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REDUCING SUBSPACES FOR A CLASS OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE OF THE BIDISK

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Abstract. In this paper, we completely characterize the nontrivial reducing subspaces of the Toeplitz operator $T_{z_1^N\bar{z}_2^M}$ on the Bergman space $A^2(\mathbb{D}^2)$, where N and M are positive integers.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. For $-1 < \alpha < \infty$, let $L^2(\mathbb{D}, dA_\alpha)$ be the Hilbert space of square integrable functions on $\mathbb{D}$ with the inner product

$$\langle f, g\rangle_\alpha = \int_{\mathbb{D}} f(z)\overline{g(z)}dA_\alpha(z), \quad f, g \in A^2_\alpha(\mathbb{D}),$$

where

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z),$$

and dA is the normalized area measure on $\mathbb{D}$.

The weighted Bergman space $A^2_\alpha(\mathbb{D})$ is the subspace of $L^2(\mathbb{D}, dA_\alpha)$ consisting of all the analytic functions in $\mathbb{D}$. We denote

$$\gamma_n = \|z^n\|_\alpha = \sqrt{\frac{n!\Gamma(2+\alpha)}{\Gamma(n+\alpha+2)}}$$

for $n = 0, 1, 2, \ldots$. Therefore,

$$\|f\|^2_\alpha = \sum_{n=0}^{+\infty} \gamma_n^2 |a_n|^2 < \infty,$$

where $f(z) = \sum_{n=0}^{+\infty} a_n z^n \in A^2_\alpha(\mathbb{D})$. Especially when $\alpha = 0$, we write $A^2(\mathbb{D}) = A^2_0(\mathbb{D})$. In this case, $\gamma_n = \sqrt{\frac{1}{n+1}}$.

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Denote by $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$ the bidisk. The Bergman space $A^2(\mathbb{D}^2)$ is the space of all holomorphic functions in $L^2(\mathbb{D}^2, d\mu)$ where $d\mu(z) = dA(z_1)dA(z_2)$. For multi-index $\beta = (\beta_1, \beta_2)$, denote $z^\beta = z_1^{\beta_1} z_2^{\beta_2}$ and

$$e_\beta = \frac{z^\beta}{\gamma_{\beta_1}\gamma_{\beta_2}}.$$

Then $\{e_\beta\}_{\beta \succeq 0}$ ($\beta \succeq 0$ means that $\beta_1 \geq 0$ and $\beta_2 \geq 0$) is an orthogonal basis in $A^2(\mathbb{D}^2)$.

For a bounded measurable function $f \in L^\infty(\mathbb{D}^2)$, the Toeplitz operator with symbol f is defined by $T_f h = P(fh)$ for every $h \in A^2(\mathbb{D}^2)$, where P is the Bergman orthogonal projection from $L^2(\mathbb{D}^2, d\mu)$ onto $A^2(\mathbb{D}^2)$.

Recall that for a bounded linear operator T on a Hilbert space H, a closed subspace $\mathcal{M}$ is called a reducing subspace of the operator T, if $T(\mathcal{M}) \subset \mathcal{M}$ and $T^*(\mathcal{M}) \subset \mathcal{M}$. A reducing subspace $\mathcal{M}$ is said to be minimal if there is no nonzero reducing subspace $\mathcal{N}$ such that $\mathcal{N}$ is properly contained in $\mathcal{M}$.

On the Bergman space over $\mathbb{D}$, it is proved that T_B has just two non-trivial reducing subspaces [13, 16], where B is the product of two Blaschke factors. In [12], M. Stessin and K. Zhu gave a complete description of the reducing subspaces of weighted unilateral shift operators of finite multiplicity. In particular, T_{z^n} has n distinct minimal reducing subspaces. If B is a finite Blaschke product (order $n \geq 2$), the number of nontrivial minimal reducing subspaces of T_B equals the number of connected components of the Riemann surface of $B^{-1} \circ B$ over $\mathbb{D}$ (see [2, 3, 4, 8, 9, 14] for details). Further, if B is an infinite Blaschke product or a covering map, the relative research can be founded in [5, 6, 7].

On the Bergman space of bidisk, Y. Lu and X. Zhou [10] characterized the reducing subspaces of $T_{z_1^N z_2^N}$, $T_{z_1^N}$ and $T_{z_2^N}$, respectively. The reducing subspaces of $T_{z_1^N z_2^M}$ on the weighted Bergman space $A_\alpha^2(\mathbb{D}^2)$ have been completely described in [11]. For $p = \alpha z^k + \beta w^l$, the minimal reducing subspaces of T_p on $A^2(\mathbb{D}^2)$ and the commutant algebra $\mathcal{V}^*(p) = \{T_p, T_p^*\}'$ was described in [1, 15].

In this paper, we mainly consider the reducing subspaces for the Toeplitz operator $T_{z_1^N \overline{z}_2^M}$ on the Bergman space $A^2(\mathbb{D}^2)$, where N and M are positive integers.

2. Main results

In this section, we will give a complete characterization of the reducing subspaces of $T_{z_1^N \overline{z}_2^M}$. To state our results, we need some notations and lemmas. Through out this paper, denote $T = T_{z_1^N \overline{z}_2^M}$, where N and M are positive integers. Denote by $[f]$ the reducing subspace of T generated by $f \in A^2(\mathbb{D}^2)$. Let $\mathbb{N}$ be the set of all the nonnegative integers.

