1, a_2, \dots, a_r, 2n, x\rangle$ equals x. By the recurrence relations given in (3.2), the $(r+1)$-th convergent of $\langle a_1, a_2, \dots, a_r, 2n, x\rangle$ is given by

$$\frac{xh_r + h_{r-1}}{xk_r + k_{r-1}}.$$

As a consequence, we have

$$x = \frac{xh_r + h_{r-1}}{xk_r + k_{r-1}},$$

and we obtain the following quadratic equation for x^{-1}:

$$x^{-2}(h_{r-1}) + x^{-1}(h_r - k_{r-1}) - k_r = 0. \tag{3.7}$$

Using Euler's rule (3.5) for the partial quotients h_i and k_i of x and then using the fact that $a_i = a_{r+1-i}$ from (2.1), we obtain

$$k_{r-1} = [a_2, a_3, \dots, a_r] = [a_r, a_{r-1}, \dots, a_2] = [a_1, a_2, \dots, a_{r-1}] = h_{r-2}. \tag{3.8}$$

On substituting in the quadratic equation (3.7), we obtain

$$x^{-2}(h_{r-1}) + x^{-1}(h_r - h_{r-2}) - k_r = 0.$$

In view of the recurrence relation (3.2), we find that $h_r = 2nh_{r-1} + h_{r-2}$. Therefore, the quadratic equation for x^{-1} simplifies to

$$x^{-2}(h_{r-1}) + x(2nh_{r-1}) - k_r = 0,$$

and we obtain its solution

$$x^{-1} = -n + \sqrt{n^2 + \frac{k_r}{h_{r-1}}} \quad \text{as } x > 0.$$

Thus we find that $\sqrt{m} = n + x^{-1} = \sqrt{n^2 + \frac{k_r}{h_{r-1}}}$ and hence

$$m = n^2 + \frac{k_r}{h_{r-1}}.$$

As m is an integer, it now follows trivially that k_r divides h_{r-1}. □

Next, we obtain a bound on the coefficients appearing in the continued fraction of $\sqrt{m}$. We will use this bound in the next section.

Proposition 3.2. *Let* $\sqrt{m} = \langle n; \overline{a, b, \dots}\rangle$ *be a continued fraction of period at least* 3. *Then* $ab < 2n$.

Proof. Let $[\sqrt{m}] = n$ so that $m = n^2 + t$ where t must be an integer satisfying $t \leq 2n$. Then

$$\sqrt{m} = n + \sqrt{m} - n = n + \frac{1}{\frac{\sqrt{m}+n}{m-n^2}} = n + \frac{1}{\frac{2n+(\sqrt{m}-n)}{t}}, \quad 0 < \sqrt{m} - n < 1.$$

Now, the next coefficient a in the continued fraction of $\sqrt{m}$ is given by

$$2n = ta + r_1, \;\; 0 < r_1 < t. \tag{3.9}$$

Note that $r_1 = 0$ would imply that $\sqrt{m}$ has continued fraction $\langle n; \overline{a, 2n} \rangle$ of period 2. Now,

$$\sqrt{m} = n + \frac{1}{\frac{ta+(\sqrt{m}-(n-r_1))}{t}} = n + \frac{1}{a + \frac{1}{\frac{t(\sqrt{m}+(n-r_1))}{m-(n-r_1)^2}}}.$$

Now, the last denominator is

$$m - (n - r_1)^2 = (m - n^2) + 2nr_1 - r_1^2 = t + (ta + r_1)r_1 - r_1^2 = t(1 + ar_1),$$

and the numerator is

$$t(\sqrt{m} + (n - r_1)) = t(2n - r_1 + \sqrt{m} - n).$$

Now, the next coefficient b in the continued fraction of $\sqrt{m}$ is given by

$$2n - r_1 = (1 + ar_1)b + r_2, \quad 0 \leq r_2 < 1 + ar_1. \tag{3.10}$$

As $r_1 \geq 1$, we deduce from the last equality that $2n > ab$. □

4. Proof of the theorem

We are now going to use the properties of the continued fraction of $\sqrt{m}$ deduced in §3 to prove theorem 2.1. It is well known (see [7]) that $nh_{r-1} + h_{r-2} + h_{r-1}\sqrt{m}$ is the fundamental unit ξ_m of $\mathbb{Q}(\sqrt{m})$ except in the case $m \equiv 5 \pmod 8$, when it equals ξ_m^3. Therefore, h_{r-1} is always a multiple of $\tilde{\beta}_0$ (see (1.2)).

