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EXTENDED HYPERGEOMETRIC FUNCTIONS OF TWO AND THREE VARIABLES

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Abstract. Extensions of some classical special functions, for example, Beta function $B(x,y)$ and generalized hypergeometric functions ${}_pF_q$ have been actively investigated and found diverse applications. In recent years, several extensions for $B(x,y)$ and ${}_pF_q$ have been established by many authors in various ways. Here, we aim to generalize Appell's hypergeometric functions of two variables and Lauricella's hypergeometric function of three variables by using the extended generalized beta type function $B_p^{(\alpha,\beta;m)}(x,y)$. Then some properties of the extended generalized Appell's hypergeometric functions and Lauricella's hypergeometric functions are investigated.

1. Introduction

Since last four decades, several extensions of the classical special functions such as Beta function $B(x,y)$ and generalized hypergeometric functions ${}_pF_q$ have been presented and their properties have been investigated by many authors (see, *e.g.*, [1, 2, 3, 4, 6, 7, 8]; see also, for very recent work, [5]). Motivated mainly by these works, here, we aim to extend Appell's hypergeometric functions of two variables and Lauricella's hypergeometric function of three variables by using the extended generalized beta type function $B_p^{(\alpha,\beta;m)}(x,y)$. Then some properties of the extended generalized Appell's hypergeometric functions and Lauricella's hypergeometric functions are investigated.

For our purpose, we begin by recalling some known functions and earlier works. Very recently, Lee *et al.* [4] introduced and studied some fundamental properties and characteristics of the more generalized Beta type function $B_p^{(\alpha,\beta;m)}(x,y)$ defined by (see [4, p. 189, Eq. (1.13)]):

$$B_p^{(\alpha,\beta;m)}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1-t)^m}\right)dt \tag{1.1}$$

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$$(\Re(p) > 0;\ \min\{\Re(x),\, \Re(y),\, \Re(\alpha),\, \Re(\beta)\} > 0 \quad \text{and} \quad \Re(m) > 0).$$

The special case of (1.1) when $m = 1$ is easily seen to reduce to the well-known generalized Beta type function defined by (see, *e.g.*, [8, p. 4602, Eq. (4)]; see also, [7, p. 32, Chapter 4]):

$$B_p^{(\alpha,\beta)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt \tag{1.2}$$

$$(\Re(p) > 0;\ \min\{\Re(x), \Re(y), \Re(\alpha), \Re(\beta)\} > 0).$$

The special case of (1.2) when $\alpha = \beta$ is also seen to reduce to the extended Beta type function due to Chaudhry *et al.* (see, *e.g.*, [2, p. 20, Eq. (1.7)]; see also [3, p. 591, Eq. (1.7)]) defined by

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\exp\left\{\frac{-p}{t(1-t)}\right\}dt \tag{1.3}$$

$$(\Re(p) > 0;\ \min\{\Re(x), \Re(y)\} > 0).$$

It is obvious that the well-known Euler's beta function $B(x,y)$ defined by

$$B(x,y) = \begin{cases} \int_0^1 t^{x-1}(1-t)^{y-1}dt & (\Re(x) > 0;\ \Re(y) > 0) \\ \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} & \left(x,\, y \in \mathbb{C} \setminus \mathbb{Z}_0^-\right) \end{cases} \tag{1.4}$$

which has the following relation:

$$B(x,y) = B_0(x,y) = B_0^{(\alpha,\beta)}(x,y) = B_0^{(\alpha,\beta;1)}(x,y).$$

Here Γ is the Eulerian Gamma function, $\mathbb{N}$, $\mathbb{C}$ and $\mathbb{Z}_0^-$ are the sets of positive integers, complex numbers and nonpositive integers, respectively.

In 2004, Chaudhry *et al.* used $B_p(x,y)$ in (1.3) to extend the Gauss hypergeometric function ${}_2F_1$ as follows (see [3, p. 591, Eqs. (2.1) and (2.2)]):

$$F_p(a,b;c;z) := \sum_{n=0}^{\infty}(a)_n\,\frac{B_p(b+n,c-b)}{B(b,c-b)}\,\frac{z^n}{n!} \tag{1.5}$$

$$(\Re(c) > \Re(b) > 0;\ \Re(p) \geq 0),$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [9, p. 2 and pp. 4–6]):

$$\begin{aligned}(\lambda)_n :&= \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases} \\ &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad \left(\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-\right).\end{aligned} \tag{1.6}$$

Recently, Özergin *et al.* appealed $B_p^{(\alpha,\beta)}(x,y)$ in (1.2) to introduce and investigate the following further extension of the Gauss hypergeometric function (see, *e.g.*, [8, p. 4606, Section 3]; see also [7, p. 39, Chapter 4]):

$$F_p^{(\alpha,\beta)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}\quad(|z|<1), \tag{1.7}$$

where $\min\{\Re(\alpha),\, \Re(\beta)\}>0;\ \Re(c)>\Re(b)>0$ and $\Re(p)\geq 0$. The extension in (1.7) proved to be (potentially) useful.

