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ZERO DIVISOR GRAPHS OF SKEW GENERALIZED POWER SERIES RINGS

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Abstract. Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. The skew generalized power series ring $R[[S, \omega]]$ is a common generalization of (skew) polynomial rings, (skew) power series rings, (skew) Laurent polynomial rings, (skew) group rings, and Mal'cev-Neumann Laurent series rings. In this paper, we investigate the interplay between the ring-theoretical properties of $R[[S, \omega]]$ and the graph-theoretical properties of its zero-divisor graph $\overline{\Gamma}(R[[S, \omega]])$. Furthermore, we examine the preservation of diameter and girth of the zero-divisor graph under extension to skew generalized power series rings.

1. Introduction

The concept of a zero-divisor graph of a commutative ring was introduced by Beck in [7]. In his work all elements of the ring were vertices of the graph (see also [3]). In [4], Anderson and Livingston introduced and studied the zero-divisor graph whose vertices are the non-zero zero-divisors of a ring. This graph turns out to best exhibit the properties of the set of zero-divisors of a commutative ring. In [32], Redmond studied the zero-divisor graph of a non-commutative ring. Several papers are devoted to studying the relationship between the zero-divisor graph and algebraic properties of rings (cf. [2], [3], [5], [7], [32], [39]).

The zero-divisors of R, denoted by $Z(R)$, is the set of elements $a \in R$ such that there exists a non-zero element $b \in R$ with $ab = 0$ or $ba = 0$. Let $Z^*(R)$ denote the (nonempty) set of nonzero zero divisors. The directed graph $\Gamma(R)$ is a graph with vertices in $Z^*(R)$, where $x \to y$ is an edge between distinct vertices x and y if and only if $xy = 0$. Recently Redmond in [32] has extended this concept to any arbitrary ring. Redmond in [32] defined an undirected zero-divisor graph of a non-commutative ring R, denoted by $\overline{\Gamma}(R)$, with vertices in the set $Z(R)^*$ and such that for distinct vertices a and b are adjacent if and

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only if $ab = 0$ or $ba = 0$. Note that for a commutative ring R, the definition of the zero-divisor graph of R in [4] coincides with the definition of $\overline{\Gamma}(R)$.

According to Cohn [12], a ring R is called *reversible* if $ab = 0$ implies that $ba = 0$ for $a, b \in R$. So, in view of [32, Theorem 2.3], over a reversible ring R, the simple (undirected) graph $\Gamma(R)$ is connected with $\text{diam}(\Gamma(R)) \leq 3$, where $\text{diam}(\Gamma(R))$ is the diameter of $\Gamma(R)$. In [32] it has been shown that for any ring R, every two vertices in $\overline{\Gamma}(R)$ are connected by a path of length at most 3. Note that using the proof of this result in commutative case, one can establish that for any arbitrary ring R, if there exists a path between two vertices x and y in the directed graph $\Gamma(R)$, then the length of the shortest path between x and y is at most 3. Moreover, in [32] it is shown that for any ring R, if $\overline{\Gamma}(R)$ contains a cycle, then the length of the shortest cycle in $\overline{\Gamma}(R)$, is at most 4.

There is considerable interest in studying if and how certain graph-theoretic properties of rings are preserved under various ring-theoretic extensions. The zero-divisor graphs offer a graphical representation of rings so that we may discover some new algebraic properties of rings that are hidden from the viewpoint of classical ring theorists. For an instance, using the notion of a zero-divisor graph, it has been proven in [33] that for any finite ring R, the sum $\sum_{x \in R} \mid r_R(x) - \ell_R(x) \mid$ is even, where $r_R(x)$ and $\ell_R(x)$ denote the right and left annihilators of the element x in R, respectively. More recently, Axtell, Coykendall and Stickles, in [6], examined the preservation of diameter and girth of zero-divisor graphs of commutative rings under extensions to polynomial and power series rings. Also, Lucas, in [22], continued the studying of the diameter of polynomial and power series of commutative rings. Moreover, Anderson and Mulay, in [5], studied the girth and diameter of a commutative ring and investigated the girth and diameter of polynomial and power series of commutative rings. For a commutative ring R with a monomorphism α and an α-derivation δ, Afkhami, Khashyarmanesh and Khorsandi, in [1], compare the diameter (and girth) of the zero-divisor graphs of R and the Ore extension $R[x; \alpha, \delta]$, when $R[x; \alpha, \delta]$ is assumed to be reversible.

In this paper, we propose a unified approach to the preservation and lack thereof of the diameter and girth of the zero-divisor graph of a non-commutative ring under extension to *skew generalized power series ring construction* $R[[S, \omega]]$, where R is a ring, S is a strictly ordered monoid, and $\omega : S \to \text{End}(R)$ is a monoid homomorphism (the definition of the ring $R[[S, \omega]]$ will be recalled in Section 2). Since (skew) polynomial rings, (skew) monoid rings, (skew) power series rings, (skew) Laurent polynomial rings, generalized power series rings, and Mal'cev-Neumann construction are particular cases of the skew generalized power series construction $R[[S, \omega]]$ (for details see the last part of Section 2), any studying interplay between the ring-theoretical properties of skew generalized power series rings and the graph-theoretical properties of its zero-divisor has its counterpart for each of the aforementioned ring constructions, and these counterparts follow immediately from a single proof. We would like to stress that using this general approach, in this paper we not only unified the already known theorems, but also obtained many new results, for several constructions simultaneously.

The paper is organized as follows. In Section 2, we recall the skew generalized power series ring construction and show that (skew) polynomial rings, (skew) Laurent polynomial rings, (skew) power series rings, (skew) Laurent series rings and (skew) monoid rings are special cases of the construction.

In Section 3, we prove that if R is a S-compatible ring which is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, then $\overline{\Gamma}(R)$ is complete if and only if $\overline{\Gamma}(R[[S, \omega]])$ is complete, where, $(S, \leq)$ is a strictly ordered a.n.u.p. monoid, $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. Furthermore, in the case when R is S-compatible, we compare the diameter (and girth) of the zero-divisor graphs $\overline{\Gamma}(R)$ and $\overline{\Gamma}(R[[S, \omega]])$. Finally we give a complete characterization for the girth of $\overline{\Gamma}(R[[S, \omega]])$.

