Bull. Korean Math. Soc. **52** (2015), No. 5, pp. 1569–1578
http://dx.doi.org/10.4134/BKMS.2015.52.5.1569

HELICOIDAL SURFACES OF THE THIRD FUNDAMENTAL FORM IN MINKOWSKI 3-SPACE

Miekyung Choi and Dae Won Yoon

Abstract. We study helicoidal surfaces with the non-degenerate third fundamental form in Minkowski 3-space. In particular, we mainly focus on the study of helicoidal surfaces with light-like axis in Minkowski 3-space. As a result, we classify helicoidal surfaces satisfying an equation in terms of the position vector field and the Laplace operator with respect to the third fundamental form on the surface.

1. Introduction

We study a (pseudo-)Riemannian manifold as a submanifold of a (pseudo-)Euclidean space via an isometric immersion by Nash's Theorem. Let $x : M \to \mathbb{E}^3$ be an isometric immersion of a connected surface M in a Euclidean 3-space $\mathbb{E}^3$. Denote by Δ the Laplacian with respect to the induced metric on M. Takahashi ([11]) proved that minimal surfaces and spheres are the only surfaces in $\mathbb{E}^3$ satisfying the condition $\Delta x = \lambda x$, $\lambda \in \mathbb{R}$. As a generalization of Takahashi's Theorem, Garay ([4]) classified the hypersurfaces whose coordinate functions in $\mathbb{E}^m$ are eigenfunctions of their Laplacian, that is, the hypersurfaces in $\mathbb{E}^m$ satisfies the condition

$$\Delta x = Ax, \quad A \in \mathrm{Mat}(m, \mathbb{R}),$$

where $\mathrm{Mat}(m, \mathbb{R})$ is the set of $m \times m$-real matrices.

On the other hand, if we suppose a hypersurface M has no parabolic points, the second fundamental form can be regarded as a new (pseudo-)Riemannian metric on M. Kaimakamis and Papantoniou ([5]) studied surfaces of revolution with non-lightlike axis in Minkowski 3-space satisfying the condition with $B = 0$

$$\Delta^{II} x = Ax + B, \quad A \in \mathrm{Mat}(3, \mathbb{R}), \quad B \in \mathbb{R}^3, \tag{1.1}$$

Received August 29, 2014; Revised December 10, 2014.
2010 *Mathematics Subject Classification.* Primary 53A35; Secondary 53B30.
Key words and phrases. helicoidal surfaces, light-like axis, third fundamental form, Minkowski space.
The second author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2012R1A1A2003994).

where Δ^{II} is the Laplacian with respect to the non-degenerate second fundamental form II. Based on the study of Kaimakamis and Papantoniou, the present authors and Kim ([3]) completely classified surfaces of revolution in Minkowski 3-space satisfying the condition (1.1) with non-zero vector B.

We also deal with the non-degenerate third fundamental form as a new (pseudo-)Riemannian metric on a (pseudo-)Riemannian submanifold in (pseudo-)Euclidean space. In fact, the third fundamental form III is expressed in terms of the first fundamental form I and the second fundamental form II, that is, $III = 2HII + KI$, where H is the mean curvature and K the Gaussian curvature. In that sense, Kaimakamis and Papantoniou ([6]) researched surfaces of revolution in Minkowski 3-space satisfying the condition

$$\Delta^{III}x = Ax, \quad A \in \mathrm{Mat}(3, \mathbb{R}), \tag{1.2}$$

where Δ^{III} is the Laplacian with respect to the non-degenerate third fundamental form III. In [8], Lee, Kim and Yoon studied ruled surfaces of the non-degenerate third fundamental form in Minkowski 3-space. Recently, Senoussi and Bekkar ([10]) investigated helicoidal surfaces with non-lightlike axis in Minkowski 3-space satisfying the condition (1.2).

As is well-known, a helicoidal surface is a kind of generalization of surfaces of revolution and ruled surfaces in (pseudo-)Euclidean space. Based on these backgrounds, it is worth studying helicoidal surfaces with the non-degenerate third fundamental form.