In the sequel to the above extensions, Lee *et al.* [4] used the generalized Beta function (1.1) to introduce and investigate a family of the following (potentially) useful Gauss hypergeometric functions (see [4, p. 197, Eq. (6.1)]):

$$F_p^{(\alpha,\beta;m)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta;m)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}\quad(|z|<1), \tag{1.8}$$

where $\min\{\Re(\alpha),\, \Re(\beta),\, \Re(m)\}>0;\ \Re(c)>\Re(b)>0$ and $\Re(p)\geq 0$. Then it is easy to see that

$$F_p^{(\alpha,\beta;1)}(a,b;c;z)=F_p^{(\alpha,\beta)}(a,b;c;z),$$
$$F_p^{(\alpha,\alpha;1)}(a,b;c;z)=F_p(a,b;c;z)$$

and

$$F_0^{(\alpha,\alpha;1)}(a,b;c;z)={}_2F_1(a,b;c;z),$$

where ${}_2F_1(\cdot)$ is a special case of the well-known generalized hypergeometric series ${}_pF_q(\cdot)$ defined by (see, *e.g.*, [9, Section 1.5]; see also [10]):

$$\begin{aligned}{}_pF_q\left[\begin{matrix}\alpha_1,\,\ldots,\,\alpha_p\,;\\ \beta_1,\,\ldots,\,\beta_q\,;\end{matrix}\ z\right]&=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}\\ &={}_pF_q(\alpha_1,\,\ldots,\,\alpha_p;\,\beta_1,\,\ldots,\,\beta_q;\,z).\end{aligned} \tag{1.9}$$

The above-mentioned detailed and systematic investigation was indeed motivated largely by the demonstrated potential for applications of the more generalized Gauss hypergeometric function $F_p^{(\alpha,\beta;m)}$ and their special cases in many diverse areas of mathematical, physical, engineering and statistical sciences (see, for details,[4] and the references cited therein). Motivated the works of Özergin and Özarslan, very recently, Liu and Wang [5] gave interesting generalizations of the Appell's hypergeometric functions and Lauricella's hypergeometric functions. In the present sequel to the aforementioned investigations, we present further extensions of Appell's hypergeometric functions and Lauricella's hypergeometric functions by using the generalized Beta type function $B_p^{(\alpha,\beta;m)}(x,y)$ in (1.1). Some interesting properties of these functions are also investigated.

2. Generalized hypergeometric functions of two and three variables and their integral representations

In this section, we *first* present further extensions of the first two Appell's hypergeometric functions of two variables

$$F_{1,p}^{\alpha,\beta;m}(a,b,c;d;x,y;m) \quad \text{and} \quad F_{2,p}^{\alpha,\beta;m}(a,b,c;d,e;x,y;m)$$

and the Lauricella's hypergeometric function of three variables

$$F_{D,p}^{3;\alpha,\beta;m}(a,b,c,d;e;x,y,z;m).$$

We *next* give certain integral representations of these extended functions.

Definition 1. We extend the Appell's hypergeometric functions of two variables $F_1(a,b,c;d;x,y)$ (see, *e.g.*, [10, p. 22, Eq. (2)]) as follows:

(2.1)
$$F_{1,p}^{(\alpha,\beta;m)}(a,b,c;d;x,y;m) := \sum_{r,s=0}^{\infty} \frac{B_p^{(\alpha,\beta;m)}\,(a+r+s,d-a)}{B\,(a,d-a)}(b)_r(c)_s\frac{x^r}{r!}\,\frac{y^s}{s!}$$
$$(\max\{|x|,\,|y|\}<1;\ \Re(p)\geq 0;\ \min\{\Re(\alpha),\,\Re(\beta),\,\Re(m)\}>0)\,.$$