For two distinct vertices a and b in the simple (undirected) graph Γ, the *distance* between a and b, denoted by $d(a, b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise we put $d(a, b) := \infty$. The *diameter* of a graph Γ is $\mathrm{diam}(\Gamma) := \sup\{d(a, b) \mid \text{a and b are distinct vertices of } \Gamma\}$. The diameter is 0 if the graph consists of a single vertex and a connected graph with more than one vertex has diameter 1 if and only if it is complete; i.e., each pair of distinct vertices forms an edge. The *girth* of a simple (undirected) connected graph Γ, denoted by $gr(\Gamma)$, is the length of the shortest cycle in Γ, provided Γ contains a cycle; otherwise $gr(\Gamma) := \infty$. We will denote by $\mathrm{End}(R)$ the monoid of ring endomorphisms of R, and by $\mathrm{Aut}(R)$ the group of ring automorphisms of R. Also, we use A^* to denote the nonzero elements of A, and $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{Z}_n$ for the integers, positive integers, rational numbers, the field of real numbers and the integers modulo n, respectively. Throughout this paper all monoids and rings are with identity element that is inherited by submonoids and subrings and preserved under homomorphisms, but neither monoids nor rings are assumed to be commutative.

2. Skew generalized power series ring

In order to recall the skew generalized power series ring construction (which was introduced in [37]) we need some definitions and facts that will also be used in further reasonings.

A partially ordered set $(S, \leq)$ is called *artinian* if every strictly decreasing sequence of elements of S is finite, and $(S, \leq)$ is called *narrow* if every subset of pairwise order-incomparable elements of S is finite. Thus, $(S, \leq)$ is artinian and narrow if and only if every nonempty subset of S has at least one but only a finite number of minimal elements. finite family of artinian and narrow subsets of an ordered set as well as any subset of an artinian and narrow set are again artinian and narrow. An *ordered monoid* is a pair $(S, \leq)$ consisting of a monoid S and an order $\leq$ on S such that for all $a, b, c \in S$, $a \leq b$ implies

$ca \leq cb$ and $ac \leq bc$. An ordered monoid $(S, \leq)$ is said to be *strictly ordered* if for all $a, b, c \in S$, $a < b$ implies $ca < cb$ and $ac < bc$.

For a strictly ordered monoid S and a ring R, Ribenboim [37] defined the ring of generalized power series $R[[S]]$ consisting of all maps from S to R whose support is artinian and narrow with the pointwise addition and the convolution multiplication. This construction provided interesting examples of rings (e.g., Elliott and Ribenboim, [13]; Ribenboim, [35], [36]) and it was extensively studied by many authors.

In [25], R. Mazurek and M. Ziembowski, introduced a "twisted" version of the Ribenboim construction and study when it produces a von Neumann regular ring. Now we recall the construction of the skew generalized power series ring introduced in [25]. Let R be a ring, $(S, \leq)$ a strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. For $s \in S$, let ω_s denote the image of s under ω, that is $\omega_s = \omega(s)$. Let A be the set of all functions $f : S \to R$ such that the support $\mathrm{supp}(f) = \{s \in S : f(s) \neq 0\}$ is artinian and narrow. Then for any $s \in S$ and $f, g \in A$ the set

$$X_s(f, g) = \{(x, y) \in \mathrm{supp}(f) \times \mathrm{supp}(g) : s = xy\}$$

is finite. Thus one can define the product $fg : S \to R$ of $f, g \in A$ as follows:

$$fg(s) = \sum_{(u,v) \in X_s(f,g)} f(u)\omega_u(g(v)),$$

(by convention, a sum over the empty set is 0). With pointwise addition and multiplication as defined above, A becomes a ring, called *the ring of skew generalized power series* with coefficients in R and exponents in S (one can think of a map $f : S \to R$ as a formal series $\sum_{s \in S} r_s s$, where $r_s = f(s) \in R$) and denoted either by $R[[S^{\leq}, \omega]]$ or by $R[[S, \omega]]$ (see [24] and [25]).

We will use the symbol 1 to denote the identity elements of the monoid S, the ring R, and the ring $R[[S, \omega]]$, as well as the trivial monoid homomorphism $1 : S \to \mathrm{End}(R)$ that sends every element of S to the identity endomorphism. To each $r \in R$ and $s \in S$, we associate elements $c_r, e_s \in R[[S, \omega]]$ defined by

$$c_r(x) = \begin{cases} r & x = 1 \\ o & x \in S \setminus \{1\}, \end{cases} \quad e_s(x) = \begin{cases} 1 & x = s \\ o & x \in S \setminus \{s\}. \end{cases}$$

It is clear that $r \to c_r$ is a ring embedding of R into $R[[S, \omega]]$ and $s \to e_s$ is a monoid embedding of S into the multiplicative monoid of the ring $R[[S, \omega]]$, and $e_s c_r = c_{\omega_s(r)} e_s$. For every nonempty subset X of R, we denote:

$$X[[S, \omega]] = \{f \in R[[S, \omega]] \mid f(s) \in X \cup \{0\} \text{ for every } s \in S\}.$$

Below we show how the classical ring aforementioned ring constructions in Section 1 can be viewed as special cases of the skew generalized power series ring construction.