By direct calculation, we know that

$$T^h(z_1^k z_2^l) = \begin{cases} \frac{\gamma_l^2}{\gamma_{l-hM}^2} z_1^{k+hN} z_2^{l-hM}, & \text{if} \quad l \geq hM \\ 0, & \text{if} \quad l < hM \end{cases};$$

$$T^{*h}(z_1^k z_2^l) = \begin{cases} \frac{\gamma_k^2}{\gamma_{k-hN}^2} z_1^{k-hN} z_2^{l+hM}, & \text{if} \quad k \geq hN \\ 0, & \text{if} \quad k < hN \end{cases}$$

for $k, l, h \in \mathbb{N}$. Set

$$E_0 = \{(k,l) \in \mathbb{N} \times \mathbb{N} : 0 \leq k < N, 0 \leq l < M\},$$
$$E_1 = \{(k,l) \in \mathbb{N} \times \mathbb{N} : k \geq 2N\},$$
$$E_2 = \{(k,l) \in \mathbb{N} \times \mathbb{N} : l \geq 2M, 0 \leq k < 2N\},$$
$$E_3 = \{(k,l) \in \mathbb{N} \times \mathbb{N} : N \leq k < 2N, M \leq l < 2M\},$$
$$E_4 = \{(k,l) \in \mathbb{N} \times \mathbb{N} : 0 \leq k < N, M \leq l < 2M\},$$
$$E_5 = \{(k,l) \in \mathbb{N} \times \mathbb{N} : 0 \leq l < M, N \leq k < 2N\}.$$

Clearly,

$$A^2(\mathbb{D}^2) = \bigoplus_{i=0}^{5} \overline{\text{span}}\{z_1^p z_2^q : (p,q) \in E_i\}.$$

Notice that $\mathcal{M}_0 = \text{span}\{z_1^p z_2^q : (p,q) \in E_0\}$ is a reducing subspace of T. To find other reducing subspaces, we first study the orthogonal decomposition of $z_1^k z_2^l$ with respect to $\mathcal{M}$.

Lemma 2.1. *Suppose $\mathcal{M} \subset \mathcal{M}_0^\perp$ is a reducing subspace of T. Let $P_\mathcal{M}$ be the orthogonal projection from $A^2(\mathbb{D}^2)$ onto $\mathcal{M}$.*

(i) *If $(k,l) \in E_1 \cup E_2 \cup E_3$, then $P_\mathcal{M} z_1^k z_2^l = \lambda z_1^k z_2^l$ with some $\lambda \in \mathbb{C}$.*

(ii) *If $(k,l) \in E_4$, then*

$$P_\mathcal{M} z_1^k z_2^l \in \text{span}\{z_1^n z_2^m : (n,m) \in E_4\}.$$

(iii) *If $(k,l) \in E_5$, then*

$$P_\mathcal{M} z_1^k z_2^l \in \text{span}\{z_1^n z_2^m : (n,m) \in E_5\}.$$

Proof. Let $k, l \in \mathbb{N}$. Since $\mathcal{M} \perp \mathcal{M}_0$, $\langle P_\mathcal{M}(z_1^k z_2^l), z_1^p z_2^q \rangle = 0$ for $(p,q) \in E_0$. In the following, we consider the inner product $\langle P_\mathcal{M}(z_1^k z_2^l), z_1^p z_2^q \rangle$ for $(p,q) \in \bigcup_{i=1}^{5} E_i$.

For every nonnegative integer h satisfying $l \geq hM$,

$$T^{h*} T^h (z_1^k z_2^l) = \frac{\gamma_l^2 \gamma_{k+hN}^2}{\gamma_{l-hM}^2 \gamma_k^2} z_1^k z_2^l. \tag{1}$$

By computation,

$$\frac{\gamma_l^2 \gamma_{k+hN}^2}{\gamma_{l-hM}^2 \gamma_k^2} \langle P_\mathcal{M}(z_1^k z_2^l), z_1^p z_2^q \rangle = \langle P_\mathcal{M} T^{h*} T^h (z_1^k z_2^l), z_1^p z_2^q \rangle$$

$$= \langle P_{\mathcal{M}}(z_1^k z_2^l), T^{h*}T^h(z_1^p z_2^q)\rangle$$

$$= \begin{cases} \frac{\gamma_q^2\gamma_{p+hN}^2}{\gamma_{q-hM}^2\gamma_p^2}\langle P_{\mathcal{M}}(z_1^k z_2^l), z_1^p z_2^q\rangle, & q \geq hM \\ 0, & q < hM. \end{cases}$$

Recall that $[s] = \max\{n \in \mathbb{Z} : n \leq s\}$ for real number s. By above equality, we get that if $\langle P_{\mathcal{M}}(z_1^k z_2^l), z_1^p z_2^q\rangle \neq 0$, then

$$\frac{\gamma_l^2\gamma_{k+hN}^2}{\gamma_{l-hM}^2\gamma_k^2} = \frac{\gamma_q^2\gamma_{p+hN}^2}{\gamma_{q-hM}^2\gamma_p^2} \tag{2}$$

for $0 \leq h \leq [\frac{l}{M}]$, $q \geq [\frac{l}{M}]M$.

Equivalently,

$$\frac{(k+1)(q+1)}{(p+1)(l+1)} = \frac{(k+1+hN)(q+1-hM)}{(p+1+hN)(l+1-hM)} \tag{3}$$

for $0 \leq h \leq [\frac{l}{M}]$, $q \geq [\frac{l}{M}]M$.

(i) If $(k,l) \in E_1 \cup E_2 \cup E_3$, we will show that the equality (2) holds if and only if $p = k$ and $q = l$.

