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WEAK AND STRONG FORMS OF sT-CONTINUOUS FUNCTIONS

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Abstract. The aim of this paper is to present some properties of sT-continuous functions. Moreover, we obtain a characterization and preserving theorems of semi-compact, S-closed and s-closed spaces.

1. Introduction

The study of semi-open sets and semi-continuity in topological spaces was initiated by Levine [10]. In 2009, Noiri et al. [13] defined the notion T-open sets and deduced some results. Quite recently, Al-omari et al. [1] have obtained some properties of T-open sets and characterizations of S-closed spaces. In this paper, we present some properties of sT-continuous functions. Moreover, we obtain characterizations and preserving theorems of semi-compact, S-closed and s-closed spaces.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) stand for topological spaces on which no separation axiom is assumed unless otherwise stated. For a subset A of X, the closure of A and the interior of A will be denoted by $Cl(A)$ and $Int(A)$, respectively. Let (X, τ) be a space and S a subset of X. A subset S of X is said to be semi-open [10] if there exists an open set U of X such that $U \subseteq S \subseteq Cl(U)$, or equivalently if $S \subseteq Cl(Int(S))$. The complement of a semi-open set is said to be semi-closed. The intersection of all semi-closed sets containing S is called the semi-closure of S and is denoted by $sCl(S)$. The semi-interior of S, denoted by $sInt(S)$, is defined by the union of all semi-open sets contained in S. It is verified in [2] that $sCl(A) = A \cup Int(Cl(A))$ and $sInt(A) = A \cap Cl(Int(A))$ for any subset $A \subseteq X$. A point $x \in X$ is said to be in the θ-semiclosure of A, denoted by $x \in \theta\text{-}sCl(A)$, if $A \cap Cl(V) \neq \phi$ for each semi-open set V containing x. A subset $A \subseteq X$ is said to be θ-semiclosed [8] if $A = \theta\text{-}sCl(A)$. The complement of a θ-semiclosed set is called a θ-semiopen

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set. The family of all semi-open sets of X is denoted by $SO(X)$. Moreover, for each $x \in X$ the family $\{U \in SO(X) : x \in U\}$ is denoted by $SO(X, x)$.

3. Weakly-sT-continuous functions

In this section, we obtain properties of weakly-sT-continuous functions.

Definition 3.1. A subset A of a space X is said to be T-open [13] if for every $x \in A$, there exists a semi-open subset $U_x \subseteq X$ containing x such that $U_x - A$ is finite. The complement of a T-open subset is said to be T-closed.

The family of all T-open (resp. regular closed) subsets of a space (X, τ) is denoted by $TO(X)$ (resp. $RC(X)$). The intersection of all T-closed sets of X containing A is called the T-closure of A and is denoted by $tCl(A)$. And the union of all T-open sets of X contained in A is called the T-interior and is denoted by $tInt(A)$.

Definition 3.2 ([7])**.** A function $f : X \to Y$ is said to be weakly θ-irresolute if for each $x \in X$ and each semi-open set V of Y containing $f(x)$, there exists $U \in SO(X, x)$ such that $f(U) \subseteq Cl(V)$.

Definition 3.3 ([13])**.** A function $f : X \to Y$ is said to be weakly-sT-continuous if for each $x \in X$ and each semi-open set V of Y containing $f(x)$, there exists $U \in TO(X, x)$ such that $f(U) \subseteq Cl(V)$.

Lemma 3.4 ([7])**.** *Let A be a subset of (X, τ). Then A is θ-semiclosed* (*resp. θ-semiopen*) *if and only if A is the intersection* (*resp. union*) *of a family of regular open* (*resp. regular closed*) *sets. In particular, any regular open* (*resp. regular closed*) *set is θ-semiclosed* (*resp. θ-semiopen*)*.*

Theorem 3.5. *The following are equivalent for a function $f : X \to Y$:*

(1) *f is weakly-sT-continuous*;
(2) *$f^{-1}(V) \subseteq tInt(f^{-1}(Cl(V)))$ for every $V \in SO(Y)$*;
(3) *the inverse image of a regular closed set of Y is T-open*;
(4) *the inverse image of a regular open set of Y is T-closed*;
(5) *the inverse image of a θ-semi-open set of Y is T-open*;
(6) *the inverse image of a θ-semi-closed set of Y is T-closed.*