In this paper, we study helicoidal surfaces with light-like axis in Minkowski 3-space satisfying the condition (1.2). As a result, we are to complete Senoussi and Bekkar's classification of helicoidal surfaces in Minkowski 3-space satisfying the condition (1.2).

2. Preliminaries

Let $\mathbb{E}^3_1$ be a Minkowski 3-space with the Lorentz metric

$$\langle \cdot, \cdot \rangle = -dx_0{}^2 + dx_1{}^2 + dx_2{}^2,$$

where (x_0, x_1, x_2) is a system of the canonical coordinates in $\mathbb{R}^3$. Let M be a 2-dimensional connected surface in $\mathbb{E}^3_1$ and $x : M \to \mathbb{E}^3_1$ a smooth isometric immersion defined by $x(u, v) = (x_0(u, v), x_1(u, v), x_2(u, v))$. For a surface M, we denote N the standard unit normal vector field on M. The first fundamental form I of a surface M is given by

$$I = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2,$$

where $g_{11} = \langle x_u, x_u \rangle$, $g_{12} = \langle x_u, x_v \rangle$, $g_{22} = \langle x_v, x_v \rangle$. And the second fundamental form II and the third fundamental form III of M are defined by, respectively,

$$II = h_{11}du^2 + 2h_{12}dudv + h_{22}dv^2,$$

$$III = t_{11}du^2 + 2t_{12}dudv + t_{22}dv^2,$$

where

$$h_{11} = \langle x_{uu}, N\rangle,\ h_{12} = \langle x_{uv}, N\rangle,\ h_{22} = \langle x_{vv}, N\rangle,$$
$$t_{11} = \langle N_u, N_u\rangle,\ t_{12} = \langle N_u, N_v\rangle,\ t_{22} = \langle N_v, N_v\rangle.$$

If the third fundamental form III is non-degenerate, then it can be regarded as a (pseudo-)Riemannian metric and the Laplacian Δ^{III} with respect to III can be defined formally on the (pseudo-)Riemannian manifold M by (cf. [9])

$$\Delta^{III} = -\frac{1}{\sqrt{|\,\mathcal{T}\,|}} \sum_{i,j} \frac{\partial}{\partial x^i}\Big(\sqrt{|\,\mathcal{T}\,|}\; t^{ij} \frac{\partial}{\partial x^j}\Big),$$

where t_{ij} are the components of III, $\mathcal{T} = \det(t_{ij})$ and $(t^{ij}) = (t_{ij})^{-1}$.

On the other hand, a *helicoidal surface* M with axis of revolution l and pitch h in $\mathbb{E}^3_1$ is a non-degenerate surface which is invariant under the action of the helicoidal motion. If a pitch h of M is zero, then it coincides with surfaces of revolution. In particular, it is called the *genuine helicoidal surface* provided h is non-zero ([1]).

Let $\gamma\ : I = (a, b) \subset \mathbb{R} \to \Pi$ be a plane curve in $\mathbb{E}^3_1$ and l a straight line in Π which does not intersect curve γ. In general, we have three types of helicoidal surfaces depending on the axis of revolution being space-like, time-like and light-like.

Case 1. The axis of revolution l is space-like.

In this case, l is transformed to the x_1-axis or x_2-axis by the Lorentz transformation. So we may consider x_2-axis as an axis of revolution. Then without loss of generality we may assume that the profile curve γ lies in the x_1x_2-plane or x_0x_2-plane. Hence, the curve γ can be represented by

$$\gamma(u) = (0, f(u), g(u)) \quad \text{or} \quad \gamma(u) = (f(u), 0, g(u))$$

for smooth functions f and g on an open interval $I = (a, b)$. Therefore, the surface M may be parameterized by

$$x(u, v) = (f(u)\sinh v, f(u)\cosh v, g(u) + hv), \quad f(u) > 0, \quad h \in \mathbb{R}$$

or

$$x(u, v) = (f(u)\cosh v, f(u)\sinh v, g(u) + hv), \quad f(u) > 0, \quad h \in \mathbb{R}.$$

Case 2. The axis of revolution l is time-like.