Case one: $l \geq 2M$. Let $g_1(\lambda) = (k+1)(q+1)(p+1+\lambda N)(l+1-\lambda M)$, $g_2(\lambda) = (p+1)(l+1)(k+1+\lambda N)(q+1-\lambda M)$ and $g(\lambda) = g_1(\lambda) - g_2(\lambda)$. Since $l \geq 2M$, we have $g(0) = g(1) = g(2) = 0$. Considering $g(\lambda)$ is a quadratic polynomial, we have $g(\lambda) \equiv 0$ on $\mathbb{C}$. Therefore, g_1 and g_2 have the same zeros, i.e.,

$$\begin{cases} (k+1)(q+1)NM = (p+1)(l+1)NM \\ (k+1)(q+1)\frac{p+1}{N} = (p+1)(l+1)\frac{k+1}{N} \\ (k+1)(q+1)\frac{l+1}{M} = (p+1)(l+1)\frac{q+1}{M}. \end{cases}$$

It follows that $p = k$ and $q = l$.

Case two: $k \geq 2N$. Replacing T^*T by TT^* in Case one, we can get the desire result. The details are listed as follows.

Since

$$T^hT^{h*}(z_1^k z_2^l) = \frac{\gamma_k^2\gamma_{l+hM}^2}{\gamma_{k-hN}^2\gamma_l^2} z_1^k z_2^l, \forall 0 \leq h \leq [\frac{k}{N}],$$

we know that

$$\frac{\gamma_k^2\gamma_{l+hM}^2}{\gamma_{k-hN}^2\gamma_l^2}\langle P_{\mathcal{M}}(z_1^k z_2^l), z_1^p z_2^q\rangle = \langle P_{\mathcal{M}}T^hT^{h*}(z_1^k z_2^l), z_1^p z_2^q\rangle$$

$$= \langle P_{\mathcal{M}}(z_1^k z_2^l), T^hT^{h*}(z_1^p z_2^q)\rangle$$

$$= \begin{cases} \frac{\gamma_p^2\gamma_{q+hM}^2}{\gamma_{p-hN}^2\gamma_q^2}\langle P_{\mathcal{M}}(z_1^k z_2^l), z_1^p z_2^q\rangle & \text{if} \quad p \geq hN \\ 0 & \text{if} \quad p < hN. \end{cases}$$

Therefore, $\langle P_{\mathcal{M}}(z_1^k z_2^l), z_1^p z_2^q\rangle \neq 0$ will give that

$$\frac{\gamma_k^2\gamma_{l+hM}^2}{\gamma_{k-hN}^2\gamma_l^2} = \frac{\gamma_p^2\gamma_{q+hM}^2}{\gamma_{p-hN}^2\gamma_q^2} \tag{4}$$

for $0 \le h \le [\frac{k}{N}]$ and $p \ge [\frac{k}{N}]N$. Equivalently,

$$\frac{(k+1)(q+1)}{(p+1)(l+1)} = \frac{(k+1-hN)(q+1+hM)}{(p+1-hN)(l+1+hM)} \tag{5}$$

for $0 \le h \le [\frac{k}{N}]$ and $p \ge [\frac{k}{N}]N$. So when $k \ge 2N$, the above equality follows for $h = 0, 1, 2$. In this case we will get $p = k$ and $q = l$ by the same arguments as the case $l \ge 2M$ has done.

Case three: $(k, l) \in E_3 = \{(n, m) \in \mathbb{N}^2 : N \le n < 2N, M \le m < 2M\}$. In this case, $[\frac{k}{N}] \ge 1$ and $[\frac{l}{M}] \ge 1$. Then equalities (3) and (5) hold for $h = 0, 1$. Recall that $g(\lambda) = g_1(\lambda) - g_2(\lambda)$, where $g_1(\lambda) = (k+1)(q+1)(p+1+\lambda N)(l+1-\lambda M)$ and $g_2(\lambda) = (p+1)(l+1)(k+1+\lambda N)(q+1-\lambda M)$. We get $g(0) = g(1) = g(-1) = 0$. Therefore, we obtain that $p = k$ and $q = l$.

(ii) Suppose that $(k, l) \in E_4$. We need only prove that

$$P_{\mathcal{M}}(z_1^k z_2^l) \perp \overline{\text{span}}\{z_1^n z_2^m : (n, m) \in (\bigcup_{i=1}^{3} E_i) \bigcup E_5\}.$$

If $(n, m) \in E_1 \cup E_2 \cup E_3$, the conclusion (i) implies that $P_{\mathcal{M}} z_1^n z_2^m = \lambda z_1^n z_2^m$ for some $\lambda \in \mathbb{C}$. Thus

$$\langle P_{\mathcal{M}} z_1^k z_2^l, z_1^n z_2^m\rangle = \langle z_1^k z_2^l, P_{\mathcal{M}} z_1^n z_2^m\rangle = \overline{\lambda}\langle z_1^k z_2^l, z_1^n z_2^m\rangle = 0.$$

That is, $P_{\mathcal{M}} z_1^k z_2^l \perp \overline{\text{span}}\{z_1^p z_2^q : (p, q) \in E_1 \cup E_2 \cup E_3\}$.

If $(n, m) \in E_5 = \{(k, l) \in \mathbb{N} \times \mathbb{N} : 0 \le l < M, N \le k < 2N\}$,

$$\begin{aligned}\langle P_{\mathcal{M}} z_1^k z_2^l, z_1^n z_2^m\rangle &= \frac{\gamma_{l-M}^2 \gamma_k^2}{\gamma_l^2 \gamma_{k+N}^2}\langle P_{\mathcal{M}} T^*T z_1^k z_2^l, z_1^n z_2^m\rangle \\ &= \frac{\gamma_{l-M}^2 \gamma_k^2}{\gamma_l^2 \gamma_{k+N}^2}\langle T P_{\mathcal{M}} z_1^k z_2^l, T z_1^n z_2^m\rangle = 0,\end{aligned}$$

where the last equality comes from $\text{span}\{z_1^p z_2^q : (p, q) \in E_5\} \subseteq \text{Ker}T$. Thus $P_{\mathcal{M}} z_1^k z_2^l \perp \text{span}\{z_1^p z_2^q : (p, q) \in E_5\}$.