Let R be a ring and σ an endomorphism of R. Then for the additive monoid $S = \mathbb{N} \cup \{0\}$ of nonnegative integers, the map $\omega : S \to \mathrm{End}(R)$ given by

$$\omega(n) = \sigma^n \text{ for any } n \in S, \tag{2.1}$$

is a monoid homomorphism. If furthermore σ is an automorphism of R, then 2.1 defines also a monoid homomorphism $\omega : S \to \mathrm{Aut}(R)$ for $S = \mathbb{Z}$, the additive monoid of integers. We can consider two different orders on each of the monoids $\mathbb{N} \cup \{0\}$ and $\mathbb{Z}$: the trivial order (i.e., the order with respect to which any two distinct elements are incomparable) and the natural linear order. In both cases these monoids are strictly ordered, and thus in each of the cases we can construct the skew generalized power series ring $R[[S, \omega]]$. As a result, we obtain the following extensions of the ring R:

(1) If $S = \mathbb{N} \cup \{0\}$ and $\leq$ is the trivial order, then the ring $R[[S, \omega]]$ is isomorphic to the skew polynomial ring $R[x, \sigma]$.
(2) If $S = \mathbb{N} \cup \{0\}$ and $\leq$ is the natural linear order, then $R[[S, \omega]]$ is isomorphic to the skew power series ring $R[[x; \sigma]]$.
(3) If $S = \mathbb{Z}$ and $\leq$ is the trivial order, and σ is an automorphism of R, then $R[[S, \omega]]$ is isomorphic to the skew Laurent polynomial ring $R[x, x^{-1}; \sigma]$.
(4) If $S = \mathbb{Z}$ and $\leq$ is the natural linear order, and σ is an automorphism of R, then $R[[S, \omega]]$ is isomorphic to the skew Laurent series ring $R[[x, x^{-1}; \sigma]]$.

By applying the above points (1)-(4) to the case where σ is the identity map of R, we can see that also the following ring extensions are special cases of the skew generalized power series ring construction: the ring of polynomials $R[x]$, the ring of power series $R[[x]]$, the ring of Laurent polynomials $R[x, x^{-1}]$, and the ring of Laurent series $R[[x, x^{-1}]]$.

Furthermore, any monoid S is a strictly ordered monoid with respect to the trivial order on S. Hence if R is a ring, S is a monoid and $\omega : S \to \mathrm{End}(R)$ is a monoid homomorphism, then we can impose the trivial order on S and construct the skew generalized power series ring $R[[S, \omega]]$, which in this case will be denoted by $R[S, \omega]$. It is clear that the ring $R[S, \omega]$ is isomorphic to the classical skew monoid ring built from R and S using the action ω of S on R. If ω is trivial, we write $R[S]$ instead of $R[S, \omega]$. Obviously the ring $R[S]$ is isomorphic to the ordinary monoid ring of S over R.

Also, the construction of skew generalized power series rings generalizes another classical ring constructions such as the Mal'cev-Neumann Laurent series rings ($(S, \leq)$ a totally ordered group and trivial ω; see [11], p. 528), the Mal'cev-Neumann construction of twisted Laurent series rings ($(S, \leq)$ a totally ordered group; see [19], p. 242), and generalized power series rings $R[[S]]$ (trivial ω; see [37], Section 4), twisted generalized power series rings (see [21, Section 2], [25]).

Recall that a monoid S is called a *unique product monoid* (or a u.p. monoid, or u.p.) if for any two nonempty finite subsets $X; Y \subseteq S$ there exist $x \in X$ and $y \in Y$ such that $xy \neq x^{'}y^{'}$ for every $(x^{'}, y^{'}) \in X \times Y \setminus \{(x, y)\}$; the element xy is called a u.p. element of $XY = \{st : s \in X, t \in Y\}$. Unique product monoids and groups play an important role in ring theory, for example providing a positive case in the zero-divisor problem for group rings (see also [9]), and their structural properties have been extensively studied (see [14]). The class of u.p. monoids includes the right and the left totally ordered monoids, submonoids of a free group, and torsionfree nilpotent groups. Every u.p.-monoid S is cancellative and has no non-unity element of finite order. For our purposes, the following more stringent conditions are needed.

Definition 2.1 ([24, Definition 4.11])**.** Let $(S, \leq)$ be an ordered monoid. Then $(S, \leq)$ is called an *artinian narrow* unique product monoid (or an a.n.u.p. monoid, or simply a.n.u.p.) if for every two artinian and narrow subsets X and Y of S there exists a u.p. element in the product XY. Also $(S, \leq)$ is called *quasitotally ordered* (and that $\leq$ is a quasitotal order on S) if $\leq$ can be refined to an order $\preceq$ with respect to which S is a strictly totally ordered monoid.

For any ordered monoid $(S, \leq)$, the following chain of implications holds:

$$\text{S is commutative, torsion-free, and cancellative}$$
$$\Downarrow$$
$$(S, \leq) \text{ is quasitotally ordered}$$
$$\Downarrow$$
$$(S, \leq) \text{ is a.n.u.p.}$$
$$\Downarrow$$
$$S \text{ is u.p.}$$

The converse of the bottom implication holds if $\leq$ is the trivial order. For more details, examples, and interrelationships between these and other conditions on ordered monoids, we refer the reader to [23].

3. Diameter and Girth of $\overline{\Gamma}(R)$ and $\overline{\Gamma}(R[[S, \omega]])$

There is considerable interest in studying if and how certain graph-theoretic properties of rings are preserved under various ring-theoretic extensions. In this section, we examine the preservation and lack thereof of the diameter and girth of the zero-divisor graph of a noncommutative ring under extension to skew generalized power series ring.

According to Krempa [18], an endomorphism α of a ring R is said to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is said to be α-*rigid* if there exists a rigid endomorphism α of R. In [16], the first author and E. Hashemi introduced α-compatible rings and studied their properties. A ring R is called α-*compatible* if for each $a, b \in R$, $ab = 0$ if and only if $a\alpha(b) = 0$.

Basic properties of rigid and compatible endomorphisms, proved by the authors in [16, Lemmas 2.2 and 2.1] are summarized here:

Lemma 3.1. *Let α be an endomorphism of a ring R. Then*:

(i) *if α is compatible, then α is injective*;
(ii) *α is compatible if and only if for all $a, b \in R$, $\alpha(a)b = 0 \Leftrightarrow ab = 0$*;
(iii) *the following conditions are equivalent*:
  (1) *α is rigid*;
  (2) *α is compatible and R is reduced*;
  (3) *for every $a \in R$, $\alpha(a)a = 0$ implies that $a = 0$.*

G. Marks, R. Mazurek and M. Ziembowski in [24] extended these in:

Definition 3.2 ([24])**.** Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. The ring R is said to be *S-compatible* (resp. *S-rigid*) if ω_s is compatible (resp. rigid) for every $s \in S$.