If the axis of revolution l is time-like, then l is transformed to the x_0-axis by the Lorentz transformation. We may assume that the profile curve γ lies in the x_0x_1-plane. So the curve γ is given by $\gamma(u) = (g(u), f(u), 0)$ for a positive function $f = f(u)$ on an open interval $I = (a, b)$. Hence, the surface M can be expressed by

$$x(u, v) = (g(u) + hv, f(u)\cos v, f(u)\sin v), \quad f(u) > 0, \quad h \in \mathbb{R}.$$

Case 3. The axis of revolution l is light-like.

In this case, we may assume that the axis of revolution l is the line spanned by the vector $(1,1,0)$. And we assume that the profile curve γ lies in the x_0x_1-plane of the form $\gamma(u)=(f(u),g(u),0)$, where $f=f(u)$ is a positive function and $g=g(u)$ is a function satisfying $p(u)=f(u)-g(u)\neq 0$ for all $u\in I$. Under the screw motion, its parametrization has the form

$$x(u,v)=(f(u)+\frac{v^2}{2}p(u)+hv,g(u)+\frac{v^2}{2}p(u)+hv,p(u)v),\quad h\in\mathbb{R}, \tag{2.1}$$

where $p(u)=f(u)-g(u)\neq 0$.

3. Helicoidal surfaces with light-like axis in $\mathbb{E}^3_1$

In this section, we study helicoidal surfaces with light-like axis in Minkowski 3-space $\mathbb{E}^3_1$ satisfying equation (1.2).

Suppose that M is a helicoidal surface with light-like axis in $\mathbb{E}^3_1$ parameterized by (2.1). Since the induced metric on M is non-degenerate, $f'(u)-g'(u)\neq 0$. Therefore we may change the variable in such a way that $p(u)=f(u)-g(u)=-2u$.

Let $k(u)=f(u)+u$. Then the functions f and g in the profile curve γ look like

$$f(u)=k(u)-u\quad\text{and}\quad g(u)=k(u)+u.$$

Thus, the parametrization of M can be written as ([2])

$$x(u,v)=(k(u)-u-uv^2+hv,k(u)+u-uv^2+hv,-2uv),\quad h\in\mathbb{R}.$$

Then the components of the first fundamental form I are given by $g_{11}=4k'$, $g_{12}=2h$, $g_{22}=4u^2$. Since M is non-degenerate, $4u^2k'-h^2\neq 0$. And the standard unit normal vector field N is defined by

$$N=\frac{1}{\sqrt{|4u^2k'-h^2|}}(uk'+u+uv^2-vh,uk'-u+uv^2-vh,2uv-h).$$

We also have the second fundamental form II with components given by

$$h_{11}=\frac{-2uk''}{\sqrt{|4u^2k'-h^2|}},\quad h_{12}=\frac{2h}{\sqrt{|4u^2k'-h^2|}},\quad h_{22}=\frac{4u^2}{\sqrt{|4u^2k'-h^2|}}.$$

Then we get the mean curvature H and the Gaussian curvature K as follows:

$$H=\frac{-u^3k''+2u^2k'-h^2}{|4u^2k'-h^2|^{3/2}}\quad\text{and}\quad K=\frac{-2u^3k''-h^2}{4u^2k'-h^2}.$$

Moreover, the components of the third fundamental form III are given by

$$t_{11}=\frac{4(u^4k''^2+uk''h^2+k'h^2)}{(4u^2k'-h^2)^2},\quad t_{12}=\frac{2h}{4u^2k'-h^2},\quad t_{22}=\frac{4u^2}{4u^2k'-h^2}.$$

Suppose that the third fundamental form on M is non-degenerate. Then we get $2u^3k''+h^2\neq 0$. By a straightforward computation, we have the Laplacian