If we can show that m does not divide h_{r-1} then there will be a prime factor p of m which does not divide h_{r-1} as m is square-free. Such a prime p will not divide $\tilde{\beta}_0$ as well. For such a prime p, it is obvious that $\theta(p)$ will be p and $\psi(p)$ will also be p in Dirichlet's formula (1.1). Therefore, such a prime p will be a conductor of relative class number 1. In the first subsection below, we will prove that if the continued fraction for $\sqrt{m}$ has period 4 then m does not divide h_{r-1}, which ensures the existence of a prime conductor p of relative class number 1. In the second subsection, we will do the same when the continued fraction for $\sqrt{m}$ has period 5.

4.1. When $\sqrt{m}$ has a continued fraction of period 4

Suppose $\sqrt{m}$ has a continued fraction of period 4 ($r = 3$ in Lemma 3.1). As outlined in §3 (2.1), the continued fraction of $\sqrt{m}$ is necessarily of the form $\sqrt{m} = \langle n, \overline{a, b, a, 2n}\rangle$. It is clear from the preceding paragraph that theorem 2.1 in this case reduces to the following proposition.

Proposition 4.1. *If $\sqrt{m} = \langle n, \overline{a, b, a, 2n}\rangle$, then m does not divide h_2.*

Proof. We first consider the case when $b > 3$. It is enough to show that $h_2 \leq n^2$ where $n = \lfloor\sqrt{m}\rfloor$ as in (2.1). By (3.3) and (3.4) we have

$$\begin{aligned} h_2 &= [a, b, a] = a^2b + 2a, \\ h_1 &= [a, b] = ab + 1 = k_2, \\ k_1 &= [b] = b. \end{aligned} \tag{4.1}$$

As $h_2k_1 - k_2h_1 = -1$ (by (3.2)), we deduce that

$$k_2^2 \equiv 1 \pmod{h_2}. \tag{4.2}$$

By Lemma 3.1, h_2 divides k_3. Using the recurrence relation (3.2) for k_3, we find that

$$2nk_2 + k_1 = k_3 \equiv 0 \pmod{h_2}.$$

Multiplying both sides by k_2 and using the relation (4.2), we obtain

$$2n \equiv 2nk_2^2 \equiv -k_2k_1 \equiv -ab^2 - b \pmod{h_2}.$$

Multiplying both sides by a, and using $h_2 = a^2b + 2a$ from (4.1), we obtain

$$2na \equiv -a^2b^2 - ab \equiv ab \pmod{h_2}.$$

By proposition 3.2, we have $2n > b$ and consequently there is a positive integer l such that

$$2na = l(h_2) + ab = la(ab + 2) + ab.$$

Consequently, $2n \geq ab + b + 2$ and

$$4n^2 > a^2b^2+4ab+4+2ab^2+4b+b^2 > a^2b^2+4ab \geq 4(a^2b+2a) = 4h_2 \ \ (\text{as } b > 3).$$

Therefore, $m > n^2 > h_2$ and in particular m can not divide h_2 when $b > 3$. For $b = 1,\ 2$ or 3, we use the corresponding expression for h_2 in terms of a, and then find a suitable upper bound for a in terms of n. As noted in (3.9), the coefficient a in the continued fraction for $\sqrt{m}$ is given by

$$2n = ta + r_1, \ \text{where } t = m - n^2, \ \text{and } r_1 > 0 \ (\text{as the period of } \sqrt{m} \text{ is } 4). \tag{4.3}$$

But t is at least 2 as otherwise,

$$m = n^2 + t = n^2 + 1 \Rightarrow \sqrt{m} = \langle n, \overline{2n}\rangle \ \ (\text{of period } 1).$$

