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BARRELLEDNESS OF SOME SPACES OF VECTOR MEASURES AND BOUNDED LINEAR OPERATORS

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Abstract. In this paper we investigate the barrellednes of some spaces of X-valued measures, X being a barrelled normed space, and provide examples of non barrelled spaces of bounded linear operators from a Banach space X into a barrelled normed space Y, equipped with the uniform convergence topology.

1. Preliminaries

The barrelledness of certain spaces of vector-valued functions has been widely studied, see [7, Chapters 8-10] and references therein. If K is a locally compact Hausdorff space, (Ω, Σ) a measurable space, $\mu \in \mathrm{ca}^+(\Sigma)$ and X a normed space over the field $\mathbb{K}$ of the real or complex numbers, the following are among the most beautiful results on this topic.

(1) The space $B(\Sigma, X)$ over $\mathbb{K}$ of all those functions $f : \Omega \to X$ that are the uniform limit of a sequence of Σ-simple X-valued functions, equipped with the supremum norm, is barrelled if and only if X is barrelled, [12].
(2) The space $C(K, X)$ over $\mathbb{K}$ of all continuous functions $f : K \to X$ endowed with the compact-open topology is barrelled if and only if $C(K)$ and X are barrelled, [13].
(3) If μ is atomless the space $L_p(\mu, X)$ over $\mathbb{K}$, with $1 \leq p \leq \infty$, of all [classes of] strongly measurable functions $f : \Omega \to X$ that are Bochner integrable if $1 \leq p < \infty$, or essentially bounded if $p = \infty$, equipped with the integral norm $\|f\|_p$ or with the essential supremum norm $\|f\|_\infty$, respectively, is barrelled ([2] and [3]), regardless X is barrelled or not.

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(4) The space $\ell_\infty(\Sigma, X)$ over $\mathbb{K}$ of all bounded Σ-measurable functions $f : \Omega \to X$, equipped with the supremum norm, is barrelled if and only if X is barrelled, [5].
(5) If X is a Banach space, the space $\mathcal{P}_1(\mu, X)$ over $\mathbb{K}$ of all [classes of scalarly equivalent] weakly μ-measurable and Pettis integrable functions $f : \Omega \to X$, equipped with the so-called Pettis norm or semivariation norm, is barrelled, as well as the subspace $P_1(\mu, X)$ of all [classes of] strongly measurable functions, [3].
(6) The space $\ell_\infty(\Omega, X)$ over $\mathbb{K}$ of all bounded functions $f : \Omega \to X$, equipped with the supremum norm, is barrelled whenever X is barrelled and either $|\Omega|$ or $|X|$ is a nonmeasurable cardinal, [4].
(7) If K is (locally compact and) normal, the space $C_0(K, X)$ over $\mathbb{K}$ of all continuous functions $f : K \to X$ vanishing at infinity, i.e., such that for each $\epsilon > 0$ there exists a compact set $K_{f,\epsilon} \subseteq K$ with the property that $\|f(\omega)\| < \epsilon$ for each $\omega \in K \setminus K_{f,\epsilon}$, provided with the supremum norm, is barrelled if and only if X is barrelled, [6].

Let us point out that $B(\Sigma, X)$ coincides with the closure in $\ell_\infty(\Omega, X)$ of the subspace $\ell_0^\infty(\Sigma, X)$ of $\ell_\infty(\Omega, X)$ consisting of all X-valued Σ-simple functions. If X is separable then $\ell_\infty(\Omega, X) = \ell_\infty\left(2^\Omega, X\right)$. In the sequel we shall write $\ell_0^\infty(\Sigma)$ instead of $\ell_0^\infty(\Sigma, \mathbb{K})$ and ℓ_0^∞ instead of $\ell_0^\infty\left(2^{\mathbb{N}}\right)$. Clearly, ℓ_0^∞ coincides with the dense subspace of ℓ_∞ of those sequences (ξ_n) of finite range. Regardless of Σ, the space $\ell_0^\infty(\Sigma)$ is always barrelled (see [7, Theorem 5.2.4]). If Γ is a nonempty set, the linear space $c_0(\Gamma, X)$ over $\mathbb{K}$ of all functions $f : \Gamma \to X$ such that for each $\epsilon > 0$ the set $\{\omega \in \Gamma : \|f(\omega)\| > \epsilon\}$ is finite, equipped with the supremum norm, coincides with $C_0(\Gamma, X)$ for discrete Γ, so that $c_0(\Gamma, X)$ is barrelled if and only if X is barrelled. We shall frequently require the following result.