Proof. (1) $\Rightarrow$ (2): Let $V \in SO(Y)$ and $x \in f^{-1}(V)$. Then $f(x) \in V$. Since f is weakly-sT-continuous, there exists a T-open set U in X containing x such that $f(U) \subseteq Cl(V)$. It follows that $x \in U \subseteq f^{-1}(Cl(V))$. Hence $x \in tInt(f^{-1}(Cl(V)))$. We have $f^{-1}(V) \subseteq tInt(f^{-1}(Cl(V)))$.

(2) $\Rightarrow$ (3): Let F be any regular closed set of Y. Since $F \in SO(Y)$, then by (2) it follows that $f^{-1}(F) \subseteq tInt(f^{-1}(Cl(F))) = tInt(f^{-1}(F))$. This shows that $f^{-1}(F)$ is T-open in X.

(3) $\Rightarrow$ (4): Let A be any regular open set of Y. Since $Y - A$ is regular closed, by (3) it follows that $f^{-1}(Y - A)$ is T-open and hence $f^{-1}(A)$ is T-closed in X.

(4) $\Rightarrow$ (5): This follows from the fact any θ-semi-open set is a union of regular closed sets.

(5) $\Rightarrow$ (6): Let A be any θ-semi-closed set of Y. Since $Y - A$ is θ-semi-open, by (5) it follows that $f^{-1}(Y - A)$ is T-open and hence $f^{-1}(A)$ is T-closed in X.

(6) $\Rightarrow$ (1): Let $x \in X$ and $V \in SO(Y, f(x))$. Then $Y - Cl(V)$ is θ-semi-closed in Y. Set $X - U = f^{-1}(Y - Cl(V))$, then U is T-open and $f(U) \subseteq Cl(V)$. Therefore f is weakly-sT-continuous. □

Recall that a space (X, τ) is said to be s-Urysohn [7] (resp. t-T_2) if for each pair x, y of distinct points in X, there exist $U, V \in SO(X)$ (resp. $U, V \in TO(X)$) such that $x \in U$, $y \in V$ and $Cl(U) \cap Cl(V) = \phi$ (resp. $U \cap V = \phi$).

Proposition 3.6. *If (Y, σ) is s-Urysohn and $f : (X, \tau) \to (Y, \sigma)$ is a weakly-sT-continuous injection, then (X, τ) is t-T_2.*

Proof. Let $x, y \in X$ and $x \neq y$. Then there exist $U, V \in SO(Y)$ such that $f(x) \in U$, $f(y) \in V$ and $Cl(U) \cap Cl(V) = \phi$. Since $Cl(U), Cl(V) \in RC(Y)$, by Theorem 3.5 $f^{-1}(Cl(U))$ and $f^{-1}(Cl(V))$ are disjoint T-open sets containing x and y, respectively. □

Definition 3.7. A topological space X is said to be S-closed [16] if for every semi-open cover $\{U_\alpha : \alpha \in \Lambda\}$ of X there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $X = \cup\{Cl(U_\alpha) : \alpha \in \Lambda_0\}$.

Recall that a space (X, τ) is said to be almost regular [14] if for each $F \in RC(X)$ and each point $x \notin F$, there exist $U, V \in \tau$ such that $x \in U$, $F \subseteq V$ and $U \cap V = \phi$. Cameron [3] has showed that (X, τ) is S-closed if and only if every cover of (X, τ) consisting of regular closed sets contains a finite subcover.

Corollary 3.8 ([1])**.** *For any space X, the following properties are equivalent:*

(1) *X is S-closed;*
(2) *For each T-open cover $\{U_\alpha : \alpha \in \Lambda\}$ of X, there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $X = \cup\{Cl(U_\alpha) : \alpha \in \Lambda_0\}$.*

Proposition 3.9 ([1])**.** *If a topological space X is a T_1-space, then every nonempty T-open set contains a nonempty semi-open set.*

Lemma 3.10 ([11])**.** *If A is a non-empty semi-open set, then $Int(A) \neq \phi$.*