(iii) Replacing T^*T by TT^* in (ii), we get the desired result. □

Remark 2.1. Let $\mathcal{M} \subset \mathcal{M}_0^\perp$ is a nonzero reducing subspace of T. In (i) of Lemma 2.1, we indeed get that $\lambda = 0$ or 1, that is $z_1^k z_2^l \in \mathcal{M}$ or $z_1^k z_2^l \in \mathcal{M}^\perp$ for each $(k, l) \in E_1 \cup E_2 \cup E_3$.

If $z_1^k z_2^l \in \mathcal{M}$, then

$$[z_1^k z_2^l] = \text{span}\{z_1^{k-hN} z_2^{l+hM} :\ k - hN \ge 0,\ l + hM \ge 0,\ h \in \mathbb{Z}\} \tag{6}$$

is a minimal reducing subspace of T, containing in $\mathcal{M}$. Moreover, if $z_1^k z_2^l, z_1^p z_2^q \in \mathcal{M}$ and $(k, l), (p, q) \in E_1 \cup E_2 \cup E_3$, then it's clear that either $[z_1^k z_2^l] \perp [z_1^p z_2^q]$ or $[z_1^k z_2^l] = [z_1^p z_2^q]$. So for any non-zero function $f(z) = \sum_{(k,l)\in E_1\cup E_2\cup E_3} a_{k,l} z_1^k z_2^l$, $[f]$ is the direct sum of some minimal reducing subspace as (6).

We define two equivalences on E_4 and E_5 respectively by:

(i) for $(p,q),(k,l)\in E_4$, $(p,q)\sim_1(k,l)\Leftrightarrow \frac{(k+1)(q+1)}{(p+1)(l+1)}=\frac{(k+1+N)(q+1-M)}{(p+1+N)(l+1-M)}$;

(ii) for $(p,q),(k,l)\in E_5$, $(p,q)\sim_2(k,l)\Leftrightarrow \frac{(k+1)(q+1)}{(p+1)(l+1)}=\frac{(k+1-N)(q+1+M)}{(p+1-N)(l+1+M)}$.

It is easy to check that

(i) $(p,q)\in E_4\Leftrightarrow(p+N,q-M)\in E_5$;

(ii) for $(p,q),(k,l)\in E_4$, $(p,q)\sim_1(k,l)\Leftrightarrow(p+N,q-M)\sim_2(k+N,l-M)$;

(iii) for $(p,q),(k,l)\in E_5$, $(p,q)\sim_2(k,l)\Leftrightarrow(p-N,q+M)\sim_1(k-N,l+M)$.

For $(n,m)\in E_4$ and $(k,l)\in E_5$, let

$$P_{n,m}:A^2(\mathbb{D}^2)\to\text{span}\{z_1^pz_2^q:(p,q)\sim_1(n,m),(p,q)\in E_4\},$$
$$Q_{k,l}:A^2(\mathbb{D}^2)\to\text{span}\{z_1^kz_2^l:(p,q)\sim_2(k,l),(p,q))\in E_5\}$$

be two orthogonal projections. For $f\in A^2(\mathbb{D}^2)$ and $P_{n,m}f\neq0$, we have

$$[P_{n,m}f]=\text{span}\{P_{n,m}f,TP_{n,m}f\},\tag{7}$$

since $T^*P_{n,m}f=0$, $T^2P_{n,m}f=0$ and $T^*TP_{n,m}f=\frac{\gamma_m^2\gamma_{n+N}^2}{\gamma_{m-M}^2\gamma_n^2}P_{n,m}f$. Similarly, if $f\in\mathcal{M}$ and $Q_{k,l}f\neq0$, then

$$[Q_{k,l}f]=\text{span}\{Q_{k,l}f,T^*Q_{k,l}f\}.\tag{8}$$

Lemma 2.2. *Let $\mathcal{M}\subset\mathcal{M}_0^\perp$ be a reducing subspace of T and $(n,m)\in E_4$. Then the following statements hold.*

(a) *If $f\in\mathcal{M}$, then $[P_{n,m}f]\subset\mathcal{M}$ and $[Q_{n+N,m-M}f]\subset\mathcal{M}$.*

(b) *If $f_1,f_2\in P_{n,m}\mathcal{M}$ and $f_1\perp f_2$, then $[f_1]\perp[f_2]$.*

(c) *$P_{n,m}T^*f=T^*Q_{n+N,m-M}f$ and $TP_{n,m}f=Q_{n+N,m-M}Tf$, $\forall f\in\mathcal{M}$.*

(d) *If $f\in\mathcal{M}$, then $[P_{n,m}f]=[Q_{n+N,m-M}Tf]$ and $[Q_{n+N,m-M}f]=[P_{n,m}T^*f]$.*

(e) *$P_{n,m}\mathcal{M}\oplus Q_{n+N,m-M}\mathcal{M}\subset\mathcal{M}$ is a reducing subspace of T.*

Proof. (a) For every $f\in\mathcal{M}$, we know that $P_\mathcal{M}P_{n,m}f=P_{n,m}f$, since $P_\mathcal{M}P_{n,m}=P_{n,m}P_\mathcal{M}$, which obtained by the following simple facts:

(i) if $(k,l)\in E_4$, then $P_\mathcal{M}z_1^kz_2^l\in\text{span}\{z_1^pz_2^q:(p,q)\in E_4\}$;

(ii) if $(k,l)\notin E_4$, then $P_\mathcal{M}z_1^kz_2^l\perp\text{span}\{z_1^pz_2^q:(p,q)\in E_4\}$.