Theorem 1 (Freniche [8]). *The space $\ell_0^\infty(\Sigma, E)$ of Σ-simple functions with values in a Hausdorff locally convex space E, where Σ is an infinite σ-algebra of subsets of a set Ω, endowed with the uniform convergence topology is barrelled if and only if $\ell_0^\infty(\Sigma)$ and E are barrelled and E is nuclear.*

Yet there are several spaces of vector-valued measures and of bounded linear operators which have received less attention. Next we investigate the barrelledness of some of them. Along this paper X will be a normed or a Banach space, Y a normed space and (Ω, Σ) a nontrivial measurable space. If X is a normed space, we denote by $\operatorname{bvca}(\Sigma, X)$ the linear space over $\mathbb{K}$ of countably additive X-valued measures $F : \Sigma \to X$ of bounded variation equipped with the variation norm $|F| = |F|(\Omega)$, where $|F|(E) = \sup \sum_{A \in \pi} \|F(A)\|$ and the supremum runs over all finite partitions π of $E \in \Sigma$ by elements of Σ. By $\operatorname{ca}(\Sigma, X)$ we represent the space of all X-valued countably additive measures provided with the semivariation norm, and by $\operatorname{cca}(\Sigma, X)$ the subspace of $\operatorname{ca}(\Sigma, X)$ of those measures of relatively compact range. We denote by $L(X, Y)$ the linear space over $\mathbb{K}$ of all bounded linear operators from X

into Y equipped with the uniform convergence topology, and by $K(X,Y)$ the subspace of $L(X,Y)$ of all those compact linear operators. By $L_{w^*}(X^*,Y)$ we denote the subspace of $L(X^*,Y)$ of all weak*-weakly continuous operators from X^* into Y. The linear space of the weakly compact linear operators from X into Y is denoted by $W(X,Y)$. Recall that spaces of vector-valued measures and spaces of linear operators are close related, and sometimes they are representable by tensor products. For example, if X is a normed space then $\ell_0^\infty(\Sigma,X) = \ell_0^\infty(\Sigma) \otimes_\varepsilon X$ and, if X is a Banach space, $L_{w^*}\left(\mathrm{ca}(\Sigma)^*, X\right)$ is linearly isomorphic to $\mathrm{ca}(\Sigma,X)$, whereas $\mathrm{cca}(\Sigma,X) = \mathrm{ca}(\Sigma)\,\widehat{\otimes}_\varepsilon X$. Naturally, if K is compact then $C(K,X) = C(K)\,\widehat{\otimes}_\varepsilon X$. If $K = \mathbb{N}\cup\{\infty\}$ is the Alexandroff compactification of the discrete space $\mathbb{N}$ and E is a linear space over $\mathbb{K}$ of uncountable dimension provided with the strongest locally convex topology, then $C(K,E)$ is no longer barrelled, [17]. Research on barrelledness conditions is still active (see [10, 15, 16]).

2. Barrelledness of some spaces of vector measures

Let X be a normed space. If $\mu \in \mathrm{ca}^+(\Sigma)$, we shall represent by $\mathrm{bvca}(\Sigma,\mu,X)$ the linear subspace of $\mathrm{bvca}(\Sigma,X)$ consisting of all those vector measures that are μ-continuous, whereas $L_1(\mu,X)$ will stand for the linear space over $\mathbb{K}$ of all (equivalence classes of) strongly measurable X-valued Bochner integrable functions defined on Ω endowed with the norm

$$\|f\|_1 = \int_\Omega \|f(\omega)\|\, d\mu(\omega).$$

The linear map $T : L_1(\mu,X) \to \mathrm{bvca}(\Sigma,\mu,X)$ defined by

$$Tf(E) = \int_E f(\omega)\, d\mu(\omega) \tag{2.1}$$

for $E \in \Sigma$ is an isometry into since $|Tf| = \|f\|_1$. If X is a Banach space, T becomes an isometry onto the whole of $\mathrm{bvca}(\Sigma,\mu,X)$ if and only if X has the Radon-Nikodým property with respect to μ.