By [24], a ring R is (S, ω)-*Armendariz* if whenever $fg = 0$ for $f, g \in R[[S, \omega]]$, then $f(s)\omega_s(g(t)) = 0$ for all $s, t \in S$, where $(S, \leq)$ is a strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. For more details and results of (S, ω)-Armendariz ring see [24] and [31].

Theorem 3.3. *Let $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ be a ring, S an a.n.u.p. monoid and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. Assume that R is S-compatible. Then $\overline{\Gamma}(R)$ is complete if and only if $\overline{\Gamma}(R[[S, \omega]])$ is complete.*

Proof. Assume that $\overline{\Gamma}(R)$ be complete, we will show that $\overline{\Gamma}(R[[S, \omega]])$ is complete. Since R is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, by [2, Theorem 5] we deduce that $Z(R)^2 = 0$ and $Z(R)$ is an ideal of R. Now, since R is S-compatible, it is easy to show that $\overline{R} := \frac{R}{Z(R)}$ is $(S, \overline{\omega})$-rigid, where $\overline{\omega} : S \to \mathrm{End}(\frac{R}{Z(R)})$ is the induced monoid homomorphism. Now, we claim that $Z(R[[S, \omega]]) \subseteq Z(R)[[S, \omega]]$. Suppose towards a contradiction that $f \in Z(R[[S, \omega]]) \setminus Z(R)[[S, \omega]]$. Then there exists a nonzero element $g \in R[[S, \omega]]$ such that $fg = 0$. We will prove that $\overline{g} \neq \overline{0}$. Assume on the contrary that our assertion fails and $g \in Z(R)[[S, \omega]]$. Now, we have the following two cases.

Case (1) Let $f(s) \notin Z(R)$ for all $s \in \mathrm{supp}(f)$.
Since $(S, \leq)$ is a strictly ordered a.n.u.p.-monoid, there exist $s \in \mathrm{supp}(f)$ and $t \in \mathrm{supp}(g)$ such that st is a u.p. element of $\mathrm{supp}(f)\mathrm{supp}(g)$. Since st is a u.p. element, it follows that $0 = (fg)(st) = f(s)\omega_s(g(t))$. Now, the S-compatibility of R implies that $f(s)g(t) = 0$. Therefore $f(s) \in Z(R)$, which is a contradiction.

Case (2) Let $\mathfrak{D} := \{s \in \mathrm{supp}(f) \mid f(s) \in Z(R)\}$ be nonempty. Setting:

$$h(s) := \begin{cases} f(s) & s \in \mathfrak{D} \\ 0 & s \notin \mathfrak{D} \end{cases} \quad \text{and} \quad k(s) := \begin{cases} f(s) & s \notin \mathfrak{D} \\ 0 & s \in \mathfrak{D}. \end{cases}$$

We obtain maps $h, k : S \to R$ with $\mathrm{supp}(h) = \mathfrak{D}$ and $\mathrm{supp}(k) = \mathfrak{D}^c \cap \mathrm{supp}(f)$. Since $\mathrm{supp}(f)$ is artinian and narrow thus $h, k \in R[[S, \omega]]$. Since $g \in Z(R)[[S, \omega]]$ and $Z(R)^2 = 0$, we have $h(u)g(v) = 0$ for each $u, v \in S$.

Now, the S-compatibility of R implies that $hg = 0$. Therefore $kg = 0$. By a similar argument, there exist $s \in \mathrm{supp}(k)$ and $t \in \mathrm{supp}(g)$ such that st is a u.p. element of $\mathrm{supp}(k)\mathrm{supp}(g)$. Since st is a u.p. element, it follows that $0 = (kg)(st) = k(s)\omega_s(g(t))$. Using the S-compatibility of R, we find that $k(s)g(t) = 0$. Therefore $k(s) = f(s) \in Z(R)$, which is a contradiction.

Therefore we conclude that $\overline{g} \neq \overline{0}$. Since $\overline{R}$ is $(S, \overline{\omega})$-rigid, the ring $\overline{R}$ is $(S, \overline{\omega})$-Armendariz, by [24, Theorem 4.12]. Now, since $\overline{fg} = \overline{0}$ and $\overline{f}, \overline{g} \neq 0$, there exist $s \in \mathrm{supp}(\overline{f})$ and $t \in \mathrm{supp}(\overline{g})$ such that $\overline{f(s)g(t)} = \overline{0}$, contrary to the fact that $\overline{R}$ is domain. Therefore $Z(R[[S,\omega]]) \subseteq Z(R) * S$.

Now, assume that f, g are two distinct elements in $Z^*(R[[S,\omega]])$. Thus $f(s), g(s) \in Z(R)$ for all $s \in S$. Since $Z(R)^2 = 0$ and R is S-compatible, $a\omega_s(b) = 0$ for every $s \in S$ and $a, b \in Z^*(R)$. Therefore

$$fg(s) = \sum_{(u,v)\in X_s(f,g)} f(u)\omega_u(g(v)) = 0$$

for every $s \in S$. So $\overline{\Gamma}(R[[S,\omega]])$ is complete. The converse follows directly from the fact that $\overline{\Gamma}(R)$ is an induced subgraph of $\overline{\Gamma}(R[[S,\omega]])$, and the proof is complete. □

Remark 3.4. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\overline{\Gamma}(R) = 1$. Now, let $\omega : S \to \mathrm{End}(R)$ be the monoid homomorphism defined by $\omega_s(a, b) = (b, a)$, for all $s \in S$. Suppose that $f = c_{-(0,1)} + c_{(0,1)}e_s$ and $g = c_{(0,1)} + c_{(0,1)}e_s$ in $R[[S,\omega]]$, for all $s \in S \setminus \{1\}$. Then $fg \neq 0$, but $c_{(1,0)}f = c_{(1,0)}g = 0$. So $f - c_{(1,0)} - g$ is a path in $R[[S,\omega]]$ and thus $\mathrm{diam}(\overline{\Gamma}(R[[S,\omega]])) \geq 2$.