Δ^{III} with respect to III of the position vector field x as following

$$\Delta^{III}x = -\frac{4u^2k' - h^2}{(2u^3k'' + h^2)^3}(X\ ,Y\ ,Z), \tag{3.1}$$

where we have put

$$\begin{aligned} X &= X(u,v) \\ &= 2h^4u - 12h^2k'u^3 - 4h^2k''u^4 - 2h^2k'''u^5 - 4k''^2u^7 + 8k'k'''u^7 \\ &\quad - 2h^4k'u + h^4k''u^2 + 12h^2k'^2u^3 + h^4k'''u^3 - 4h^2k'k''u^4 \\ &\quad - 6h^2k''^2u^5 - 2h^2k'k'''u^5 + 12k'k''^2u^7 - 8k'^2k'''u^7 - 4k''^3u^8 \\ &\quad + (-2h^5 + 4h^3k'u^2 - 12h^3k''u^3 - 2h^3k'''u^4 + 32hk'k''u^5 \\ &\quad - 4hk''^2u^6 + 8hk'k'''u^6)v + (2h^4u - 12h^2k'u^3 - 4h^2k''u^4 \\ &\quad - 2h^2k'''u^5 - 4k''^2u^7 + 8k'k'''u^7)v^2 \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} Y &= Y(u,v) \\ &= -2h^4u + 12h^2k'u^3 + 4h^2k''u^4 + 2h^2k'''u^5 + 4k''^2u^7 - 8k'k'''u^7 \\ &\quad - 2h^4k'u + h^4k''u^2 + 12h^2k'^2u^3 + h^4k'''u^3 - 4h^2k'k''u^4 \\ &\quad - 6h^2k''^2u^5 - 2h^2k'k'''u^5 + 12k'k''^2u^7 - 8k'^2k'''u^7 - 4k''^3u^8 \\ &\quad + (-2h^5 + 4h^3k'u^2 - 12h^3k''u^3 - 2h^3k'''u^4 + 32hk'k''u^5 \\ &\quad - 4hk''^2u^6 + 8hk'k'''u^6)v + (2h^4u - 12h^2k'u^3 - 4h^2k''u^4 \\ &\quad - 2h^2k'''u^5 - 4k''^2u^7 + 8k'k'''u^7)v^2. \end{aligned} \tag{3.3}$$

Equation (3.2) can be written as

$$X(u,v) = X_1(u) + X_2(u)v + X_3(u)v^2$$

by factorizing out by v. Here we put

$$\begin{aligned} X_1(u) &= 2h^4u - 12h^2k'u^3 - 4h^2k''u^4 - 2h^2k'''u^5 - 4k''^2u^7 + 8k'k'''u^7 \\ &\quad - 2h^4k'u + h^4k''u^2 + 12h^2k'^2u^3 + h^4k'''u^3 - 4h^2k'k''u^4 - 6h^2k''^2u^5 \\ &\quad - 2h^2k'k'''u^5 + 12k'k''^2u^7 - 8k'^2k'''u^7 - 4k''^3u^8, \\ X_2(u) &= -2h^5 + 4h^3k'u^2 - 12h^3k''u^3 - 2h^3k'''u^4 + 32hk'k''u^5 - 4hk''^2u^6 \\ &\quad + 8hk'k'''u^6, \\ X_3(u) &= 2h^4u - 12h^2k'u^3 - 4h^2k''u^4 - 2h^2k'''u^5 - 4k''^2u^7 + 8k'k'''u^7. \end{aligned}$$

In fact, $X - Y = 2X_3$ and we also get

$$uX_2 - hX_3 = 4hu(2u^3k'' + h^2)(4u^2k' - h^2). \tag{3.4}$$

Moreover, the third component Z in (3.1) is expressed by $Z(u,v) = X_2(u) + 2X_3(u)v$.

Now we suppose that M satisfies the condition (1.2). Then we obtain

$$-\frac{4u^2k' - h^2}{(2u^3k'' + h^2)^3}X = a_{11}(k-u-uv^2+hv)+a_{12}(k+u-uv^2+hv)-2a_{13}uv, \tag{3.5}$$

$$-\frac{4u^2k' - h^2}{(2u^3k'' + h^2)^3}Y = a_{21}(k-u-uv^2+hv)+a_{22}(k+u-uv^2+hv)-2a_{23}uv, \tag{3.6}$$

$$-\frac{4u^2k' - h^2}{(2u^3k'' + h^2)^3}Z = a_{31}(k-u-uv^2+hv)+a_{32}(k+u-uv^2+hv)-2a_{33}uv, \tag{3.7}$$

where a_{ij} are the components of the matrix A and $i, j = 1, 2, 3$.