Theorem 2. *Assume that the completion $\widehat{X}$ of X has the Radon-Nikodým property with respect to each $\mu \in \mathrm{ca}^+(\Sigma)$. Then* $\mathrm{bvca}(\Sigma,X)$ *is barrelled if and only if X is barrelled.*

Proof. If X is barrelled and $\omega \in \Omega$, the standard map $P_\omega : \mathrm{bvca}(\Sigma,X) \to \mathrm{bvca}(\Sigma,X)$ defined by $P_\omega F = F(\Omega)\,\delta_\omega$ is a bounded linear projection from $\mathrm{bvca}(\Sigma,X)$ onto the copy $\{x\delta_\omega : x \in X\}$ of X within $\mathrm{bvca}(\Sigma,X)$. Since P_ω is a quotient map, then X is barrelled if $\mathrm{bvca}(\Sigma,X)$ does [9, 11.3.1 Proposition (a)].

For the converse let us fix $\mu \in \mathrm{ca}^+(\Sigma)$. If $S_1(\mu)$ denotes the barrelled linear subspace of $L_1(\mu)$ of all (classes of) scalarly valued μ-simple functions and $S_1(\mu,X)$ stands for the subspace of $L_1(\mu,X)$ consisting of the X-valued μ-simple functions, the mapping $\varphi : S_1(\mu)\otimes_\pi X \to S_1(\mu,X)$ obtained by linearizing the ansatz $\varphi(\chi_E \otimes x) = \chi_E x$ with $E \in \Sigma$ and $x \in X$ is an isometry. This implies that the composition $T \circ \varphi$ is a linear isometry from $S_1(\mu) \otimes_\pi X$ into a subspace of $\mathrm{bvca}(\Sigma, \mu, \widehat{X})$. But if $x_i \in X$ and $E_i \in \Sigma$ for $1 \leq i \leq n$ then

$$(T \circ \varphi)\left(\sum_{i=1}^{n} \chi_{E_i} \otimes x_i\right)(A) = \sum_{i=1}^{n} \int_A \chi_{E_i}(\omega)\, x_i \, d\mu(\omega) = \sum_{i=1}^{n} \mu(E_i \cap A)\, x_i \in X$$

for every $A \in \Sigma$, so that $\mathrm{Im}(T \circ \varphi) \subseteq X$. Hence actually $T \circ \varphi$ is a linear isometry from $S_1(\mu) \otimes_\pi X$ into a subspace of $\mathrm{bvca}(\Sigma, \mu, X)$.

Denote by S the canonical map (2.1) from $L_1(\mu, \widehat{X})$ into $\mathrm{bvca}(\Sigma, \mu, \widehat{X})$ and reserve the letter T for the restriction of S to the subspace $L_1(\mu, X)$. Since $\widehat{X}$ is supposed to have the Radon-Nikodým property with respect to μ, then S maps isometrically $L_1(\mu, \widehat{X})$ onto $\mathrm{bvca}(\Sigma, \mu, \widehat{X})$. Given that $S_1(\mu, X)$ is a dense subspace of $S_1(\mu, \widehat{X})$ and $S_1(\mu, \widehat{X})$ is dense in $L_1(\mu, \widehat{X})$, then $S(S_1(\mu, X)) = (T \circ \varphi)(S_1(\mu) \otimes_\pi X)$ is a dense subspace of $\mathrm{bvca}(\Sigma, \mu, \widehat{X})$ contained in $\mathrm{bvca}(\Sigma, \mu, X)$. So we conclude that $S_1(\mu) \otimes_\pi X$ is linearly isometric to a dense subspace of $\mathrm{bvca}(\Sigma, \mu, X)$.

On the other hand, since each $F \in \mathrm{bvca}(\Sigma, X)$ is $|F|$-continuous we have

$$\mathrm{bvca}(\Sigma, X) = \bigcup \left\{ \mathrm{bvca}(\Sigma, \mu, X) : \mu \in \mathrm{ca}^+(\Sigma) \right\}.$$