Now, we provide the following example to support given Theorem 3.3.

Example 3.5. Assume that D be a domain. Put:

$$R := \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D \right\}.$$

Let S be either $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{Q}^+ = \{a \in \mathbb{Q} \mid a \geq 0\}$ or $\mathbb{R}^+ = \{a \in \mathbb{R} \mid a \geq 0\}$ where $\leq$ is the usual order. Suppose that $\omega : S \to \mathrm{End}(D)$ be a monoid homomorphism such that ω_s is injective for all $s \in S \setminus \{1\}$. Then $\overline{\Gamma}(R)$ is complete, since $Z(R)^* = \{\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \mid b \in D^*\}$. Also, it is easy to show that R is $(S, \overline{\omega})$-compatible, where $\overline{\omega} : S \to \mathrm{End}(R)$ is a monoid homomorphism given by $\overline{\omega_s}((a_{ij})) = (\omega_s(a_{ij}))$ for all $s \in S$. Therefore $\overline{\Gamma}(R[[\mathbb{Q}, \overline{\omega}]])$, $\overline{\Gamma}(R[[\mathbb{R}, \overline{\omega}]])$, $\overline{\Gamma}(R[[\mathbb{Q}^+, \overline{\omega}]])$, and $\overline{\Gamma}(R[[\mathbb{R}^+, \overline{\omega}]])$ are complete, by Theorem 3.3.

Theorem 3.3 extends and unifies [2, Theorem 6] and [6, Theorem 3.2]. Furthermore, the following corollaries leads to interplay between the ring-theoretical properties in more extensions of rings and the graph-theoretical properties of its zero-divisor.

Corollary 3.6. *Let $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ be a ring and S a u.p. monoid. Then $\overline{\Gamma}(R)$ is complete if and only if $\overline{\Gamma}(R[S])$ is complete, where $R[S]$ is the monoid ring.*

If R is a ring, $(S, \leq)$ a totally ordered group and $\omega : S \to \mathrm{Aut}(R)$ a group homomorphism, then the skew generalized power series ring $R[[S,\omega]]$ is called the Mal'cev-Neumann ring of twisted Laurent series and denoted by $R((S,\omega))$ (see [19], p. 242).

Corollary 3.7. *Let R be a ring which is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let $(S, \leq)$ be a totally ordered group and $\omega : S \to \mathrm{Aut}(R)$ a group homomorphism such that R is S-compatible. Then $\overline{\Gamma}(R)$ is complete if and only if $\overline{\Gamma}(R((S,\omega)))$ is complete.*

Let R be a ring, and consider the multiplicative monoid $\mathbb{N}^{\geq 1}$, endowed with the usual order $\leq$. Then $A = R[[\mathbb{N}^{\geq 1}]]$ is the *ring of arithmetical functions* with values in R, endowed with the Dirichlet convolution:

$$fg(n) = \sum_{d|n} f(d)g(n/d) \quad \text{for each } n \geq 1.$$

Corollary 3.8. *Let R be a ring which is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Then $\overline{\Gamma}(R)$ is complete if and only if $\overline{\Gamma}(R[[\mathbb{N}^{\geq 1}]])$ is complete.*

Corollary 3.9. *Let $(S_1, \leq_1), \ldots, (S_n, \leq_n)$ be strictly totally ordered monoids. Denote by $(lex \leq)$ and $(relex \leq)$ the lexicographic order, the reverse lexicographic order, respectively, on the monoid $S_1 \times \cdots \times S_n$. Let $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ be a ring and $\omega : S_1 \times \cdots \times S_n \to \mathrm{End}(R)$ is a monoid homomorphism such that R is $S_1 \times \cdots \times S_n$-compatible. Then the following statements are equivalent.*

(1) *$\overline{\Gamma}(R[[S_1 \times \cdots \times S_n, \omega, lex \leq]])$ is complete*;
(2) *$\overline{\Gamma}(R[[S_1 \times \cdots \times S_n, \omega, relex \leq]])$ is complete*;
(3) *$\overline{\Gamma}(R)$ is complete.*

Let α and β be endomorphisms (resp. automorphisms) of R such that $\alpha\beta = \beta\alpha$. Let $S = (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$ (resp. $\mathbb{Z} \times \mathbb{Z}$) be endowed the lexicographic order, or the reverse lexicographic order, or the product order of the usual order of $\mathbb{N} \cup \{0\}$ (resp. $\mathbb{Z}$), and define $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism via $\omega(m,n) = \alpha^m \beta^n$ for any $m, n \in \mathbb{N} \cup \{0\}$ (resp. $m, n \in \mathbb{Z}$). Then $R[[S,\omega]] \cong R[[x,y;\alpha,\beta]]$ (resp. $R[[S,\omega]] \cong R[[x,y,x^{-1},y^{-1};\alpha,\beta]]$), in which $(ax^m y^n)(bx^p y^q) = a\alpha^m\beta^n(b)x^{m+p}y^{n+q}$ for any $m,n,p,q \in \mathbb{N} \cup \{0\}$ (resp. $m,n,p,q \in \mathbb{Z}$).