Combining (3.5) and (3.6), we get the following equations:

$$\begin{aligned} -\frac{2(4u^2k' - h^2)}{(2u^3k'' + h^2)^3}X_3 &= (a_{11} + a_{12} - a_{21} - a_{22})(k - uv^2 + hv) \\ &\quad - (a_{11} - a_{12} - a_{21} + a_{22})u - 2(a_{13} - a_{23})uv, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &-\frac{2(4u^2k' - h^2)}{(2u^3k'' + h^2)^3}(X_1 - X_3 + X_2v + X_3v^2) \\ &= (a_{11} + a_{12} + a_{21} + a_{22})(k - uv^2 + hv) \\ &\quad - (a_{11} - a_{12} + a_{21} - a_{22})u - 2(a_{13} + a_{23})uv. \end{aligned} \tag{3.9}$$

If we consider that (3.8) is a polynomial in the parameter v with functions of u as coefficients, we have

$$a_{11} + a_{12} - a_{21} - a_{22} = 0 \quad \text{and} \quad a_{13} - a_{23} = 0. \tag{3.10}$$

Therefore, equation (3.8) can be reduced as

$$\frac{2(4u^2k' - h^2)}{(2u^3k'' + h^2)^3}X_3 = (a_{11} - a_{12} - a_{21} + a_{22})u. \tag{3.11}$$

Similarly, if we consider that (3.9) is a polynomial in the parameter v with functions of u as coefficients, we obtain as follows:

$$\frac{2(4u^2k' - h^2)}{(2u^3k'' + h^2)^3}X_3 = (a_{11} + a_{12} + a_{21} + a_{22})u, \tag{3.12}$$

$$\frac{2(4u^2k' - h^2)}{(2u^3k'' + h^2)^3}X_2 = -h(a_{11} + a_{12} + a_{21} + a_{22}) + 2(a_{13} + a_{23})u, \tag{3.13}$$

$$\frac{2(4u^2k' - h^2)}{(2u^3k'' + h^2)^3}(X_1 - X_3) = -(a_{11} + a_{12} + a_{21} + a_{22})k + (a_{11} - a_{12} + a_{21} - a_{22})u. \tag{3.14}$$

If we compare the right hand sides of equations (3.11) and (3.12), we easily get $a_{12}+a_{21}=0$. Hence equations (3.12), (3.13) and (3.14) with the help of (3.10) are reduced as follows:

$$\frac{2(4u^2k'-h^2)}{(2u^3k''+h^2)^3}X_3=(a_{11}+a_{22})u, \tag{3.15}$$

$$\frac{2(4u^2k'-h^2)}{(2u^3k''+h^2)^3}X_2=-h(a_{11}+a_{22})+4a_{13}u, \tag{3.16}$$

$$\frac{2(4u^2k'-h^2)}{(2u^3k''+h^2)^3}(X_1-X_3)=-(a_{11}+a_{22})k-4a_{12}u. \tag{3.17}$$

Combining (3.15) and (3.16) and using (3.4), we obtain

$$\frac{4h(4u^2k'-h^2)^2}{(2u^3k''+h^2)^2}=-h(a_{11}+a_{22})+2a_{13}u. \tag{3.18}$$

Similarly as above, we have $a_{31}+a_{32}=0$ from (3.7). Since $Z=X_2+2X_3v$, we also obtain

$$\frac{4u^2k'-h^2}{(2u^3k''+h^2)^3}X_2=2a_{31}u \quad \text{and} \quad \frac{4u^2k'-h^2}{(2u^3k''+h^2)^3}X_3=a_{33}u. \tag{3.19}$$

If we examine (3.15), (3.16) and (3.19), we have

$$a_{11}+a_{22}=2a_{33}, \quad a_{13}=a_{31}, \quad h(a_{11}+a_{22})=0.$$

Thus we obtain $ha_{33}=0$.