Let us show that $\mathrm{bvca}(\Sigma, X)$ is the locally convex hull of $\{\mathrm{bvca}(\Sigma, \mu, X) : \mu \in \mathrm{ca}^+(\Sigma)\}$. Let U be an absolutely convex set of $\mathrm{bvca}(\Sigma, X)$ which meets each $\mathrm{bvca}(\Sigma, \mu, X)$ in a neighborhood of the origin in $\mathrm{bvca}(\Sigma, \mu, X)$. We claim that U is a neighborhood of the origin of $\mathrm{bvca}(\Sigma, X)$. Otherwise there exists a normalized sequence $\{F_n\}_{n=1}^{\infty}$ in $\mathrm{bvca}(\Sigma, X)$ such that $F_n \notin nU$ for each $n \in \mathbb{N}$. Since $\{F_n : n \in \mathbb{N}\}$ is bounded in $\mathrm{bvca}(\Sigma, X)$, then the scalar measure $\nu := \sum_{n=1}^{\infty} 2^{-n} |F_n|$ belongs to $\mathrm{ca}^+(\Sigma)$ and, consequently, $F_n \in \mathrm{bvca}(\Sigma, \nu, X)$ for every $n \in \mathbb{N}$. But since $U \cap \mathrm{bvca}(\Sigma, \nu, X)$ is a neighborhood of the origin in $\mathrm{bvca}(\Sigma, \nu, X)$, there must exist $m \in \mathbb{N}$ such that $F_m \in mU$, a contradiction.

Since $S_1(\mu)$ and X are barrelled normed spaces, we have that $S_1(\mu) \otimes_\pi X$ is barrelled too [7, Theorem 1.6.6], and since $S_1(\mu) \otimes_\pi X$ is linearly isometric to a dense subspace of $\mathrm{bvca}(\Sigma, \mu, X)$, then this latter subspace is also barrelled [11, 27.1.(2)]. Finally, the conclusion follows from the fact that the locally convex hull of a family of barrelled spaces is barrelled [11, 27.1.(3)]. □

Remark 3. An alternative proof. The proof of the previous theorem solves Problem 6 of [7, Chapter 8]. Another approach may be the following. If $\widehat{X}$ has the Radon-Nikodým property with respect to each $\mu \in \mathrm{ca}^+(\Sigma)$, it can be shown (cf. [14, Corollary 5.23]) that $\mathrm{ca}(\Sigma) \widehat{\otimes}_\pi X = \mathrm{ca}(\Sigma) \widehat{\otimes}_\pi \widehat{X} = \mathrm{bvca}(\Sigma, \widehat{X})$ isometrically. But a careful reading of the proof of [14, Theorem 5.22] shows that (under the assumption that $\widehat{X}$ has the Radon-Nikodým property with respect to each $\mu \in \mathrm{ca}^+(\Sigma)$) even for normed spaces the projective product space $\mathrm{ca}(\Sigma) \otimes_\pi X$ is in fact linearly isometric to a dense subspace of $\mathrm{bvca}(\Sigma, X)$.

Since $\mathrm{ca}(\Sigma) \otimes_\pi X$ is barrelled if X is barrelled (cf. [7, Theorem 1.6.6]), it follows that $\mathrm{bvca}(\Sigma, X)$ is barrelled if and only if X is barrelled.

Corollary 4. *Let X be a normed space and suppose that each $\mu \in \mathrm{ca}^+(\Sigma)$ is purely atomic. Then $\mathrm{bvca}(\Sigma, X)$ is barrelled if and only if X is barrelled.*

Proof. Since each $\mu \in \mathrm{ca}^+(\Sigma)$ is purely atomic, the Banach space $\widehat{X}$ has the Radon-Nikodým property with respect to every $\mu \in \mathrm{ca}^+(\Sigma)$. So the previous theorem applies. □

Theorem 5. *Assume that the σ-algebra Σ is infinite. Then $\mathrm{ca}(\Sigma, \ell_0^\infty) = \mathrm{cca}(\Sigma, \ell_0^\infty)$ and neither $\mathrm{ca}(\Sigma, \ell_0^\infty)$ nor $\mathrm{cca}(\Sigma, \ell_0^\infty)$ are barrelled, despite the fact that ℓ_0^∞ is barrelled.*