So $P_{n,m}f\in\mathcal{M}$, which implies that $[P_{n,m}f]\subset\mathcal{M}$.

Similarly, we have $P_\mathcal{M}Q_{n+N,m-M}f=Q_{n+N,m-M}f$, which shows that $Q_{n+N,m-M}f\in\mathcal{M}$. Thus $[Q_{n+N,m-M}f]\subset\mathcal{M}$.

(b) It is clear that $Tf_1,Tf_2\in\text{span}\{z_1^kz_2^l:(k,l)\in E_5\}$ and

$$\langle Tf_1,Tf_2\rangle=\langle T^*Tf_1,f_2\rangle=\frac{\gamma_{n+N}^2\gamma_m^2}{\gamma_n^2\gamma_{m-M}^2}\langle f_1,f_2\rangle=0.$$

Equality (7) shows that

$$[f_1]=\text{span}\{f_1,Tf_1\},\ [f_2]=\text{span}\{f_2,Tf_2\}.$$

So $[f_1]\perp[f_2]$.

(c) For every $(n,m)\in E_4$, let

$$\mathcal{M}_{n,m}=\operatorname{span}\{z_1^kz_2^l:(k,l)\sim_1(n,m),(k,l)\in E_4\},$$
$$\mathcal{M}_{n+N,m-M}=\operatorname{span}\{z_1^kz_2^l:(k,l)\sim_2(n+N,m-M),(k,l)\in E_5\}.$$

Then $\mathcal{M}_{n,m}$ and $\mathcal{M}_{n+N,m-M}$ are finite dimension, and the following statements hold:

(i) $T\mathcal{M}_{n,m}=\mathcal{M}_{n+N,m-M}$ and $T^*\mathcal{M}_{n+N,m-M}=\mathcal{M}_{n,m}$;
(ii) $T(\mathcal{M}_{n,m}^\perp)\subset\mathcal{M}_{n+N,m-M}^\perp$ and $T^*(\mathcal{M}_{n+N,m-M}^\perp)\subset\mathcal{M}_{n,m}^\perp$.

Therefore, $TP_{n,m}f=Q_{n+N,m-M}Tf$ and $P_{n,m}T^*f=T^*Q_{n+N,m-M}f$ for any $f\in\mathcal{M}$.

(d) By equality (7), conclusion (c) and

$$T^*TP_{n,m}f=\frac{\gamma_{n+N}^2\gamma_m^2}{\gamma_n^2\gamma_{m-M}^2}P_{n,m}f, \tag{9}$$

we have

$$\begin{aligned}[Q_{n+N,m-M}Tf]&=\operatorname{span}\{Q_{n+N,m-M}Tf,T^*Q_{n+N,m-M}Tf\}\\&=\operatorname{span}\{TP_{n,m}f\}\oplus\operatorname{span}\{T^*TP_{n,m}f\}\\&=\operatorname{span}\{TP_{n,m}f\}\oplus\operatorname{span}\{P_{n,m}f\}\\&=[P_{n,m}f].\end{aligned}$$

Similarly, $[Q_{n+N,m-M}f]=[P_{n,m}T^*f]$ comes from equality (8), conclusion (c) and

$$TT^*Q_{n+n,m-M}f=\frac{\gamma_{n+N}^2\gamma_m^2}{\gamma_n^2\gamma_{m-M}^2}Q_{n+N,m-M}f. \tag{10}$$

(e) By equalities (9), (10) and conclusion (c), we have

$$\begin{aligned}Q_{n+N,m-M}\mathcal{M}&=TT^*(Q_{n+N,m-M}\mathcal{M})=TP_{n,m}T^*\mathcal{M},\\P_{n,m}\mathcal{M}&=T^*T(P_{n,m}\mathcal{M})=T^*Q_{n+N,m-M}T\mathcal{M}.\end{aligned} \tag{11}$$

Therefore, we only need to show that $P_{n,m}\mathcal{M}\oplus Q_{n+N,m-M}\mathcal{M}$ is an invariant subspace of T and T^*. In fact,

$$T(P_{n,m}\mathcal{M}\oplus Q_{n+N,m-M}\mathcal{M})=TP_{n,m}\mathcal{M}=Q_{n+N,m-M}\mathcal{M},$$

where the last equality comes from $TP_{n,m}f=Q_{n+N,m-M}Tf\in Q_{n+N,m-M}\mathcal{M}$ and $Q_{n+N,m-M}f\in TP_{n,m}T^*\mathcal{M}\subset TP_{n,m}\mathcal{M}$ for all $f\in\mathcal{M}$. Therefore,

$$T(P_{n,m}\mathcal{M}\oplus Q_{n+N,m-M}\mathcal{M})\subset P_{n,m}\mathcal{M}\oplus Q_{n+N,m-M}\mathcal{M}.$$

Similarly, we can prove that

$$T^*(P_{n,m}\mathcal{M}\oplus Q_{n+N,m-M}\mathcal{M})=T^*Q_{n+N,m-M}\mathcal{M}=P_{n,m}\mathcal{M}.$$

So we finish the proof. □

Remark 2.2. In the prove of (e), we also get that

$$[P_{n,m}\mathcal{M}] = P_{n,m}\mathcal{M} \oplus Q_{n+N,m-M}\mathcal{M} = [Q_{n+N,m-M}\mathcal{M}],$$

where $[P_{n,m}\mathcal{M}]$ and $[Q_{n+N,m-M}\mathcal{M}]$ are the reducing subspaces generated by $P_{n,m}\mathcal{M}$ and $Q_{n+N,m-M}\mathcal{M}$, respectively.