Now we suppose that $h\neq 0$. Then $a_{33}=0$ and so $a_{11}+a_{22}=0$. From (3.19), $X_3=0$. Moreover, if we investigate (3.18), a_{13} cannot be zero. Since $a_{11}-a_{22}=-2a_{12}$, $a_{12}=-a_{11}$. Hence A is given by

$$A=\begin{pmatrix} a_{11} & -a_{11} & a_{13} \\ a_{11} & -a_{11} & a_{13} \\ a_{13} & -a_{13} & 0 \end{pmatrix}, \quad a_{13}\neq 0.$$

Furthermore, from $X_3=0$, we have the following differential equation

$$h^4-6h^2k'u^2-2h^2k''u^3-h^2k'''u^4-2k''^2u^6+4k'k'''u^6=0.$$

If we put $y=4k'-\frac{h^2}{u^2}$, the above equation can be written as

$$yy''-\frac{1}{2}y'^2=0. \tag{3.20}$$

If $y=c$ for some constant c, then we get $2u^3k''+h^2=0$ by differentiating the function y with respect to the parameter u. It contradicts the non-degeneracy of the third fundamental form on M. Thus, y is a non-constant function and the general solution of (3.20) is given by

$$y=\frac{1}{4}(c_1u+c_2)^2 \tag{3.21}$$

for some constants $c_1 \neq 0$ and c_2. On the other hand, if we represent (3.18) in terms of y, it is given by

$$8hy^2 - a_{13}u^3{y'}^2 = 0. \tag{3.22}$$

Putting (3.21) into (3.22), we have $c_1 = 0$ and $h = 0$ because of $a_{13} \neq 0$, a contradiction. Therefore, we conclude that $h = 0$.

Thus we have the following theorem.

Theorem 3.1. *Let M be a helicoidal surface with light-like axis in Minkowski 3-space $\mathbb{E}^3_1$. Then there exists no genuine helicoidal surface satisfying the condition* (1.2).

In general, if a pitch h is zero, then M is a surface of revolution. From now on, we investigate surfaces of revolution with light-like axis in $\mathbb{E}^3_1$ satisfying the condition (1.2).

In this case, it easily seen that X_2 vanishes. Hence $a_{31} = 0$ in (3.19) and so $a_{13} = a_{23} = a_{31} = a_{32} = 0$. Therefore, the matrix A has the form

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ -a_{12} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}, \quad a_{11} + a_{22} = 2a_{33}, \quad a_{11} - a_{22} = -2a_{12}.$$

On the other hand, from (3.15) and (3.17), we have the system of differential equations

$$\begin{cases} \frac{-2k'}{u{k''}^3}({k''}^2 - 2k'k''') = \lambda, \\ \\ \frac{-2k'}{{k''}^3}(3k'{k''}^2 - 2{k'}^2k''' - {k''}^3u) = \lambda k + 2\mu u, \end{cases} \tag{3.23}$$

where $\lambda = a_{33}$ and $\mu = a_{12}$.

We now consider two cases separately according to λ.

Case 1. $\lambda = 0$. In this case, we have ${k''}^2 - 2k'k''' = 0$. Then the solution is given by

$$k(u) = \frac{1}{12a}u^3 + c, \quad a > 0$$

for some constant c. Putting it in the second equation of (3.23), we get $\mu = 0$. Moreover, we easily lead to $X = 0$ in (3.2). Then it implies $a_{11} = 0$ in (3.5). Thus, all the components of A are zero. Consequently, the surface M is parameterized by

$$x(u, v) = (\frac{1}{12a}u^3 - u - uv^2 + c, \frac{1}{12a}u^3 + u - uv^2 + c, -2uv), \quad a > 0$$

for some constant c. It is the *Enneper surface of the second kind* according to [7].

Case 2. $\lambda \neq 0$. From the system of differential equations (3.23), we have

$$4{k'}^2 - (\lambda + 2)uk'k'' + \lambda kk'' + 2\mu uk'' = 0. \tag{3.24}$$

It could be hard to get a general solution but not analytical solution.

Therefore, we obtain the following theorem:

Theorem 3.2. *Let M be a surface of revolution with light-like axis in Minkowski 3-space $\mathbb{E}^3_1$. Suppose that M satisfies the condition* (1.2). *Then M is a part of Enneper surface of the second kind or the surface determined by the function $k = k(u)$ satisfying equation* (3.24).