Proof. Let $F \in \mathrm{ca}(\Sigma, \ell_0^\infty)$. Let us see first that $F(\Sigma)$ is contained in a finite-dimensional subspace of ℓ_0^∞. Indeed, assume by contradiction that $F(\Sigma)$ is infinite-dimensional. In this case there is a sequence $\{E_n : n \in \mathbb{N}\} \subseteq \Sigma$ such that the linear space $\mathrm{span}\{F(E_n) : n \in \mathbb{N}\}$ is infinite-dimensional. Setting $A_1 := E_1$ and $A_n := E_n \setminus \bigcup_{i=1}^{n-1} A_i$ for $n \geq 2$ as is frequently done, then $\{A_n : n \in \mathbb{N}\}$ is a countable family of pairwise disjoint sets of Σ such that $F(E_n) = \sum_{i=1}^n F(A_i)$. Thus we have $\mathrm{span}\{F(E_n) : n \in \mathbb{N}\} \subseteq \mathrm{span}\{F(A_n) : n \in \mathbb{N}\}$. But the series $\sum_{n=1}^\infty F(A_n)$ is subseries convergent in ℓ_0^∞ as a consequence of the fact that $\sum_{i=1}^\infty F(A_{n_i}) = F\left(\bigcup_{i=1}^\infty A_{n_i}\right) \in \ell_0^\infty$ for every increasing sequence $\{n_i\}_{i=1}^\infty$ of positive integers. Thus, according to [1, Theorem 1(b)], the linear subspace $\mathrm{span}\{F(A_n) : n \in \mathbb{N}\}$ of ℓ_0^∞ must be finite-dimensional, a contradiction.

Since $F(\Sigma)$ is contained in a finite-dimensional subspace of ℓ_0^∞ and (because of F is countably additive) the set $F(\Sigma)$ is weakly compact, it follows that $F(\Sigma)$ is relatively compact in ℓ_0^∞, which ensures that $\mathrm{ca}(\Sigma, \ell_0^\infty) = \mathrm{cca}(\Sigma, \ell_0^\infty)$.

On the other hand, the fact that the range $F(\Sigma)$ of F is finite-dimensional also tells us that there is a finite family $\{B_1, \ldots, B_p\}$ of pairwise disjoints elements of Σ, which depends on F, such that $F(\Sigma) \subseteq \mathrm{span}\{F(B_1), \ldots, F(B_p)\}$. Consequently, the vector measure F must be of the form

$$F(E) = \sum_{i=1}^p \mu_i(E) F(B_i),$$

where each $\mu_i : \Sigma \to \mathbb{K}$ is clearly a countably additive scalar measure, i.e., $\mu_i \in \mathrm{ca}(\Sigma)$. Setting $x_i := F(B_i)$ for $1 \leq i \leq p$, we see that we can represent the measure F as a tensor product of the form $F = \sum_{i=1}^p \mu_i \otimes x_i$, so that clearly $\mathrm{ca}(\Sigma, \ell_0^\infty) = \mathrm{cca}(\Sigma, \ell_0^\infty)$ can be represented as a (topological) subspace of $\mathrm{ca}(\Sigma) \otimes_\varepsilon \ell_0^\infty$. Since $\mathrm{ca}(\Sigma) \otimes_\varepsilon \ell_0^\infty$ embeds linearly into $\mathrm{cca}(\Sigma, \ell_0^\infty)$, it follows that

$$\mathrm{ca}(\Sigma, \ell_0^\infty) = \mathrm{cca}(\Sigma, \ell_0^\infty) = \mathrm{ca}(\Sigma) \otimes_\varepsilon \ell_0^\infty = \ell_0^\infty\left(2^{\mathbb{N}}, \mathrm{ca}(\Sigma)\right).$$

Now, given that $\mathrm{ca}(\Sigma)$ is an infinite-dimensional normed space, and a normed space is nuclear if and only if is finite-dimensional, Theorem 1 assures that

$\ell_0^\infty\left(2^{\mathbb{N}}, \mathrm{ca}\,(\Sigma)\right)$ is not a barrelled space. So we conclude that neither $\mathrm{ca}\,(\Sigma, \ell_0^\infty)$ nor $\mathrm{cca}\,(\Sigma, \ell_0^\infty)$ are barrelled. □

3. Barrelled and non-barrelled $L\,(X, Y)$ spaces

If X is a Banach space and Y is a non complete barrelled normed space, it turns out that there are non barrelled spaces of bounded linear operators $T : X \to Y$, as the following propositions shows.