Theorem 2.1. *Let $\mathcal{M} \subset \mathcal{M}_0^{\perp}$ be a non-zero reducing subspace of T on the bidisk. Then $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, where*

(i) *$\mathcal{M}_1$ is a direct sum of minimal reducing subspace $[z_1^p z_2^q]$ with $z_1^p z_2^q \in \mathcal{M}$ for some $(p,q) \in E_1 \cup E_2 \cup E_3$;*

(ii) *$\mathcal{M}_2$ is a direct sum of minimal reducing subspace $[f]$ with $f \in P_{n,m}\mathcal{M}$ for some $(n,m) \in E_4$.*

Proof. Firstly, we prove that

$$\mathcal{M} = \mathcal{M}_1 \bigoplus \bigoplus_{(n,m)\in E} (P_{n,m}\mathcal{M} \bigoplus Q_{n+N,m-M}\mathcal{M}), \tag{12}$$

where $\mathcal{M}_1 = \bigoplus_{(p,q)\in\Lambda} [z_1^p z_2^q]$ with $\Lambda = \{(p,q) \in E_1 \cup E_2 \cup E_3 : z_1^p z_2^q \in \mathcal{M}\}$, and E is the partition of E_4 by the equivalence $\sim_1$. Set $\mathcal{H}_{n,m} = P_{n,m}\mathcal{M} \bigoplus Q_{n+N,m-M}\mathcal{M}$.

On the one hand, $\mathcal{M}_1 \bigoplus \bigoplus_{(n,m)\in E} \mathcal{H}_{n,m} \subset \mathcal{M}$, since $\mathcal{M}_1 \subset \mathcal{M}$ is a reducing subspace of T, and conclusion (e) in Lemma 2.2 implies that $\bigoplus_{(n,m)\in E} \mathcal{H}_{n,m} \subset \mathcal{M}$. On the other hand, for $g = g_1 + g_2 \in \mathcal{M}$ with

$$g_1(z) = \sum_{(p,q)\in E_1\cup E_2\cup E_3} a_{p,q} z_1^p z_2^q, \; g_2(z) = \sum_{(p,q)\in E_4\cup E_5} a_{p,q} z_1^p z_2^q. \tag{13}$$

Remark 2.1 shows that $g_1 \in \mathcal{M}_1 \subset \mathcal{M}$, which implies that $g_2 = g - g_1 \in \mathcal{M}$. Therefore, $g_2 = \sum_{(n,m)\in E} (P_{n,m} g_2 + Q_{n+N,m-M} g_2) \in \bigoplus_{(n,m)\in E} \mathcal{H}_{n,m}$. It follows that $\mathcal{M}$ is in the direct sum of $\mathcal{M}_1$ and $\{\mathcal{H}_{n,m}\}$ with $(n,m) \in E$. So we have equality (12) holds.

Secondly, for each $(n,m) \in E_4$, we prove that $\mathcal{H}_{n,m}$ is the direct sum of minimal reducing subspaces as $[f] = \text{span}\{f, Tf\}$ with $f \in P_{n,m}\mathcal{M}$. There are some steps in the proof.

Step 1. Take $0 \neq f_1 \in P_{n,m}\mathcal{M}$. Then $[f_1] = \text{span}\{f_1, Tf_1\} \subset \mathcal{H}_{n,m}$.

Step 2. If $P_{n,m}\mathcal{M} \neq \mathbb{C}f_1$, take $0 \neq f_2 \in P_{n,m}\mathcal{M} \ominus \mathbb{C}f_1$. Then

$$[f_2] = \text{span}\{f_2, Tf_2\} \subset \mathcal{H}_{n,m} \ominus [f_1].$$

Step 3. If $P_{n,m}\mathcal{M} \neq \text{span}\{f_1, f_2\}$, take $0 \neq f_3 \in P_{n,m}\mathcal{M} \ominus \text{span}\{f_1, f_2\}$. Then

$$[f_3] = \text{span}\{f_3, Tf_3\} \subset \mathcal{H}_{n,m} \ominus [f_1] \ominus [f_2].$$

If $P_{n,m}\mathcal{M} \neq \text{span}\{f_1, f_2, f_3\}$, continue this process. This process will stop in finite steps, since the dimension of $\mathcal{H}_{n,m}$ is finite. Thus, we finish the proof. □

Remark 2.3. In particular, if $\mathcal{M}$ is a reducing subspace generated by $g = g_1 + g_2 \in A^2(\mathbb{D}^2)$ as in (13), then $[g] = [g_1] \oplus [g_2]$ and

$$[g_2] = \bigoplus_{(n,m)\in E} [P_{n,m}g, Q_{n+N,m-M}g],$$

where $[P_{n,m}g, Q_{n+N,m-M}g]$ is the reducing subspace generated by $P_{n,m}g$ and $Q_{n+N,m-M}g$. By conclusions (a) and (d) in Lemma 2.2 and equalities in (11), we get $[P_{n,m}g, Q_{n+N,m-M}g] = [P_{n,m}g, P_{n,m}T^*g] = \operatorname{span}\{P_{n,m}g, P_{n,m}T^*g\} \oplus \operatorname{span}\{Q_{n+N,m-M}g, Q_{n+N,m-M}Tg\}$.