Corollary 3.10. *Let $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ be a ring, α and β be automorphisms of R such that $\alpha\beta = \beta\alpha$ and $T = R[[x,y;\alpha,\beta]]$ or $T = R[[x,y,x^{-1},y^{-1};\alpha,\beta]]$. Assume that R is (α,β)-compatible. Then $\overline{\Gamma}(R)$ is complete if and only if $\overline{\Gamma}(T)$ is complete.*

Let $(S, \leq)$ be a strictly commutative totally ordered monoid which is also artinian. Then the set $X_s = \{(u,v) \mid uv = s,\ u,v \in S\}$ is finite for any $s \in S$. Let V be a free abelian additive group with the base consisting of elements of S. It was noted in [20, Remark 1.2] that V is a coalgebra over $\mathbb{Z}$ with the comultiplication map and the counit map as follows:

$$\triangle(s) = \sum_{(u,v)\in X_s} u \otimes v, \qquad \epsilon(s) = \begin{cases} 1 & s = 1 \\ o & s \neq 1, \end{cases}$$

and $R[[S]] \cong \mathrm{Hom}(V, R)$, the dual algebra with multiplication

$$f * g = (f \otimes g) \triangle \quad \text{for each } f, g \in \mathrm{Hom}(V, R).$$

Corollary 3.11. *Let $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ be a ring, $(S, \leq)$ be a strictly commutative totally ordered monoid which is also artinian and $\mathrm{Hom}(V, R)$ defined as above. Then $\overline{\Gamma}(R)$ is complete if and only if $\overline{\Gamma}(\mathrm{Hom}(V, R))$ is complete.*

Assume that $S = \mathbb{Z}^n$ (resp. $(\mathbb{N} \cup \{0\})^n$) be endowed the trivial order and for each i, let σ_i be a ring automorphism (resp. endomorphism) of R. Suppose that $\sigma_i\sigma_j = \sigma_j\sigma_i$ for all i, j. Define $\omega : S \to \mathrm{End}(R)$ via $\omega(k_1, k_2, \ldots, k_n) = \sigma_1^{k_1}\sigma_2^{k_2}\cdots\sigma_n^{k_n}$. Then we have $R[[S, \omega]] \cong R[x_1, x_2, \ldots, x_n, x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}; \sigma_1, \sigma_2, \ldots, \sigma_n]$ (resp. $R[[S, \omega]] \cong R[x_1, x_2, \ldots, x_n; \sigma_1, \sigma_2, \ldots, \sigma_n]$).

Corollary 3.12. *Let $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ be a ring and $\sigma_1, \sigma_2, \ldots, \sigma_n$ are automorphisms of R such that $\sigma_i\sigma_j = \sigma_j\sigma_i$ for all i, j. If R is σ_i-compatible for each i, then the following are equivalent*:

(1) $\overline{\Gamma}(R)$ *is complete*;
(2) $\overline{\Gamma}(R[x_1, x_2, \ldots, x_n, x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}; \sigma_1, \sigma_2, \ldots, \sigma_n])$ *is complete*;
(3) $\overline{\Gamma}(R[x_1, x_2, \ldots, x_n; \sigma_1, \sigma_2, \ldots, \sigma_n])$ *is complete.*

Proof. Note that $S = \mathbb{Z}^n$ (resp. $(\mathbb{N} \cup \{0\})^n$) is an a.n.u.p. monoid. □

Let R be a ring and G a group acting on R by means of a homomorphism into the automorphism group of R. We define $\omega : G \to \mathrm{End}(R)$ via $\omega(g) = g$ for each $g \in G$. Let $\leq$ be the trivial order of G. Then it is easy to see that $R[[G, \omega]] = R * G$, the skew group ring of G with coefficients in R.

Skew group ring $R * G$ is an important tool for Galois theory because it is related to the fixed ring R^G. The skew group ring $R * G$ and the fixed ring R^G have been extensively studied in ([8], [10], [15], [26]) when G is X-outer or R has no $|G|$-torsion.

Corollary 3.13. *Let $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ be a ring, G any abelian, torsion-free group acting on R as a group of automorphisms. Assume that R is G-compatible. Then $\overline{\Gamma}(R)$ is complete if and only if $\overline{\Gamma}(R * G)$ is complete.*

Remark 3.14. For a commutative ring with identity R, the collection of zero-divisors $Z(R)$ of R is the set-theoretic union of prime ideals $\bigcup_{i\in\Lambda} \mathcal{P}_i$. We will also assume that these primes are maximal with respect to being contained in $Z(R)$. So if $\mathrm{diam}(\Gamma(R)) \leq 2$ and Λ is a finite set (in particular if R is Noetherian), in view of [6, Corollary 3.5], $|\Lambda| \leq 2$.

Proposition 3.15 ([6, Proposition 3.6])**.** *Let R be a commutative ring such that $\mathrm{diam}(\Gamma(R)) = 2$. If $Z(R) = \mathcal{P}_1 \cup \mathcal{P}_2$ with $\mathcal{P}_1$ and $\mathcal{P}_2$ distinct maximal* *primes in* $Z(R)$*, then* $\mathcal{P}_1 \cap \mathcal{P}_2 = \{0\}$ *(in particular, for all* $p_1 \in \mathcal{P}_1$ *and* $p_2 \in \mathcal{P}_2$*,* $p_1 p_2 = 0$*).*

Theorem 3.16 ([34, 3.6])**.** *Let* S *be commutative, torsion-free, and cancellative monoid,* R *be a commutative Noetherian ring and the zero ideal has an irredundant primary decomposition* $0 = Q_1 \cap \cdots \cap Q_n$ *with* $\sqrt{Q_i} = P_i$ *for every* $i = 1, \ldots, n$*. If* $f \in R[[S]]$*, then the following conditions are equvalent*:

(1) f *is zero divisor in* $R[[S]]$;
(2) *There exists* i *such that* $f \in P_i[[S]]$;
(3) *There exists* $0 \neq r \in R$ *such that* $rf(s) = 0$ *for every* $s \in S$*.*

Theorem 3.17. *Let* R *be a commutative Noetherian ring,* S *a commutative, torsion-free, and cancellative monoid. If* $\operatorname{diam}(\Gamma(R)) = 2$*, then*

$$\operatorname{diam}(\Gamma(R[[S]])) = 2.$$

Proof. By Remark 3.14, either $Z(R) = \mathcal{P}_1 \cup \mathcal{P}_2$ union of precisely two maximal primes in $Z(R)$, or $Z(R) = \mathcal{P}$ is a prime ideal.