Definition 2. We extend the Appell's hypergeometric functions of two variables $F_2(a,b,c;d,e;x,y)$ (see, *e.g.*, [10, p. 23, Eq. (3)]) as follows:

$$\begin{aligned} &F_{2,p}^{(\alpha,\beta,\alpha',\beta';m)}(a,b,c;d,e;x,y;m) \\ &:= \sum_{r,s=0}^{\infty} \frac{(a)_{r+s}B_p^{(\alpha,\beta;m)}\,(b+r,d-b)\ B_p^{(\alpha',\beta';m)}\,(c+s,e-c)}{B\,(b,d-b)\ B\,(c,e-c)}\frac{x^r}{r!}\,\frac{y^s}{s!} \end{aligned} \tag{2.2}$$

$$(|x|+|y|<1;\ \Re(p)\geq 0;\ \min\{\Re(\alpha),\,\Re(\beta),\Re(\alpha'),\,\Re(\beta'),\,\Re(m)\}>0)\,.$$

Definition 3. We extend the Lauricella's hypergeometric functions of three variables $F_D^{(3)}(a,b,c,d;e;x,y,z)$ (see, *cf.*, [10, p. 33, Eq. (4)]) as follows:

$$\begin{aligned} &F_{D,p}^{(3;\alpha,\beta;m)}(a,b,c,d;e;x,y,z;m) \\ &:= \sum_{r,s,t=0}^{\infty} \frac{B_p^{(\alpha,\beta;m)}\,(a+r+s+t,e-a)\ (b)_r\,(c)_s\,(d)_t}{B\,(a,e-a)}\frac{x^r}{r!}\,\frac{y^s}{s!}\,\frac{z^t}{t!} \end{aligned} \tag{2.3}$$

$$(\max\{|x|,|y|,|z|\}<1;\ \Re(p)\geq 0;\ \min\{\Re(\alpha),\,\Re(\beta),\,\Re(m)\}>0)\,.$$

Then it is obvious to see that

$$F_{1,p}^{(\alpha,\beta;0)}(a,b,c;d;x,y;0) = F_{1,p}^{(\alpha,\beta)}(a,b,c;d;x,y),$$
$$F_{2,p}^{(\alpha,\beta,\alpha',\beta';0)}(a,b,c;d,e;x,y;0) = F_{2,p}^{(\alpha,\beta,\alpha',\beta')}(a,b,c;d,e;x,y)$$

and

$$F_{D,p}^{(3;\alpha,\beta;0)}(a,b,c,d;e;x,y,z;0) = F_{D,p}^{(3;\alpha,\beta)}(a,b,c,d;e;x,y,z),$$

where $F_{1,p}^{(\alpha,\beta)}(a,b,c;d;x,y)$, $F_{2,p}^{(\alpha,\beta,\alpha',\beta')}(a,b,c;d,e;x,y)$ and $F_{D,p}^{(3;\alpha,\beta)}(a,b,c,d;e;x,y,z)$ are the generalizations of Appell's hypergeometric functions of two variables and Lauricella's hypergeometric functions of three variables, respectively, given by Liu and Wang [5].

Now we provide certain integral representations of our extended functions $F_{1,p}^{(\alpha,\beta;m)}(\cdot)$, $F_{2,p}^{(\alpha,\beta,\alpha',\beta';m)}(\cdot)$ and $F_{D,p}^{(3;\alpha,\beta;m)}(\cdot)$ as in the following theorems.

Theorem 1. *Each of the following integral representations of* $F_{1,p}^{(\alpha,\beta;m)}$ *holds true*:

$$\begin{aligned} F_{1,p}^{(\alpha,\beta;m)}(a,b,c;d;x,y;m) &= \frac{\Gamma(d)}{\Gamma(a)\,\Gamma(d-a)} \int_0^1 t^{a-1}\,(1-t)^{d-a-1} \\ &\times\,(1-xt)^{-b}\,(1-yt)^{-c}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t^m\,(1-t)^m}\right)dt; \end{aligned} \tag{2.4}$$

$$\begin{aligned} &F_{1,p}^{(\alpha,\beta;m)}(a,b,c;d;x,y;m) \\ &= \frac{1}{B(a,d-a)} \int_0^\infty u^{a-1}\,(1+u)^{b+c-d}\,(1+u(1-x))^{-b} \\ &\quad\times\,(1+u(1-y))^{-c}\,{}_1F_1\left(\alpha;\beta;-p\left(2+u+\frac{1}{u}\right)^m\right)du; \end{aligned} \tag{2.5}$$

$$\begin{aligned} F_{1,p}^{(\alpha,\beta;m)}(a,b,c;d;x,y;m) &= \frac{2}{B(a,d-a)} \int_0^{\frac{\pi}{2}} \sin^{2a-1}\theta\,\cos^{2d-2a-1}\theta \\ &\times (1-x\,\sin^2\theta)^{-b}\,(1-y\,\sin^2\theta)^{-c}\,{}_1F_1\left(\alpha;\beta;-p\,\csc^{2m}\theta\,\sec^{2m}\theta\right)d\theta, \end{aligned} \tag{2.6}$$

where $\max\{|x|,\,|y|\}<1$, $\Re(p)\ge 0$, *and* $\min\{\Re(\alpha),\,\Re(\beta),\,\Re(m)\}>0$.