Proposition 6. *If X is an infinite-dimensional Banach space, the space $L\,(X, \ell_0^\infty)$ equipped with the operator norm is not barrelled.*

Proof. If $T \in L\,(X, \ell_0^\infty)$ then, according to [1, Theorem 3(a)], the range of T is finite-dimensional. This forces to conclude that $L\,(X, \ell_0^\infty)$ coincides with $X^* \otimes_\varepsilon \ell_0^\infty$. Indeed, on the one hand $X^* \otimes_\varepsilon \ell_0^\infty$ can be identified with the subspace of $L\,(X, \ell_0^\infty)$ of all those linear operators T such that $\operatorname{Im} T$ is a finite-dimensional subspace of ℓ_0^∞ and, on the other hand, given $T \in L\,(X, \ell_0^\infty)$, since the range of T is finite-dimensional and the family $\left\{\chi_A : A \in 2^{\mathbb{N}}\right\}$ contains a Hamel basis of ℓ_0^∞, even a discrete one, there is a finite partition $\{A_1, \dots, A_p\}$ of $\mathbb{N}$ such that $\operatorname{Im} T = \operatorname{span}\left\{\chi_{A_i} : 1 \leq i \leq p\right\}$, so that

$$Tx = \sum_{i=1}^{p} \alpha_i\,(x)\,\chi_{A_i}$$

for every $x \in X$, where $\alpha_i : X \to \mathbb{K}$ is a bounded linear form for $1 \leq i \leq p$. In fact α_i is clearly linear and there is $K > 0$ such that

$$|\alpha_i\,(x)| \leq \sup_{1\leq j\leq p} |\alpha_j\,(x)| = \sup_{n\in\mathbb{N}} \left|\sum_{j=1}^{p} \alpha_j\,(x)\,\chi_{A_j}\,(n)\right| = \|Tx\|_\infty \leq K\,\|x\|_\infty\,.$$

So we can write $T = \sum_{i=1}^{p} x_i^* \otimes \chi_{A_i}$, with $x_i^* \in X^*$ for $1 \leq i \leq p$, verifying that

$$\|T\| = \max\left\{\|x_1^*\|, \dots, \left\|x_p^*\right\|\right\} = \left\|\sum_{i=1}^{p} x_i^* \otimes \chi_{A_i}\right\|_\varepsilon.$$

Thus we have the following linear isometries

$$L\,(X, \ell_0^\infty) = X^* \otimes_\varepsilon \ell_0^\infty = \ell_0^\infty\left(2^{\mathbb{N}}, X^*\right).$$

Since X^* is an infinite-dimensional Banach space and the family $2^{\mathbb{N}}$ of all the subsets of $\mathbb{N}$ is an infinite σ-algebra, it follows again from Theorem 1 that the space $\ell_0^\infty\left(2^{\mathbb{N}}, X^*\right)$ is not barrelled. Hence $L\,(X, \ell_0^\infty)$ is a non barrelled operator space. □

Proposition 7. *If X is an infinite-dimensional Banach space, then the operator space $L_{w^*}\,(X^*, \ell_0^\infty)$ is not barrelled.*

Proof. Since each operator $T \in L_{w^*}\,(X^*, \ell_0^\infty)$ is weak*-weakly continuous, standing for Q the closed unit ball of X^* then $T\,(Q)$ is an absolutely convex weakly compact subset of ℓ_0^∞, whence $T\,(Q)$ is a Banach disk of $(\ell_0^\infty,$

$\sigma(\ell_0^\infty, \mathrm{ba}(2^{\mathbb{N}})))$, hence of $(\ell_0^\infty, \sigma(\ell_0^\infty, \mathbb{K}^{(\mathbb{N})}))$. Since the linear span of every Banach disk of $(\ell_0^\infty, \sigma(\ell_0^\infty, \mathbb{K}^{(\mathbb{N})}))$ is finite-dimensional [1, Theorem 3(b)] (see also [7, Corollary 6.2.5]), it follows that the range of T is a finite-dimensional subspace of ℓ_0^∞. This implies that $L_{w^*}(X^*, \ell_0^\infty) = X \otimes_\varepsilon \ell_0^\infty = \ell_0^\infty(\mathbb{N}, X)$. Indeed $T \in L_{w^*}(X^*, \ell_0^\infty)$ if and only if there is a partition $\{A_1, \ldots, A_p\}$ of $\mathbb{N}$ such that $Tx^* = \sum_{i=1}^p \alpha_i(x^*)\chi_{A_i}$, each $\alpha_i : X^*(\text{weak}^*) \to \mathbb{K}$ being linear and continuous. Hence we can write $T = \sum_{i=1}^p x_i \otimes \chi_{A_i}$, with $x_i \in X$ for $1 \le i \le p$. Since X is infinite-dimensional, $\ell_0^\infty\left(2^{\mathbb{N}}, X\right)$ cannot barrelled. Therefore $L_{w^*}(X^*, \ell_0^\infty)$ is not barrelled. □