Notice that $\operatorname{span}\{P_{n,m}g, P_{n,m}T^*g\}$ has an orthonormal basis $\{e_1, \ldots, e_k\}$, since the dimension of $\operatorname{span}\{P_{n,m}g, P_{n,m}T^*g\}$ is finite. Conclusion (b) in Lemma 2.2 shows that $[e_i]\perp[e_j]$ for $i \neq j$. Then we get

$$[P_{n,m}g, P_{n,m}T^*g] = \bigoplus_{j=1}^{k}[e_j] = \bigoplus_{j=1}^{k}\operatorname{span}\{e_j, Te_j\}.$$

Similarly, we can prove that

$$[g_2] = \bigoplus_{(n,m)\in E} [Q_{n+N,m-M}g, Q_{n+N,m-M}Tg],$$

and

$$[Q_{n+N,m-M}g, Q_{n+N,m-M}Tg] = \bigoplus_{j=1}^{l}[h_j] = \bigoplus_{j=1}^{l}\operatorname{span}\{h_j, T^*h_j\},$$

where $\{h_1, \ldots, h_l\}$ is an orthonormal basis of

$$\operatorname{span}\{Q_{n+N,m-M}g, Q_{n+N,m-M}Tg\}.$$

In the last part of this paper, we give some examples of the reducing subspaces of $T_{z_1^N \overline{z}_2^M}$ for the case that $N = M$ and $N \neq M$, respectively.

Example 2.1. Fix $a, b, c, d, e \in \mathbb{C}$ with $e \neq 0$. Let

$$f(z_1, z_2) = az_1^9 z_2^{14} + bz_1^7 z_2^{15} + cz_1^5 z_2^{17} + dz_1^4 z_2^{19} + ez_1^{11} z_2^{12},$$

and $[f]$ be the reducing subspace of $T_{z_1^{10}\overline{z}_2^{10}}$ generated by f. Then

$$[f] = \operatorname{span}\{f_1, f_2\} \oplus \operatorname{span}\{z_1^{11+10h} z_2^{12-10h} : h = -1, 0, 1\},$$

where

$$f_1(z_1, z_2) = az_1^9 z_2^{14} + bz_1^7 z_2^{15} + cz_1^5 z_2^{17} + dz_1^4 z_2^{19},$$
$$f_2(z_1, z_2) = \tfrac{a}{3} z_1^{19} z_2^4 + \tfrac{3b}{8} z_1^{17} z_2^5 + \tfrac{4c}{9} z_1^{15} z_2^7 + \tfrac{d}{2} z_1^{14} z_2^9.$$

Proof. Notice that $(11, 12) \in E_3$ and $(9, 14) \in E_4$. A direct computation shows that $(9, 14) \sim_1 (7, 15) \sim_1 (5, 17) \sim_1 (4, 19)$. Remark 2.1 implies that $f_1 = P_{4,19}f$ and $z_1^{11}z_2^{12}$ are in $\mathcal{M}$. As in Remark 2.3, there is $\operatorname{span}\{P_{4,19}f, P_{4,19}T^*f\} = [f_1] = \operatorname{span}\{f_1, f_2\}$. Therefore we get the desired result. □

Example 2.2. Let $f(z_1,z_2) = z_1^4z_2^{14} + z_1^7z_2^7 + z_1^3z_2^{15}$ and $[f]$ be the reducing subspace of $T_{z_1^5\overline{z}_2^{10}}$ generated by f. Then

$$[f] = \mathrm{span}\{z_1^4z_2^{14} + z_1^3z_2^{15}, \tfrac{1}{3}z_1^9z_2^4 + \tfrac{3}{8}z_1^8z_2^5\} \oplus \mathrm{span}\{z_1^7z_2^7, z_1^2z_2^{17}\}.$$

Proof. Notice that $(7,7) \in E_5$, $(4,14),(3,15) \in E_4$ and $(4,14) \sim_1 (3,15)$. Let $f_1 = P_{4,14}f = z_1^4z_2^{14} + z_1^3z_2^{15}$ and $f_2 = Q_{7,7}f = z_1^7z_2^7$. Then $[P_{4,14}f, P_{4,14}T^*f] = [f_1] = \mathrm{span}\{z_1^4z_2^{14} + z_1^3z_2^{15}, \frac{1}{3}z_1^9z_2^4 + \frac{3}{8}z_1^8z_2^5\}$, $[P_{2,17}f, P_{2,17}T^*f] = [Q_{7,7}f, Q_{7,7}Tf] = [f_2] = \mathrm{span}\{z_1^7z_2^7, z_1^2z_2^{17}\}$. Then we finish the proof. □

Example 2.3. Let $f(z_1,z_2) = z_1^3z_2^8 + z_1^7z_2^3$, and $[f]$ be the reducing subspace of $T_{z_1^4\overline{z}_2^5}$ generated by f. Then

$$[f] = \mathrm{span}\{z_1^3z_2^8, z_1^7z_2^3\}.$$

Proof. Notice that $(3,8) \in E_4$, $(7,3) \in E_5$. It is easy to check that $T_{z_1^4\overline{z}_2^5}z_1^3z_2^8 = \frac{4}{9}z_1^7z_2^3$ and $T^*_{z_1^4\overline{z}_2^5}z_1^7z_2^3 = \frac{1}{2}z_1^3z_2^8$. So $[z_1^3z_2^8] = [z_1^7z_2^3] = \mathrm{span}\{z_1^3z_2^8, z_1^7z_2^3\}$. It means that $[f] = \mathrm{span}\{z_1^3z_2^8, z_1^7z_2^3\}$. □