Proof. For simplicity, let $\mathcal{I}$ denote the left-hand side of (2.4). Then, using (2.1) yields

$$\mathcal{I} \;=\; \sum_{r,s=0}^{\infty} \frac{B_p^{(\alpha,\beta;m)}\,(a+r+s,d-a)}{B\,(a,d-a)}(b)_r(c)_s\frac{x^r}{r!}\,\frac{y^s}{s!}. \tag{2.7}$$

By applying (1.1) to the integrand of (2.7), after a little simplification, we have

$$\begin{aligned} \mathcal{I} \;=\; \sum_{r,s=0}^{\infty} \int_0^1 t^{a+r+s-1}\,(1-t)^{d-a-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t^m\,(1-t)^m}\right) \\ \times\,\frac{(b)_r(c)_s}{B\,(a,d-a)}\,\frac{x^r}{r!}\,\frac{y^s}{s!}\,dt. \end{aligned} \tag{2.8}$$

Interchanging the order of summation and integration in (2.8), which is guaranteed, we get

$$\mathcal{I} = \frac{1}{B\,(a,d-a)} \int_0^1 t^{a-1}\,(1-t)^{d-a-1}$$

$$\times\, {}_1F_1\left(\alpha;\beta;\frac{-p}{t^m\,(1-t)^m}\right)\left\{\sum_{r=0}^{\infty}\frac{(b)_r\,(xt)^r}{r!}\right\}\left\{\sum_{s=0}^{\infty}\frac{(c)_s\,(yt)^s}{s!}\right\}dt,$$
$$=\frac{\Gamma(d)}{\Gamma(a)\,\Gamma(d-a)}\int_0^1 t^{a-1}\,(1-t)^{d-a-1}\,(1-xt)^{-b}\,(1-yt)^{-c}$$
$$\times\, {}_1F_1\left(\alpha;\beta;\frac{-p}{t^m\,(1-t)^m}\right)dt,$$

which proves the integral representation (2.4).

Replacing t by $\frac{u}{1+u}$ and setting $t=\sin^2\theta$ in (2.4) is, after a little simplification, seen to produce the integral representations in (2.5) and (2.6), respectively. This completes the proof of Theorem 1. □

To establish Theorem 2, we need to recall the following elementary series identity involving the bounded sequence of $\{f(N)\}_{N=0}^{\infty}$ stated in the lemma below (see [5]).

Lemma 1. *For a bounded sequence $\{f(N)\}_{N=0}^{\infty}$ of essentially arbitrary complex numbers, we have*

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!}=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f(r+s)\frac{x^r}{r!}\frac{y^s}{s!}. \tag{2.9}$$

Theorem 2. *Each of the following integral representations of $F_{2,p}^{(\alpha,\beta,\alpha',\beta';m)}$ holds true*:

$$F_{2,p}^{(\alpha,\beta,\alpha',\beta';m)}(a,b,c;d,e;x,y;m) = \frac{\Gamma(d)\,\Gamma(e)}{\Gamma(b)\,\Gamma(d-b)\,\Gamma(c)\,\Gamma(e-c)}$$
$$\times\int_0^1\int_0^1 t^{b-1}\,(1-t)^{d-b-1}\,w^{c-1}\,(1-w)^{e-c-1}\,(1-xt-yw)^{-a} \tag{2.10}$$
$$\times\, {}_1F_1\left(\alpha;\beta;\frac{-p}{t^m\,(1-t)^m}\right)\,{}_1F_1\left(\alpha';\beta';\frac{-p}{w^m\,(1-w)^m}\right)dt\,dw;$$

(2.11)
$$F_{2,p}^{(\alpha,\beta,\alpha',\beta';m)}(a,b,c;d,e;x,y;m)=\frac{1}{B(b,d-b)\,B(c,e-c)}\int_0^\infty\int_0^\infty u^{b-1}$$
$$\times\,(1+u)^{a-d}\,v^{c-1}\,(1+v)^{a-e}\,(1+u(1-x)+v(1-y)+uv(1-x-y))^{-a}$$
$$\times\, {}_1F_1\left(\alpha;\beta;-p\left(2+u+\frac{1}{u}\right)^m\right)\,{}_1F_1\left(\alpha';\beta';-p\left(2+v+\frac{1}{v}\right)^m\right)du\,dv;$$