If $T \in W(X, \ell_0^\infty)$ or $T \in K(X, \ell_0^\infty)$, as before the range of T is a finite-dimensional subspace of ℓ_0^∞, which implies that $W(X, \ell_0^\infty) = K(X, \ell_0^\infty) = X^* \otimes_\varepsilon \ell_0^\infty = \ell_0^\infty\left(2^{\mathbb{N}}, X^*\right)$. So if X is infinite-dimensional, again $\ell_0^\infty\left(2^{\mathbb{N}}, X^*\right)$ is not barrelled, whence nor $W(X, \ell_0^\infty)$ neither $K(X, \ell_0^\infty)$ is barrelled. However, the following positive result holds.

Theorem 8. *Let X be a Banach space such that X^* is an $\mathcal{L}^\infty$-space with the approximation property. If Y is the locally convex hull of a sequence of Banach subspaces (which cover it), then $K(X, Y)$ is barrelled.*

Proof. Assume that Y is the locally convex hull $\mathrm{ind}_{n\in\mathbb{N}} Y_n$ of a sequence $\{Y_n : n \in \mathbb{N}\}$ of Banach subspaces of $Y = \bigcup_{n=1}^\infty Y_n$. First observe that $K(X, Y) = \bigcup_{n=1}^\infty K(X, Y_n)$.

Let $T \in K(X, Y)$. If B_X denotes the unit ball of X, then $\overline{T(B_X)}^Y$ is an absolutely convex compact set of Y, hence a Banach disk of Y. Since $\{Y_n : n \in \mathbb{N}\}$ is a countable covering of Y by closed sets, the Baire category theorem provides $\epsilon > 0$ and $n_0 \in \mathbb{N}$ such that $\epsilon\overline{T(B_X)}^Y \subseteq Y_{n_0}$. Consequently $T \in K(X, Y_{n_0})$, so that $K(X, Y) = \bigcup_{n=1}^\infty K(X, Y_n)$.

Let us show that this implies that $K(X, Y)$ is barrelled. Indeed, according to [9, 16.5 Proposition], the fact that X^* is a gDF-space ensures that $X^* \otimes_\varepsilon (\oplus_{n=1}^\infty Y_n)$ is canonically isomorphic to $\oplus_{n=1}^\infty (X^* \otimes_\varepsilon Y_n)$. So, since X^* is assumed to be an $\mathcal{L}_\infty$-space, this yields a topological isomorphism from $\mathrm{ind}_{n\in\mathbb{N}} (X^* \otimes_\varepsilon Y_n)$ onto $X^* \otimes_\varepsilon Y$ in the canonical manner [9, 16.3.6 Remark]. Thus we have that

$$\mathrm{ind}_{n\in\mathbb{N}} (X^* \otimes_\varepsilon Y_n) = X^* \otimes_\varepsilon Y. \tag{3.1}$$

But since X^* has the approximation property, then $X^* \widehat{\otimes}_\varepsilon Y_n = K(X, Y_n)$ whereas $X^* \otimes_\varepsilon Y$ is isometric to a dense linear subspace of $K(X, Y)$. Let U be a barrel of $K(X, Y)$, i.e., a closed absolutely convex and absorbing set. Clearly U meets each subspace $K(X, Y_n)$ in a neighborhood of the origin in $K(X, Y_n)$, consequently U meets each $X^* \otimes_\varepsilon Y_n$ in a neighborhood of the origin in $X^* \otimes_\varepsilon Y_n$. But due to (3.1) this implies that U meets $X^* \otimes_\varepsilon Y$ in a neighborhood of the origin of $X^* \otimes_\varepsilon Y$. Since $X^* \otimes_\varepsilon Y$ is dense in $K(X, Y)$ and U is closed in $K(X, Y)$, it follows that U is a neighborhood of the origin in

$K(X,Y)$. In other words, since $\operatorname{ind}_{n\in\mathbb{N}} K(X,Y_n)$ is an ultrabornological (hence barrelled) dense subspace of $K(X,Y)$, then $K(X,Y)$ is itself barrelled. □

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