Theorem 3.11. *Let $f : (X, \tau) \to (Y, \sigma)$ be a weakly-sT-continuous surjection and let (Y, σ) be almost regular and X a T_1-space. If (X, τ) is S-closed, then (Y, σ) is S-closed.*

Proof. Let $\{F_i : i \in I\}$ be a cover of (Y, σ), where $F_i \in RC(Y)$ for each $i \in I$. Then $\{f^{-1}(F_i) : i \in I\}$ is a T-open cover of (X, τ). By Corollary 3.8, there exists a finite subset $I_0 \subseteq I$ such that $X = \cup\{Cl(f^{-1}(F_i)) : i \in I_0\}$. We claim that $Y = \cup\{F_i : i \in I_0\}$. Suppose there exists $y \in Y - \cup\{F_i : i \in I_0\}$. Since (Y, σ) be almost regular and $\cup\{F_i : i \in I_0\} \in RC(Y)$, there exist $V, W \in \sigma$

with $y \in V$, $\cup\{F_i : i \in I_0\} \subseteq W$ and $V \cap W = \phi$. Hence $Cl(V) \cap W = \phi$ and $f^{-1}(Cl(V))$ is nonempty and T-open. Since X is T_1-space, every nonempty T-open set contains a nonempty open set by Proposition 3.9 and Lemma 3.10. This is contrary to the fact that $\cup\{f^{-1}(F_i) : i \in I_0\}$ is dense in (X, τ). Thus $Y = \cup\{F_i : i \in I_0\}$. □

Proposition 3.12. *If f, $g : X \to Y$ are weakly-sT-continuous, X is an extremally disconnected space and Y is s-Urysohn, then $E = \{x \in X : f(x) = g(x)\}$ is T-closed in X.*

Proof. If $x \in X - E$, then it follows that $f(x) \neq g(x)$. Since Y is s-Urysohn, there exist $V \in SO(Y, f(x))$ and $W \in SO(Y, g(x))$ such that $Cl(V) \cap Cl(W) = \phi$. Since f and g are weakly-sT-continuous, there exist $U \in TO(X, x)$ and $G \in TO(X, x)$ such that $f(U) \subseteq Cl(V)$ and $g(G) \subseteq Cl(W)$. Set $D = U \cap G$. Then D is T-open in X [1, Theorem 2.8] since X is extremally disconnected. Therefore $D \cap E = \phi$ and $D \cap tCl(E) = \phi$. It follows $tCl(E) \subseteq E$. This shows that E is T-closed in X. □

Proposition 3.13. *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function of f defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is weakly-sT-continuous, then f is weakly-sT-continuous.*

Proof. Let $F \in RC(Y)$. Then $X \times F = X \times Cl(Int(F)) = Cl(Int(X)) \times Cl(Int(F)) = Cl(Int(X \times F))$. Therefore $X \times F \in RC(X \times Y)$. It follows from Theorem 3.5 that $f^{-1}(F) = g^{-1}(X \times F)$ is T-open in X. Thus, f is weakly-sT-continuous. □

Definition 3.14 ([5])**.** A function $f : X \to Y$ is said to be irresolute if $f^{-1}(V)$ is semi-open in X for each semi-open set V in Y.

Proposition 3.15 ([13])**.** *If $f : X \to Y$ is irresolute injective and A is T-open in Y, then $f^{-1}(A)$ is T-open in X.*

Definition 3.16 ([9])**.** A function $f : X \to Y$ is said to be θ-irresolute if for each $x \in X$ and each $V \in SO(Y, f(x))$, there exists $U \in SO(X, x)$ such that $f(Cl(U)) \subseteq Cl(V)$.

Proposition 3.17. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the following hold*:

(1) *If f is irresolute and g is weakly-sT-continuous, then $g \circ f : X \to Z$ is weakly-sT-continuous.*
(2) *If f is weakly-sT-continuous and g is θ-irresolute, then $g \circ f : X \to Z$ is weakly-sT-continuous.*

Proof. (1) Let $x \in X$ and W be a semi-open set in Z containing $(g \circ f)(x)$. Since g is weakly-sT-continuous, there exists $V \in TO(Y, f(x))$ such that $g(V) \subseteq Cl(W)$. Since f is irresolute, by Proposition 3.15 there exists $U \in TO(X, x)$

such that $f(U) \subseteq V$. This shows that $(g \circ f)(U) \subseteq Cl(W)$. Therefore, $g \circ f$ is weakly-sT-continuous.