(2.12)
$$F_{2,p}^{(\alpha,\beta,\alpha',\beta';m)}(a,b,c;d,e;x,y;m)=\frac{4}{B(b,d-b)\,B(c,e-c)}$$
$$\times\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin^{2b}\theta\,\cos^{2d-2b}\theta\,\sin^{2c}\phi\,\cos^{2e-2c}\phi\,(1-x\sin^2\theta-y\cos^2)^{-a}$$
$$\times\, {}_1F_1\left(\alpha;\beta;-p\,\csc^{2m}\theta\,\sec^{2m}\theta\right)\,{}_1F_1\left(\alpha';\beta';-p\,\csc^{2m}\phi\,\sec^{2m}\phi\right)d\theta\,d\phi,$$

where $|x| + |y| < 1$, $\Re(p) \geq 0$, *and* $\min\{\Re(\alpha), \Re(\beta), \Re(\alpha'), \Re(\beta'), \Re(m)\} > 0$.

Proof. Let $\mathcal{L}$ denote the left-hand side of (2.10). Then, using (2.1) yields

$$\mathcal{L} = \sum_{r,s=0}^{\infty} \frac{(a)_{r+s} B_p^{(\alpha,\beta;m)}(b+r, d-b)\ B_p^{(\alpha',\beta';m)}(c+s, e-c)}{B(b, d-b)\ B(c, e-c)} \frac{x^r}{r!} \frac{y^s}{s!}. \tag{2.13}$$

An application of (1.1) to (2.13), after a little simplification, gives

$$\begin{aligned}\mathcal{L} = &\frac{1}{B(b, d-b)\ B(c, e-c)} \\ &\times \sum_{r,s=0}^{\infty} \left\{ \int_0^1 t^{b+r-1}(1-t)^{d-b-1}\, {}_1F_1\left(\alpha; \beta; \frac{-p}{t^m (1-t)^m}\right) dt \right\} \\ &\times \left\{ \int_0^1 w^{c+s-1}(1-w)^{e-c-1}\, {}_1F_1\left(\alpha'; \beta'; \frac{-p}{w^m (1-w)^m}\right) dw \right\} (a)_{r+s} \frac{x^r}{r!} \frac{y^s}{s!}.\end{aligned} \tag{2.14}$$

Next interchanging the order of summation and integration in (2.14), which is guaranteed, yields

$$\begin{aligned}\mathcal{L} = &\frac{1}{B(b, d-b)\ B(c, e-c)} \left\{ \int_0^1 \int_0^1 t^{b-1} w^{c-1} (1-t)^{d-b-1} (1-w)^{e-c-1} \right. \\ &\times {}_1F_1\left(\alpha; \beta; \frac{-p}{t^m (1-t)^m}\right) {}_1F_1\left(\alpha'; \beta'; \frac{-p}{w^m (1-w)^m}\right) \\ &\left. \times \left(\sum_{r,s=0}^{\infty} (a)_{r+s} \frac{(xt)^r}{r!} \frac{(yw)^s}{s!} \right) dt\, dw \right\}.\end{aligned} \tag{2.15}$$

Finally, applying (2.9) to the double series in (2.15), we obtain the right-hand side of (2.10).

Setting $u = \frac{t}{1-t}$ and $v = \frac{w}{1-w}$ in (2.10), after a simplification, yields (2.11).

Similarly, taking $t = \sin^2\theta$ and $w = \sin^2\phi$ in (2.10) proves (2.11).

This completes the proof of Theorem 2. □

Theorem 3. *The following integral representation of* $F_{D,p}^{(3;\alpha,\beta;m)}$ *holds true*:

$$\begin{aligned}&F_{D,p}^{(3;\alpha,\beta;m)}(a, b, c, d; e; x, y, z; m) \\ &= \frac{\Gamma(e)}{\Gamma(a)\,\Gamma(e-a)} \int_0^1 t^{(a-1)} (1-t)^{e-a-1} (1-xt)^{-b} \\ &\quad \times (1-yt)^{-c} (1-zt)^{-d} \cdot {}_1F_1\left(\alpha; \beta; \frac{-p}{t^m (1-t)^m}\right) dt,\end{aligned} \tag{2.16}$$

where $\max\{|x|, |y|, |z|\} < 1$, $\Re(p) \geq 0$, and $\min\{\Re(\alpha), \Re(\beta), \Re(m)\} > 0$.