Example 2.4. Let $f(z_1,z_2) = z_1^2z_2^{17} + z_1^4z_2^{14} + z_1^9z_2^4 + z_1^3z_2^{15} + z_1^8z_2^5$ and $[f]$ be the reducing subspace of $T_{z_1^5\overline{z}_2^{10}}$ generated by f. Then

$$\begin{aligned}[f] &= [z_1^2z_2^{17}] \oplus [z_1^4z_2^{14}] \oplus [z_1^3z_2^{15}]\\ &= [z_1^2z_2^{17}] \oplus [z_1^4z_2^{14} + z_1^3z_2^{15}] \oplus [z_1^4z_2^{14} - \tfrac{64}{75}z_1^3z_2^{15}]\\ &= [z_1^7z_2^7] \oplus [z_1^9z_2^4 + z_1^8z_2^5] \oplus [z_1^9z_2^4 - \tfrac{27}{25}z_1^8z_2^5].\end{aligned}$$

Proof. Notice that $(2,17),(4,14),(3,15) \in E_4$, $(9,4),(8,5) \in E_5$ and

$$(4,14) \sim_1 (3,15),\ (9,4) \sim_2 (8,5).$$

(i) Since $P_{4,14}T^*f = T^*(z_1^9z_2^4 + z_1^8z_2^5) = \frac{1}{2}z_1^4z_2^{14} + \frac{4}{9}z_1^3z_2^{15}$, we have

$$\mathrm{span}\{P_{4,14}f, P_{4,14}T^*f\} = \mathrm{span}\{z_1^4z_2^{14}, z_1^3z_2^{15}\}.$$

Therefore,

$$\begin{aligned}[f] &= [z_1^2z_2^{17}] \oplus [z_1^4z_2^{14}] \oplus [z_1^3z_2^{15}]\\ &= \mathrm{span}\{z_1^2z_2^{17}, z_1^7z_2^7\} \oplus \mathrm{span}\{z_1^4z_2^{14}, z_1^9z_2^4\} \oplus \mathrm{span}\{z_1^3z_2^{15}, z_1^8z_2^5\}.\end{aligned}$$

(ii) It is easy to check that $\langle z_1^4z_2^{14} - \frac{64}{75}z_1^3z_2^{15}, z_1^4z_2^{14} + z_1^3z_2^{15}\rangle = 0$ and

$$\mathrm{span}\{P_{4,14}f, P_{4,14}T^*f\} = \mathrm{span}\{z_1^4z_2^{14} + z_1^3z_2^{15}, z_1^4z_2^{14} - \tfrac{64}{75}z_1^3z_2^{15}\}.$$

So $[f] = [z_1^4z_2^{14} + z_1^3z_2^{15}] \oplus [z_1^4z_2^{14} - \frac{64}{75}z_1^3z_2^{15}] \oplus [z_1^2z_2^{17}]$.

(iii) Notice that

$$\begin{aligned}\mathrm{span}\{Q_{9,4}f, Q_{9,4}Tf\} &= \mathrm{span}\{z_1^9z_2^4 + z_1^8z_2^5, \tfrac{1}{3}z_1^9z_2^4 + \tfrac{3}{8}z_1^8z_2^5\}\\ &= \mathrm{span}\{z_1^9z_2^4 + z_1^8z_2^5, z_1^9z_2^4 - \tfrac{27}{25}z_1^8z_2^5\},\end{aligned}$$

where $z_1^9z_2^4 - \frac{27}{25}z_1^8z_2^5 \perp Q_{9,4}f$. Then

$$[f] = [z_1^7z_2^7] \oplus [z_1^9z_2^4 + z_1^8z_2^5] \oplus [z_1^9z_2^4 - \tfrac{27}{25}z_1^8z_2^5].$$ □

Remark 2.4. In Example 2.4, since $T^*(z_1^9z_2^4 + z_1^8z_2^5) = \frac{1}{2}z_1^4z_2^{14} + \frac{4}{9}z_1^3z_2^{15}$ and $T^*(z_1^9z_2^4 - \frac{27}{25}z_1^8z_2^5) = \frac{1}{2}z_1^4z_2^{14} - \frac{12}{25}z_1^3z_2^{15}$, conclusion (d) in Lemma 2.2 implies that $[f] = [z_1^2z_2^{17}] \oplus [\frac{1}{2}z_1^4z_2^{14} + \frac{4}{9}z_1^3z_2^{15}] \oplus [\frac{1}{2}z_1^4z_2^{14} - \frac{12}{25}z_1^3z_2^{15}]$.

Moreover, let $T = T_{z_1^5\bar{z}_2^{10}}$ and $g = z_1^4z_2^{14} + z_1^9z_2^4 + z_1^3z_2^{15}$, then $[g] = [g + az_1^8z_2^5] = [z_1^4z_2^{14}] \oplus [z_1^3z_2^{15}]$ for $a \neq \frac{9}{8}$. In fact, $\mathrm{span}\{P_{4,14}(g + az_1^8z_2^5), P_{4,14}T^*(g + az_1^8z_2^5)\} = \mathrm{span}\{z_1^4z_2^{14}, z_1^3z_2^{15}\}$, since $T^*(z_1^9z_2^4 + az_1^8z_2^5)$ and $z_1^4z_2^{14} + z_1^3z_2^{15}$ are linearly independent.

For the case that $a = \frac{9}{8}$, we have

$$\begin{aligned}[g + \tfrac{9}{8}z_1^8z_2^5] &= \mathrm{span}\{z_1^4z_2^{14} + z_1^3z_2^{15}, z_1^9z_2^4 + \tfrac{9}{8}z_1^8z_2^5\} \\ &= [z_1^4z_2^{14} + z_1^9z_2^4]\end{aligned}$$

since $T^*(g + \frac{9}{8}z_1^8z_2^5) = \frac{1}{2}P_{4,14}g$.

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