(2) Let $x \in X$ and W be a semi-open set in Z containing $(g \circ f)(x)$. Since g is θ-irresolute, there exists $V \in SO(Y, f(x))$ such that $g(Cl(V)) \subseteq Cl(W)$. Since f is weakly-sT-continuous, there exists $U \in TO(X, x)$ such that $f(U) \subseteq Cl(V)$. Therefore, we have $(g \circ f)(U) \subseteq Cl(W)$. This shows that $g \circ f$ is weakly-sT-continuous. □

Definition 3.18 ([5]). A function $f : X \to Y$ is said to be pre-semi-open if $f(V)$ is semi-open in Y for each semi-open set V in X.

Lemma 3.19 ([13]). *If $f : X \to Y$ is pre-semi-open, then the image of a T-open set of X is T-open in Y.*

Proposition 3.20. *Let $f : X \to Y$ be a surjective pre semi-open function and $g : Y \to Z$ a function such that $g \circ f : X \to Z$ is weakly-sT-continuous, then g is weakly-sT-continuous.*

Proof. Suppose that y is any point of Y. Thus, since f is surjective, there exists a point $x \in X$ such that $f(x) = y$. Let $W \in SO(Z, (g \circ f)(x))$. Then there exists $U \in TO(X, x)$ such that $g(f(U)) \subseteq Cl(W)$. Since f is pre-semi-open, then by Lemma 3.19 $f(U) \in TO(Y, y)$ such that $g(f(U)) \subseteq Cl(W)$. This implies that g is weakly-sT-continuous. □

Lemma 3.21 ([13]). *Let (X, τ) be a topological space. Then the intersection of an α-open set and a T-open set is T-open.*

Lemma 3.22 ([1]). *Let A and X_0 be subsets of X such that $A \subseteq X_0$ and $X_0 \in \alpha O(X)$. Then $A \in TO(X)$ if and only if $A \in TO(X_0)$.*

Proposition 3.23. *If $f : X \to Y$ is weakly-sT-continuous and X_0 is an α-open set in X, then the restriction $f_{|X_0} : X_0 \to Y$ is weakly-sT-continuous.*

Proof. Since f is weakly-sT-continuous, for any regular closed set V in Y, $f^{-1}(V)$ is T-open in X. Hence by Lemma 3.21, $f^{-1}(V) \cap X_0$ is T-open in X. Therefore, by Lemma 3.22, $f_{|X_0}(V) = f^{-1}(V) \cap X_0$ is T-open in X_0. This implies that $f_{|X_0}$ is weakly-sT-continuous. □

A subset S of X is said to be S-closed relative to X [12] if for every cover $\{U_\alpha : \alpha \in \Lambda\}$ of S by semi-open sets in X, there exists a finite subfamily Λ_0 of Λ such that $S \subseteq \cup\{Cl(U_\alpha) : \alpha \in \Lambda_0\}$.

Definition 3.24. (1) [4] A space X is said to be semi-compact if every semi-open cover of X has a finite subcover.

(2) A subset A of a space X is said to be semi-compact relative to X if every cover of A by semi-open sets of X has a finite subcover.

Theorem 3.25. *A subset A of a space X is semi-compact relative to X if and only if for any cover $\{V_\alpha : \alpha \in \Lambda\}$ of A by T-open sets of X, there exists a finite subset Λ_0 of Λ such that $A \subseteq \cup\{V_\alpha : \alpha \in \Lambda_0\}$.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of A and $V_\alpha \in TO(X)$. For each $x \in A$, there exists $\alpha(x) \in \Lambda$ such that $x \in V_{\alpha(x)}$. Since $V_{\alpha(x)}$ is T-open, there exists a semi-open set $U_{\alpha(x)} \in SO(X)$ such that $x \in U_{\alpha(x)}$ and $U_{\alpha(x)} \backslash V_{\alpha(x)}$ is finite. The family $\{U_{\alpha(x)} : x \in A\}$ is a semi-open cover of A and A is semi-compact relative to X. There exists a finite subset, says, $x_1, x_2, \ldots, x_n$ such that $A \subseteq \cup\{U_{\alpha(x_i)} : i \in F\}$ where $F = \{1, 2, \ldots, n\}$. Now, we have