Proof. A similar argument in the proof of Theorem 1 will be able to establish the integral representation in (2.16). Therefore, details of the proof are omitted. □

3. Mellin transforms and further results

In this section, we obtain the Mellin transforms and some transformation formulas of the extended generalized Appell's hypergeometric functions and Lauricella's hypergeometric functions.

Theorem 4. *The Mellin transform of* $F_{1,p}^{(\alpha,\beta;m)}$ *is given as follows*:

$$\begin{aligned}&\mathfrak{M}\left[F_{1,p}^{(\alpha,\beta;m)}(a,b,c;d;x,y;m);s\right]\\ &= \frac{\Gamma(s)B\left(\alpha-s,s\right)B\left(a+ms+m-1,d-a+ms+m-1\right)}{B\left(\beta-s,s\right)B\left(a,d-a\right)}\\ &\quad\times F_1(a+ms+m-1,b,c;d+2ms+2m-2;x,y),\end{aligned} \tag{3.1}$$

where $\Re(s)\geq 0$, $\Re(m)\geq 0$ and $\Re(p)\geq 0$.

Proof. Taking Mellin transform for the function $F_{1,p}^{(\alpha,\beta;m)}$ gives

$$\begin{aligned}&\mathfrak{M}\left[F_{1,p}^{(\alpha,\beta;m)}(a,b,c;d;x,y;m);s\right] = \frac{\Gamma(d)}{\Gamma(a)\,\Gamma(d-a)}\int_0^\infty p^{s-1}\left\{\int_0^1 t^{a-1}\right.\\ &\quad\left.\times\left(1-t\right)^{d-a-1}\left(1-xt\right)^{-b}\left(1-yt\right)^{-c}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^m\left(1-t\right)^m}\right)dt\right\}dp\\ &= \frac{\Gamma(d)}{\Gamma(a)\,\Gamma(d-a)}\int_0^1 t^{a-1}\left(1-t\right)^{d-a-1}\left(1-xt\right)^{-b}\left(1-yt\right)^{-c}\\ &\quad\times\left\{\int_0^\infty p^{s-1}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^m\left(1-t\right)^m}\right)dp\right\}dt.\end{aligned} \tag{3.2}$$

Applying the following formula (see [5])

$$\int_0^\infty b^{s-1}\,{}_1F_1\left(\alpha;\beta;-b\right)db = \frac{\Gamma(\beta)\Gamma(\alpha-s)\Gamma(s)}{\Gamma(\beta-s)\Gamma(\alpha)}, \tag{3.3}$$

to the last integral in (3.2), we get

$$\begin{aligned}&\mathfrak{M}\left[F_{1,p}^{(\alpha,\beta;m)}(a,b,c;d,e;x,y;m);s\right] = \frac{\Gamma(d)\Gamma(\beta)\Gamma(\alpha-s)\Gamma(s)}{\Gamma(a)\,\Gamma(d-a)\Gamma(\beta-s)\Gamma(\alpha)}\\ &\quad\times\int_0^1 t^{a+ms+m-2}\left(1-t\right)^{d+ms+m-a-2}\left(1-xt\right)^{-b}\left(1-yt\right)^{-c}dt\\ &= \frac{\Gamma(s)B\left(\alpha-s,s\right)B\left(a+ms+m-1,d-a+ms+m-1\right)}{B\left(\beta-s,s\right)B\left(a,d-a\right)}\\ &\quad\times F_1(a+ms+m-1,b,c;d+2ms+2m-2;x,y),\end{aligned}$$

which proves the formula (3.2). □

Theorem 5. *The Mellin transform of* $F_{2,p}^{(\alpha,\beta,\gamma,\gamma;m)}$ *is given as follows*:

$$\begin{aligned}
&\mathfrak{M}\left[F_{2,p}^{(\alpha,\beta,\gamma,\gamma;m)}(a,b,c;d,e;x,y;m);s\right] \\
&= \Gamma(s)\sum_{n=0}^{\infty}\frac{(\alpha)_n(s)_n}{n!(-1)^n(\beta)_n B\left(b,d-b\right)B\left(c,e-c\right)} \\
&\quad\times B\left(b-mn+2ms+m-1,d-b-mn+2ms+m-1\right) \\
&\quad\times B\left(c-ms+mn,e-c-ms+mn\right) \\
&\quad\times F_2(a,b-mn+2ms+m-1,c-ms+mn; \\
&\qquad\qquad d+4ms-2mn+2m-2,e-2ms+2mn;x,y),
\end{aligned} \tag{3.4}$$

where $\Re(s)\geq 0$*,* $\Re(m)\geq 0$ and $\Re(p)\geq 0$.