Case (1) Suppose that $Z(R) = \mathcal{P}_1 \cup \mathcal{P}_2$ is the union of precisely two maximal primes in $Z(R)$. Let f and g be two distinct elements in $Z^*(R[[S]])$. By Theorem 3.16, $f(s), g(s) \in Z(R)$ for all $s \in S$. Then it is necessary for f (and hence g) to be contained in $\mathcal{P}_1[[S]]$ or $\mathcal{P}_2[[S]]$. Because otherwise there exist $f(s) \in \mathcal{P}_1 \setminus \mathcal{P}_2$ and $f(t) \in \mathcal{P}_2 \setminus \mathcal{P}_1$ such that $f(s)r = 0$ and $f(t)r = 0$ for some nonzero element r of R and $s, t \in S$. Thus $r \in \mathcal{P}_1 \cap \mathcal{P}_2$, contrary to the fact that $\mathcal{P}_1 \cap \mathcal{P}_2 = \{0\}$, by Proposition 3.15. So, we have two cases. Firstly, suppose that $f \in \mathcal{P}_1[[S]]$ and $g \in \mathcal{P}_2[[S]]$. Then, by Proposition 3.15, $f(s)g(t) = 0$ for all $s, t \in S$. Hence $fg = 0$. Now, consider the case that $f, g \in \mathcal{P}_1[[S]]$. Then any element of $\mathcal{P}_2$ suffices as a mutual annihilator. Thus $\operatorname{diam}(\Gamma(R[[S]])) = 2$.

Case (2) Assume that $Z(R) = \mathcal{P}$ is a prime ideal. By [17, Theorem 82], $\mathcal{P}$ is annihilated by a single element (say z). Suppose that f, g are zero-divisors. If $fg = 0$, we are done. If $fg \neq 0$, then z is a mutual annihilator of f and g, by Theorem 3.16. ☐

Proposition 3.18. *Let* $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ *be a ring,* S *an a.n.u.p. monoid and* $\omega : S \to \operatorname{End}(R)$ *a monoid homomorphism. If* R *is* S*-compatible and*

$$\operatorname{diam}(\overline{\Gamma}(R[[S, \omega]])) = 2,$$

then $\operatorname{diam}(\overline{\Gamma}(R)) = 2$*.*

Proof. It is easy to show that $\operatorname{diam}(\overline{\Gamma}(R)) \leq \operatorname{diam}(\overline{\Gamma}(R[[S, \omega]]))$. Now, the result follows from Theorem 3.3. ☐

As an immediate consequence of Theorem 3.17 and Proposition 3.18, we obtain the following generalization of [6, Theorem 3.11].

Corollary 3.19. *Let* $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ *be a commutative Noetherian ring with nontrivial zero-divisors. Then the following conditions are equivalent*:

(1) $\operatorname{diam}(\Gamma(R)) = 2$;

(2) $\operatorname{diam}(\Gamma(R[x])) = 2$;
(3) $\operatorname{diam}(\Gamma(R[x_1, x_2, \ldots, x_n])) = 2$;
(4) $\operatorname{diam}(\Gamma(R[x, x^{-1}])) = 2$;
(5) $\operatorname{diam}(\Gamma(R[x_1, x_2, \ldots, x_n, x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}])) = 2$;
(6) $\operatorname{diam}(\Gamma(R[S])) = 2$, *where S is a commutative, torsion-free, and cancellative monoid*;
(7) $\operatorname{diam}(\Gamma(R[[x]])) = 2$;
(8) $\operatorname{diam}(\Gamma(R[[x_1, x_2, \ldots, x_n]])) = 2$;
(9) $\operatorname{diam}(\Gamma(R[[x, x^{-1}]])) = 2$;
(10) $\operatorname{diam}(\Gamma(R[[x_1, x_2, \ldots, x_n, x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}]])) = 2$;
(11) $\operatorname{diam}(\Gamma(R[[\mathbb{N}^{\geq 1}]])) = 2$, *where $R[[\mathbb{N}^{\geq 1}]]$ is the ring of arithmetical functions with values in R*;
(12) $\operatorname{diam}(\Gamma(R((S)))) = 2$, *where $R((S))$ is the Mal'cev-Neumann construction and $(S, \leq)$ is a totally ordered abelian group.*

Recall from [30] that a ring R is called *right (S,ω)-McCoy* if whenever elements $f, g \in R[[S,\omega]] \setminus \{0\}$ satisfy $fg = 0$, then there exists $0 \neq r \in R$ such that $fr = 0$, where $(S, \leq)$ is a strictly ordered monoid, and $\omega : S \to \operatorname{End}(R)$ a monoid homomorphism. Left (S,ω)-McCoy rings is defined similarly. If R is both left and right (S,ω)-McCoy, then we say R is (S,ω)-McCoy ring. Note that every S-compatible and (S,ω)-Armendariz ring is a (S,ω)-McCoy ring. For more details and results of (S,ω)-McCoy ring see [30].

Proposition 3.20. *Let $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ be a ring, S a monoid and $\omega : S \to \operatorname{End}(R)$ a monoid homomorphism. If R is (S,ω)-McCoy, S-compatible and for some $n \in \mathbb{Z}$ with $n > 2$, $(Z(R))^n = 0$ such that $(Z(R))^{n-1} \neq 0$, then*

$$\operatorname{diam}(\overline{\Gamma}(R)) = \operatorname{diam}(\overline{\Gamma}(R[[S,\omega]])) = 2.$$