$$\begin{aligned} A &\subseteq \underset{i\in F}{\cup} \big((U_{\alpha(x_i)} \backslash V_{\alpha(x_i)}) \cup V_{\alpha(x_i)}\big) \\ &= \left(\underset{i\in F}{\cup}(U_{\alpha(x_i)} \backslash V_{\alpha(x_i)})\right) \cup \left(\underset{i\in F}{\cup} V_{\alpha(x_i)}\right). \end{aligned}$$

For each $\alpha(x_i)$, $U_{\alpha(x_i)} \backslash V_{\alpha(x_i)}$ is a finite set and there exists a finite subset $\Lambda_{\alpha(x_i)}$ of Λ such that $(U_{\alpha(x_i)} \backslash V_{\alpha(x_i)}) \cap A \subseteq \cup\{V_\alpha : \alpha \in \Lambda_{\alpha(x_i)}\}$. Therefore, we have $A \subseteq \left(\underset{i\in F}{\cup} \big(\cup \big(V_\alpha : \alpha \in \Lambda_{\alpha(x_i)}\big)\big)\right) \cup \left(\underset{i\in F}{\cup} V_{\alpha(x_i)}\right)$. Hence A is semi-compact relative to X.

Conversely since every semi-open set is T-open, the proof is obvious. □

Theorem 3.26. *If $f : X \to Y$ is weakly-sT-continuous and A is semi-compact relative to X, then $f(A)$ is S-closed relative to Y.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of $f(A)$ by semi-open sets of Y. Since f is weakly-sT-continuous and $Cl(V_\alpha)$ is regular closed in Y, $f^{-1}(Cl(V_\alpha))$ is T-open in X and $\{f^{-1}(Cl(V_\alpha)) : \alpha \in \Lambda\}$ is a cover of A by T-open sets of X. By Theorem 3.25, there exists a finite subset Λ_0 of Λ such that $A \subseteq \cup\{f^{-1}(Cl(V_\alpha)) : \alpha \in \Lambda_0\}$. Therefore, we obtain $f(A) \subseteq \cup\{Cl(V_\alpha) : \alpha \in \Lambda_0\}$. This shows that $f(A)$ is S-closed relative to Y. □

Corollary 3.27. *Let $f : X \to Y$ be a weakly-sT-continuous surjection. If X is semi-compact, then Y is S-closed.*

4. *T*-closed graphs

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\} \subseteq X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 4.1. The graph $G(f)$ of a function $f : X \to Y$ is said to be T-closed if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in TO(X, x)$ and $V \in SO(Y, y)$ such that $[U \times Cl(V)] \cap G(f) = \phi$.

Lemma 4.2. *The following properties are equivalent for the graph $G(f)$ of a function $f : X \to Y$:*

(1) *The graph $G(f)$ is T-closed in $X \times Y$;*
(2) *For each point $(x, y) \in (X \times Y) - G(f)$, there exist $U \in TO(X, x)$ and $V \in SO(Y, y)$ such that $f(U) \cap Cl(V) = \phi$;*
(3) *For each point $(x, y) \in (X \times Y) - G(f)$, there exist $U \in TO(X, x)$ and $F \in RC(Y, y)$ such that $f(U) \cap F = \phi$.*

Proof. The proof follows immediately from Definition 4.1. □

Proposition 4.3. *If $f : X \to Y$ is weakly-sT-continuous and Y is s-Urysohn, $G(f)$ is T-closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. It follows that $f(x) \neq y$. Since Y is s-Urysohn, there exist $V \in SO(Y, f(x))$ and $W \in SO(Y, y)$ such that $Cl(V) \cap Cl(W) = \phi$. Since f is weakly-sT-continuous, there exists $U \in TO(X, x)$ such that $f(U) \subseteq Cl(V)$. Therefore $f(U) \cap Cl(W) = \phi$ and by Lemma 4.2, $G(f)$ is T-closed in $X \times Y$. □

A space X is weakly Hausdorff [15] if each point of X is an intersection of regular closed sets of X.