Proof. Taking Mellin transform of the integral representation (2.10), we obtain

$$\begin{aligned}
&\mathfrak{M}\left[F_{2,p}^{(\alpha,\beta,\gamma,\gamma;m)}(a,b,c;d,e;x,y;m);s\right] \\
&= \int_0^1\int_0^1\frac{t^{b-1}(1-t)^{d-b-1}w^{c-1}(1-w)^{e-c-1}}{(1-xt-yw)^a B(b,d-b)B(c,e-c)} \\
&\quad\times\left\{\int_0^{\infty}p^{s-1}e^{\left(\frac{-p}{w^m(1-w)^m}\right)}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1-t)^m}\right)dp\right\}dtdw.
\end{aligned} \tag{3.5}$$

Using the known result

$$\int_0^{\infty}t^{s-1}\,e^{-ct}\,{}_1F_1\left(a;b;-t\right)dt = c^{-s}\Gamma(s)\,{}_2F_1\left(a,s;b;-\frac{1}{c}\right), \tag{3.6}$$

in (3.5) and setting $u=\frac{-p}{t^m(1-t)^m}$ in the resulting identity, we have

$$\begin{aligned}
&\mathfrak{M}\left[F_{2,p}^{(\alpha,\beta;m)}(a,b,c;d,e;x,y;m);s\right] \\
&= \frac{\Gamma(s)}{B\left(b,d-b\right)B\left(c,e-c\right)}\int_0^1\int_0^1 t^{b+2ms+m-2}\left(1-t\right)^{d-b+2ms+m-2}w^{c+ms-1} \\
&\quad\times\left(1-w\right)^{e-c+ms-1}\left(1-xt-yw\right)^{-a}\,{}_2F_1\left(\alpha,s;\beta;-\frac{w^m(1-w)^m}{t^m(1-t)^m}\right)dtdw \\
&= \frac{\Gamma(s)}{B\left(b,d-b\right)B\left(c,e-c\right)}\sum_{n=0}^{\infty}\frac{(\alpha)_n\,(s)_n}{(-1)^n(\beta)_n n!} \\
&\quad\times\left\{\int_0^1\int_0^1 t^{b-mn+2ms+m-2}\left(1-t\right)^{d-b-mn+2ms+m-2}\right. \\
&\qquad\left.\times\, w^{c-ms+mn-1}\left(1-w\right)^{e-c-ms+mn-1}\left(1-xt-yw\right)^{-a}dtdw\right\}.
\end{aligned}$$

Finally, by using definition of F_2, we get the desired result. □

Considering the parameters b and d are symmetric to c and e in (2.11), we get the following Mellin transform of $F_{2,p}^{(\gamma,\gamma,\alpha',\beta';m)}(a,b,c;d;x,y;m)$ given in Theorem 6.

Theorem 6. *The following Mellin transform holds true*:

$$\begin{aligned}
&\mathfrak{M}\left[F_{2,p}^{(\gamma,\gamma,\alpha,\beta;m)}(a,b,c;d,e;x,y;m);s\right]\\
&=\Gamma(s)\sum_{n=0}^{\infty}\frac{(\alpha)_n(s)_n}{n!(-1)^n(\beta)_n B\left(d,b-d\right)B\left(e,c-e\right)}\\
&\quad\times B\left(d-mn+2ms+m-1,b-d-mn+2ms+m-1\right)\\
&\quad\times B\left(e-ms+mn,c-e-ms+mn\right)\\
&\quad\times F_2(a,d-mn+2ms+m-1,e-ms+mn;\\
&\qquad\qquad b+4ms-2mn+2m-2,c-2ms+2mn;x,y),
\end{aligned}\tag{3.7}$$

where $\Re(s)\geq 0$, $\Re(m)\geq 0$ *and* $\Re(p)\geq 0$.

Proof. A similar argument as in the proof of Theorem 5 will be able to establish (3.7). So details of the proof are omitted. □

Theorem 7. *The following Mellin transform holds true*:

$$\begin{aligned}
&\mathfrak{M}\left[F_{D,p}^{(3;\alpha,\beta;m)}(a,b,c,d;e;x,y,z;m)\,;s\right]\\
&=\Gamma(s)\frac{B\left(\alpha-s,s\right)B\left(a+ms,e-a+ms\right)}{B\left(\beta-s,s\right)B\left(a,e-a\right)}\\
&\quad\times F_D^{(3)}(a+ms,b,c,d;e+2ms;x,y,z),
\end{aligned}\tag{3.8}$$

where $\Re(s)\geq 0$, $\Re(m)\geq 0$ *and* $\Re(p)\geq 0$.