Proof. Since the ring R is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $(Z(R))^2 \neq 0$, by [2, Theorem 5], $\overline{\Gamma}(R)$ is not complete. Hence, there exist distinct $a, b \in Z(R)$ with $ab \neq 0$ and $ba \neq 0$. On the other hand, since $(Z(R))^{n-1} \neq 0$, there exist $d_1, d_2, \ldots, d_n \in Z(R)$ with $d = \prod_{i=1}^{n-1} d_i \neq 0$. Therefore, $ad = 0 = bd$, since $(Z(R))^n = 0$. So $a - d - b$ is a path in R and hence $\operatorname{diam}(\overline{\Gamma}(R)) = 2$. Now, it is sufficient to prove that $\operatorname{diam}(\overline{\Gamma}(R[[S,\omega]])) = 2$. Let f and g be two distinct element in $Z^*(R[[S,\omega]])$. Since R is (S,ω)-McCoy, thus $f(s), g(s) \in Z(R)$ for all $s \in S$. Hence the S-compatibility of R yields either $f - g$ or $f - c_d - g$. Therefore $\operatorname{diam}(\overline{\Gamma}(R[[S,\omega]])) = 2$, and the proof is complete. □

Proposition 3.21. *Let R be a ring which is not a domain, S a nontrivial monoid and $\omega : S \to \operatorname{End}(R)$ a monoid homomorphism. Assume that R is S-compatible. Then $gr(\overline{\Gamma}(R[[S,\omega]]))$ is either 3 or 4. In particular, if R is not reduced and S is a.n.u.p. monoid, then $gr(\overline{\Gamma}(R[[S,\omega]])) = 3$.*

Proof. Let $ab = 0$ for distinct elements $a, b \in Z^*(R)$. Using the S-compatibility of R, we find that $a\omega_s(b) = 0$ for all $s \in S \setminus \{1\}$. Hence $c_a - c_b - c_a e_s - c_b e_s -$

c_a is a 4-cycle in $R[[S,\omega]]$. Let $a^2 = 0$ for some $a \in Z^*(R)$. Then the S-compatibility of R and the hypothesis that $(S,\leq)$ is a.n.u.p. monoid, yields $c_a - c_a e_s - c_a e_{s^2} - c_a$ is a 3-cycle in $R[[S,\omega]]$, for all $s \in S \setminus \{1\}$. □

Theorem 3.22. *Let R be a ring, S a nontrivial a.n.u.p. monoid and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. If R is S-rigid and $\overline{\Gamma}(R)$ contains a cycle, then $gr(\overline{\Gamma}(R)) = gr(\overline{\Gamma}(R[[S,\omega]]))$.*

Proof. If $Z^*(R) = \emptyset$, then $gr(\overline{\Gamma}(R)) = \infty = gr(\overline{\Gamma}(R[[S,\omega]]))$. So we may assume $Z^*(R) \neq \emptyset$. Since the graph $\overline{\Gamma}(R)$ is an induced subgraph of $\overline{\Gamma}(R[[S,\omega]])$, we have that $gr(\overline{\Gamma}(R)) \geq gr(\overline{\Gamma}(R[[S,\omega]]))$. Also, by Proposition 3.21,

$$gr(\overline{\Gamma}(R[[S,\omega]])) \leq 4.$$

Furthermore, since $\overline{\Gamma}(R)$ contains a cycle, by [32], $gr(\overline{\Gamma}(R)) \leq 4$. So it suffices to show that $gr(\overline{\Gamma}(R)) = 3$, whenever $gr(\overline{\Gamma}(R[[S,\omega]])) = 3$. So suppose that $f - g - h - f$ is a cycle in $R[[S,\omega]]$. Since $fg = gh = hf = 0$ thus by [24, Theorem 4.12], we have $f(u)g(v) = g(v)h(w) = h(w)f(v) = 0$ for all $u, v, w \in S$. We may assume $f(u_0)$, $g(v_0)$ and $h(w_0)$ are non-zero elements in R. Therefore $f(u_0)g(v_0) = g(v_0)h(w_0) = h(w_0)f(v_0) = 0$. Moreover, the elements $f(u_0)$, $g(v_0)$ and $h(w_0)$ are distinct, since R is reduced, by Lemma 3.1. Now consider the cycle $f(u_0) - g(v_0) - h(w_0) - f(u_0)$ of length three in $\overline{\Gamma}(R)$. Therefore $gr(\overline{\Gamma}(R)) = 3$, and hence $gr(\overline{\Gamma}(R)) = gr(\overline{\Gamma}(R[[S,\omega]]))$, and the result follows. □

A complete characterization for the girth of $gr(\Gamma(R[x]))$ and $gr(\Gamma(R[[x]]))$ in terms of $gr(\Gamma(R))$ is given in [5, Theorem 3.2]. In the following we explain Theorem 3.2 in [5] in the context of skew generalized power series extension rings.

Corollary 3.23. *Let R be a ring, S a nontrivial a.n.u.p. monoid and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. Assume that R is S-compatible.*

(1) *Suppose that $\overline{\Gamma}(R)$ is nonempty with $gr(\overline{\Gamma}(R)) = \infty$.*
- (i) *If R is reduced, then $gr(\overline{\Gamma}(R[[S,\omega]])) = 4$;*
- (ii) *If R is not reduced, then $gr(\overline{\Gamma}(R[[S,\omega]])) = 3$.*

(2) *If $gr(\overline{\Gamma}(R)) = 3$, then $gr(\overline{\Gamma}(R[[S,\omega]])) = 3$.*

(3) *Suppose that $gr(\overline{\Gamma}(R)) = 4$.*
- (i) *If R is reduced, then $gr(\overline{\Gamma}(R[[S,\omega]])) = 4$;*
- (ii) *If R is not reduced, then $gr(\overline{\Gamma}(R[[S,\omega]])) = 3$.*

Proof. We have already observed in Proposition 3.21 that $gr(\overline{\Gamma}(R[[S,\omega]]))) = 3$ if R is not reduced. Thus (1)(ii) and (3)(ii) hold. By using the proof of Theorem 3.22, if R is reduced and $gr(\overline{\Gamma}(R[[S,\omega]])) = 3$, then $gr(\overline{\Gamma}(R)) = 3$. Now, since $gr(\overline{\Gamma}(R[[S,\omega]])) \leq 4$, by Proposition 3.21, and thus (1)(i) and (3)(i) hold. Clearly (2) holds since $gr(\overline{\Gamma}(R)) \geq gr(\overline{\Gamma}(R[[S,\omega]]))$. □

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