Together with Theorem 3.1 and Theorem 3.2, we have:

Theorem 3.3. *Let M be a helicoidal surface with light-like axis in Minkowski 3-space $\mathbb{E}^3_1$. Suppose that M satisfies the condition $\Delta^{III}x = Ax$ for some $A \in \mathrm{Mat}(3, \mathbb{R})$. Then M is a part of Enneper surface of the second kind or the surface determined by the function $k = k(u)$ satisfying equation* (3.24).

Remark. If we put $\lambda = -2$ and $\mu = 0$ in (3.24) and it becomes

$$4k'^2 - 2kk'' = 0,$$

then the general solution is given by

$$k(u) = \frac{1}{au}$$

for some non-zero constant a. Hence, the surface M is parameterized by

$$x(u, v) = (\frac{1}{au} - u - uv^2, \frac{1}{au} + u - uv^2, -2uv), \quad a \neq 0.$$

In this case, we have $a_{11} = a_{22} = a_{33}$ because of $\mu = a_{12} = 0$. Therefore, A is a diagonal matrix with diagonal component $\lambda = -2$.

As a consequence, putting together with the results described above and theorems in [10], we have the following result:

Theorem 3.4. *Let M be a helicoidal surface in Minkowski 3-space $\mathbb{E}^3_1$. Suppose that M satisfies the condition $\Delta^{III}x = Ax$ for some $A \in \mathrm{Mat}(3, \mathbb{R})$. Then M is a part of right helicoid of type I, right helicoid of type II, Enneper surface of the second kind or the surface determined by the function $k = k(u)$ satisfying the equation* (3.24).

References

[1] M. Choi, Y. H. Kim, H. Liu, and D. W. Yoon, *Helicoidal surfaces and their Gauss map in Minkowski 3-space*, Bull. Korean Math. Soc. **47** (2010), no. 4, 859–881.

[2] M. Choi, Y. H. Kim, and G. Park, *Helicoidal surfaces and their Gauss map in Minkowski 3-space II*, Bull. Korean Math. Soc. **46** (2009), no. 3, 567–576.

[3] M. Choi, Y. H. Kim, and D. W. Yoon, *Some classification of surfaces of revolution in Minkowski 3-space*, J. Geom. **104** (2013), no. 1, 85–106.

[4] O. J. Garay, *An extension of Takahashi's Theorem*, Geom. Dedicata **34** (1990), no. 2, 105–112.

[5] G. Kaimakamis and B. Papantoniou, *Surfaces of revolution in the 3-dimensional Lorentz-Minkowski space satisfying $\Delta^{II}\vec{r} = A\vec{r}$*, J. Geom. **81** (2004), no. 1-2, 81–92.

[6] ———, *Surfaces of revolution in the 3-dimensional Lorentz-Minkowski space* $\mathbb{E}_1^3$ *satisfying* $\Delta^{III}\overrightarrow{r} = A\overrightarrow{r}$, Bull. Greek Math. Soc. **50** (2005), 75–90.

[7] O. Kobayashi, *Maximal surfaces in the* 3*-dimensional Minkowski space* $\mathbb{L}^3$, Tokyo J. Math. **6** (1983), no. 2, 297–309.

[8] C. W. Lee, Y. H. Kim, and D. W. Yoon, *Ruled surfaces of non-degenerate third fundamental forms in Minkowski* 3*-space*, Appl. Math. Comput. **216** (2010), no. 11, 3200–3208.

[9] B. O'Neill, *Semi-Riemannian Geometry and its applications to Relativity*, Academic Press, New York, 1983.

[10] B. Senoussi and M. Bekkar, *Helicoidal surfaces in the 3-dimensional Lorentz-Minkowski space* $\mathbb{E}_1^3$ *satisfying* $\Delta^{III}r = Ar$, Tsukuba J. Math. **37** (2013), no 2, 339–353.

[11] T. Takahashi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan **18** (1966), 380–385.

Miekyung Choi
Department of Mathematics education
Gyeongsang National University
Jinju 660-701, Korea
E-mail address: mkchoi@gnu.ac.kr

Dae Won Yoon
Department of Mathematics education and RINS
Gyeongsang National University
Jinju 660-701, Korea
E-mail address: dwyoon@gnu.ac.kr