Proof. The proof will be able to proceed as in the proof of Theorems 4 and 5. So details of the proof are omitted. □

The following three theorems are concerned with certain transformation formulas involving extended hypergeometric functions of two and three variables.

Theorem 8. *A transformation formula for the extended Appell's hypergeometric functions* $F_{1,p}^{(\alpha,\beta;m)}(a,b,c;d;x,y;m)$ *is given as follows*:

$$\begin{aligned}
&F_{1,p}^{(\alpha,\beta;m)}(a,b,c;d;x,y;m)\\
&=(1-x)^{-b}\,(1-y)^{-c}F_{1,p}^{(\alpha,\beta;m)}(d-a,b,c;d;\frac{x}{1-x},\frac{y}{1-y};p;m).
\end{aligned}\tag{3.9}$$

Proof. Setting $t = 1 - u$ in (2.4), we have

$$\begin{aligned}
&F_{1,p}^{(\alpha,\beta;m)}(a,b,c;d;x,y;m)\\
&= \frac{\Gamma(d)}{\Gamma(a)\,\Gamma(d-a)} \int_0^1 (1-u)^{a-1}\, u^{d-a-1}\, (1-x+ux)^{-b}\, (1-y+yu)^{-c}\\
&\qquad \times {}_1F_1\left(\alpha;\beta;\frac{-p}{u^m\,(1-u)^m}\right) du\\
&= (1-x)^{-b}\,(1-y)^{-c}\frac{\Gamma(d)}{\Gamma(a)\,\Gamma(d-a)} \int_0^1 (1-u)^{a-1}\, u^{d-a-1}\, (1-\frac{ux}{x-1})^{-b}\\
&\qquad \times (1-\frac{yu}{y-1})^{-c}\, {}_1F_1\left(\alpha;\beta;\frac{-p}{u^m\,(1-u)^m}\right) du.
\end{aligned}$$

Using a representation in Theorem 1 proves (3.9). □

Theorem 9. *Each of the following transformation formulas for the extended Appell's hypergeometric functions* $F_{2,p}^{(\alpha,\beta;m)}(a,b,c;d,e;x,y;m)$ *holds true*:

$$\begin{aligned}
&F_{2,p}^{(\alpha,\beta,\alpha',\beta';m)}(a,b,c;d,e;x,y;m)\\
&= (1-y)^{-a} F_{2,p}^{(\alpha,\beta,\alpha',\beta';m)}(a,b,e-c;d,e;\frac{x}{1-y},\frac{y}{y-1};p;m);
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&F_{2,p}^{(\alpha,\beta,\alpha',\beta';m)}(a,b,c;d,e;x,y;m)\\
&= (1-x)^{-a} F_{2,p}^{(\alpha,\beta,\alpha',\beta';m)}(a,d-b,c;d,e;\frac{x}{x-1},\frac{y}{1-x};p;m);
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&F_{2,p}^{(\alpha,\beta,\alpha',\beta';m)}(a,b,c;d,e;x,y;m) = (1-x-y)^{-a}\\
&\times F_{2,p}^{(\alpha,\beta,\alpha',\beta';m)}(a,d-b,e-c;d,e;\frac{x}{x+y-1},\frac{y}{x+y-1};p;m).
\end{aligned} \tag{3.12}$$

Proof. Setting $w = 1 - v$ and $t = 1 - u$ in (2.10), respectively, after a little simplification, is seen to yield (3.10) and (3.11). Setting *both* $w = 1 - v$ *and* $t = 1 - u$ in (2.10), after a little simplification, proves (3.12). □

Theorem 10. *The following transformation formula for the extended Lauricella's hypergeometric functions* $F_{D,p}^{(3;\alpha,\beta;m)}(a,b,c,d;e;x,y,z;m)$ *holds true*:

$$\begin{aligned}
&F_{D,p}^{(3;\alpha,\beta;m)}(a,b,c,d;e;x,y,z;m) = (1-x)^{-b}\,(1-y)^{-c}\,(1-z)^{-d}\\
&\times F_{D,p}^{(3;\alpha,\beta;m)}\left(e-a,b,c,d;e;\frac{x}{x-1},\frac{y}{y-1},\frac{z}{z-1};m\right).
\end{aligned} \tag{3.13}$$

Proof. Setting $t = 1 - u$ in (2.16), after some algebra, yields (3.13). □

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