Consequently $2a < 2n$ from (4.3) and $a \leq n - 1$. Observe also that $n > 1$ as $\sqrt{m} = \sqrt{2}$ or $\sqrt{3}$ is of period strictly less than 4.
For $b = 1$, we now have

$$h_2 = [a, 1, a] = a^2 + 2 \leq (n - 1)^2 + 2 \leq n^2 < m \text{ as } a \leq n - 1.$$

For $b = 2$, we have

$$h_2 = [a, 2, a] = 2a^2 + 2a \leq 2(n-1)^2 + 2(n-1) \leq 2n^2 < 2m.$$

If m divides h_2, then that would mean $h_2 = m$. But then $m = h_2 = 2a(a+1)$ will not be square free as 4 divides $2a(a+1)$.

For $b = 3$, we have $h_2 = 3a^2 + 2a$ and we need a sharper upper bound for a than just $n-1$. By (4.3), it amounts to finding a sharper lower bound for t. We have

$$t = \frac{2n - r_1}{a} > \frac{2n - (r_1 + r_2)}{1 + ar_1} = b,$$

where the last equality above is due to (3.10). Thus $t \geq 4$ and by (4.3)

$$a = \frac{2n - r_1}{t} \leq \frac{2n-1}{4} = \frac{n}{2} - \frac{1}{4} \Rightarrow a \leq \frac{n-1}{2}$$

as a is an integer. Finally,

$$h_2 = 3a^2 + 2a \leq \frac{3}{4}(n-1)^2 + (n-1) \leq n^2 < m. \qquad \square$$

4.2. When $\sqrt{m}$ has a continued fraction of period 5

Suppose $\sqrt{m}$ has a continued fraction of period 5 ($r = 4$ in Lemma 3.1). As outlined in §3 (2.1), it is clear that the continued fraction of $\sqrt{m}$ is necessarily of the form $\sqrt{m} = \langle n, \overline{a, b, b, a, 2n} \rangle$. By the paragraph preceding §4.1, we need only to show that m does not divide h_3. Therefore, Theorem 2.1 in this case reduces to the following proposition.

Proposition 4.2. *If $\sqrt{m} = \langle n, \overline{a, b, b, a, 2n} \rangle$, then $m > h_3$.*

Proof. Recall that $\sqrt{m} = \sqrt{n^2 + \frac{k_r}{h_{r-1}}}$ when $\sqrt{m} = \langle n, \overline{a_1, a_2, \ldots, a_r, 2n} \rangle$ where, $a_i = a_{r+1-i}$, and $\frac{h_i}{k_i}$ denotes the i-th convergent. Here, $r = 4$. By (3.3) and (3.4), we have

$$\begin{aligned} h_3 &= [a, b, b, a] = (ab+1)^2 + a^2 = (a^2b^2 + a^2 + ab) + (ab+1) \\ k_3 &= [b, b, a] = ab^2 + a + b \\ h_2 &= [a, b, b] = k_3 \\ k_2 &= [b, b] = b^2 + 1 \end{aligned} \tag{4.4}$$

and we obtain

$$k_3^2 \equiv (-1)^3 = -1 \pmod{h_3} \text{ (as in (4.2))}. \tag{4.5}$$

By Lemma 3.1 h_2 divides k_3. Using the recurrence relation (3.2) for k_2, we find that

$$2nk_3 + k_2 = k_4 \equiv 0 \ (\text{mod } h_3).$$

By (4.5), we obtain

$$2n \equiv -2nk_3^2 \equiv k_3 k_2 \equiv (ab^2 + a + b)(b^2 + 1) \pmod{h_3}.$$

Multiplying both sides by a and then using the expression for h_3 in (4.4) we obtain

$$2na \equiv (a^2b^2 + a^2 + ab)(b^2 + 1) \equiv -(ab + 1)(b^2 + 1) \pmod{h_3}.$$