Proposition 4.4. *If $f : X \to Y$ is surjective and $G(f)$ is T-closed, then Y is weakly Hausdorff.*

Proof. Let y_1 and y_2 be any distinct points of Y. Since f is surjective, $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y) - G(f)$. By Lemma 4.2, there exist $U \in TO(X, x)$ and $F \in RC(Y, y_2)$ such that $f(U) \cap F = \phi$, hence $y_1 \notin F$. This implies that Y is weakly Hausdorff. □

Theorem 4.5. *Let X be an extremally disconnected topological space. If a function $f : X \to Y$ has a T-closed graph, then $f^{-1}(S)$ is T-closed in X for every subset S which is S-closed relative to Y.*

Proof. Let S be S-closed relative to Y and $x \notin f^{-1}(S)$. For each $y \in S$, we have $(x, y) \in (X \times Y) - G(f)$ and there exist $U_y \in TO(X, x)$ and $V_y \in SO(Y, y)$ such that $f(U_y) \cap Cl(V_y) = \phi$. The family $\{V_y : y \in S\}$ is a cover of S by semi-open sets of Y and there exists a finite number of points say, $y_1, y_2, \dots, y_n$ of S such that $S \subseteq \{Cl(V_{y_i}) : i = 1, 2, \dots, n\}$. Put $U = \cap\{U_{y_i} : i = 1, 2, \dots, n\}$. Since X is extremally disconnected, by Theorem 2.8 of [1] U is a T-open neighborhood of x and $f(U) \cap S = \phi$. Therefore, we obtain $U \cap f^{-1}(S) = \phi$. This shows that $f^{-1}(S)$ is T-closed in X. □

5. Quasi-sT-continuous functions

A subset S of X is said to be semi-regular [6] (resp. T-regular) if it is both semi-open and semi-closed (resp. T-open and T-closed) in X. The family of all semi-regular (resp. T-regular) sets of X is denoted $SR(X)$ (resp. $TR(X)$). For each $x \in X$, the family of all semi-regular (resp. T-regular) sets containing x is denoted by $SR(X, x)$ (resp. $TR(X, x)$).

In [6], the authors showed the following propositions.

Proposition 5.1. *If $U \in SO(X)$, then $sCl(U) \in SR(X)$.*

A pint $x \in X$ is called a semi-θ-adherent point of a subset S of X if $sCl(U) \cap S \neq \phi$ for every $U \in SO(X, x)$. The set of all semi-θ-adherent points of S is called the semi-θ-closure of S and is denoted by $sCl_\theta(S)$. A subset S is said to be semi-θ-closed if $sCl_\theta(S) = S$. The complement of a semi-θ-closed set is said to be semi-θ-open.

Proposition 5.2. *Let A be a subset of a space X. Then we have*

(1) *If $A \in SO(X)$, then $sCl(A) = sCl_\theta(A)$.*
(2) *If $A \in SR(X)$, then A is semi-open and semi-closed.*

Definition 5.3. A function $f : X \to Y$ is said to be quasi-sT-continuous if for each $x \in X$ and each semi-open set V of Y containing $f(x)$, there exists $U \in TO(X, x)$ such that $f(U) \subseteq sCl(V)$.

Theorem 5.4. *For a function $f : X \to Y$, the following conditions are equivalent*:

(1) *f is quasi-sT-continuous*;
(2) *$tCl(f^{-1}(B)) \subseteq f^{-1}(sCl_\theta(B))$ for every subset B of Y;*
(3) *$f(tCl(A)) \subseteq sCl_\theta(f(A))$ for every subset A of X;*
(4) *$f^{-1}(F) \in TC(X)$ for every semi-θ-closed set F in Y;*
(5) *$f^{-1}(V) \in TO(X)$ for every semi-θ-open set V in Y.*

Proof. (1) $\Rightarrow$ (2): Let $B \subseteq Y$ and $x \notin f^{-1}(sCl_\theta(B))$. Then $f(x) \notin sCl_\theta(B)$ and there exists $V \in SO(Y, f(x))$ such that $sCl(V) \cap B = \phi$. By (1), there exists $U \in TO(X, x)$ such that $f(U) \subseteq sCl(V)$. Hence, $f(U) \cap B = \phi$ and $U \cap f^{-1}(B) = \phi$. Consequently, we obtain $x \notin tCl(f^{-1}(B))$.