Multiplying both sides by $(1 + ab)$ then using the expression for h_3 in (4.4) we get

$$2na(ab + 1) \equiv -(ab + 1)^2(b^2 + 1) \equiv a^2(b^2 + 1) \pmod{h_3},$$

and consequently,

$$2n(ab + 1) \equiv a(b^2 + 1) \pmod{h_3} \qquad (\text{as } \gcd(a, h_3) = 1 \text{ from } (4.4)).$$

Therefore, there is an integer l such that

$$2n(ab + 1) = lh_3 + a(b^2 + 1) = l((ab + 1)^2 + a^2) + a(b^2 + 1). \tag{4.6}$$

By Proposition 3.2, $2n > ab$ and hence l must be a positive integer. We now claim that l must be even from parity considerations in the last equality. When a is even or a, b both are odd, it is clear from (4.1) that h_3 is odd and then it follows from (4.6) that l must be even. If a is odd and b is even, h_3 is even but $a(b^2 + 1)$ is odd which is ruled out by parity consideration in the first equality of (4.6). Putting $l \geq 2$ in (4.6), we obtain

$$2n \geq 2(ab + 1) + \frac{2a^2 + a(b^2 + 1)}{ab + 1},$$

and by squaring both sides, we get

$$4n^2 \geq 4(ab + 1)^2 + 4(2a^2 + a(b^2 + 1)) + \left(\frac{2a^2 + a(b^2 + 1)}{ab + 1}\right)^2.$$

It follows that

$$4n^2 > 4(ab + 1)^2 + 4a^2 = 4.h_3$$

and as a result

$$m > n^2 > h_3. \qquad \square$$

The following result also comes out in the proof.

Corollary 4.3. *There does not exist square free positive integer m such that $\sqrt{m} = \langle n, \overline{a, b, b, a, 2n}\rangle$ and a is odd and b is even.*

5. Certain interesting implications

We conclude with two interesting observations that follow from our results. The first observation is that we can easily construct an infinite family real quadratic fields of class number 1. The second observation is that there are infinitely many real quadratic fields with any given power of 2 as relative class number provided there are infinitely many Mersenne primes. We explain these two observations below.

If m is a square-free integer of the form $n^2 + n$ then it follows that $\sqrt{m} = \langle n, \overline{2, 2n}\rangle$. Here, $h_{r-1} = 2 \Rightarrow \theta(2) = 1$ and $\psi(2) = 2$ as 2 always divides $n^2 + n$. So, $H_d(2)$ is 2. As $n^2 + n$ can not be power of 2, it has an odd prime divisor p

dividing m. As p does not divide $2 = h_{r-1}$, we must have $H_d(p) = 1$. Note that there exist infinitely many square-free consecutive integers n and $(n+1)$: if N is a sufficiently large natural number and we sieve out all the integers between $2kN + 1$ to $(2k + 2)N$ which are divisible by squares of prime, we will be left more than N square-free integers as

$$\sum \frac{1}{p^2} < 0.4522474200 \cdots < \frac{1}{2}$$

(see [5]) and hence at least two of those square-free integers will be consecutive. Hence there exist infinitely many square free positive integers of the from $m = n^2 + n$ giving relative class number 1 for any odd prime divisor of m as conductor.

Now suppose $m > 6$ is an integer which is twice a Mersenne prime. In other words, let $m = 2(2^p - 1)$ where $p = 2k + 1$ is a prime. Then, m is square free and we have

$$m = (2^k)^2 - 2 = (n+1)^2 - 2, \quad \sqrt{m} = \langle n, \overline{1, n-1, 1, 2n} \rangle, \quad h_{r-1} = n + 1 = 2^k.$$

As $\xi_m \in 2^i \mathcal{O}_K$ for all $0 \leq i \leq k$, $\theta(2^i) = 1$. Now,

$$\psi(2^i) = 2^i \Big(1 - \Big(\frac{d}{p}\Big)\frac{1}{p}\Big)\Big(1 - \Big(\frac{d}{2}\Big)\frac{1}{2}\Big) = 2^i.$$

Therefore $H_d(2^i) = 2^i$. If we assume that the number of Mersenne primes is infinite, there will be infinitely many primes of the form $2^{2k-1} - 1$, and hence we can demonstrate infinitely many real quadratic fields with any power of 2 as relative class number.

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