(2) $\Rightarrow$ (3): For any subset A of X, the inclusion $tCl(A) \subseteq tCl(f^{-1}(f(A)))$ hold. By (2), we have $tCl(f^{-1}(f(A))) \subseteq f^{-1}(sCl_\theta(f(A)))$ and hence $f(tCl(A)) \subseteq sCl_\theta(f(A))$.

(3) $\Rightarrow$ (4): Let F be semi-θ-closed in Y. We have $sCl_\theta(f(f^{-1}(F))) \subseteq sCl_\theta(F)$. By (3) we obtain $f(tCl(f^{-1}(F)) \subseteq sCl_\theta(f(f^{-1}(F)))$, and hence $tCl(f^{-1}(F)) \subseteq f^{-1}(sCl_\theta(F)) = f^{-1}(F)$. Therefore, $f^{-1}(F)$ is T-closed in X.

(4) $\Rightarrow$ (5): If V is semi-θ-open in Y, then $Y - V$ is semi-θ-closed. By (4), $f^{-1}(Y - V) = X - f^{-1}(V)$ is T-closed in X. Thus, $f^{-1}(V) \in TO(X)$.

(5) $\Rightarrow$ (1): Let $x \in X$ and $V \in SO(Y, f(x))$. It follows from Propositions 5.1 and 5.2 that $sCl(V)$ is semi-θ-open in Y. Set $U = f^{-1}(sCl(V))$. By (5) we observe that $U \in TO(X)$ and $f(U) \subseteq sCl(V)$. The proof is complete. □

Theorem 5.5. *For a function $f : X \to Y$, the following conditions are equivalent*:

(1) *f is quasi-sT-continuous*;
(2) *$f^{-1}(V) \subseteq tInt(f^{-1}(sCl(V)))$ for every subset $V \in SO(Y)$;*
(3) *$tCl(f^{-1}(V)) \subseteq f^{-1}(sCl(V))$ for every subset $V \in SO(Y)$.*

Proof. (1) $\Rightarrow$ (2): Assume that $V \in SO(Y)$ and let $x \in f^{-1}(V)$. Then there exists $U \in TO(X, x)$ such that $f(U) \subseteq sCl(V)$. Hence we have $U \subseteq f^{-1}(sCl(V))$ and $x \in U \subseteq tInt(f^{-1}sCl(V)))$. This shows that $f^{-1}(V) \subseteq tInt(f^{-1}(sCl(V)))$.

(2) $\Rightarrow$ (3): Assume that $V \in SO(Y)$ and $x \notin f^{-1}(sCl(V))$. Then $f(x) \notin sCl(V)$. There exists $H \in SO(Y, f(x))$ such that $H \cap V = \phi$. Since $V \in SO(Y)$,

we have $sCl(H) \cap V = \phi$ and hence $tInt(f^{-1}(sCl(H))) \cap f^{-1}(V) = \phi$. It follows from (2) that $x \in f^{-1}(H) \subseteq tInt(f^{-1}(sCl(H))) \in TO(X)$. Therefore, $x \notin tCl(f^{-1}(V))$. This shows that $tCl(f^{-1}(V)) \subseteq f^{-1}(sCl(V))$.

(3) ⇒ (1): Let $x \in X$ and $V \in SO(Y, f(x))$. By Proposition 5.1, $sCl(V) \in SR(X)$ and $f(x) \in Y - sCl(Y - sCl(V))$. Hence we have $x \notin f^{-1}(sCl(Y - sCl(V)))$. Since $Y - sCl(V) \in SO(Y)$, it follows from (3) that $x \notin tCl(f^{-1}(Y - sCl(V)))$. Thus, there exists $U \in TO(X, x)$ such that $U \cap f^{-1}(Y - sCl(V)) = \phi$. Therefore, we obtain $f(U) \cap (Y - sCl(V)) = \phi$ and hence $f(U) \subseteq sCl(V)$. □

Theorem 5.6. *For a function $f : X \to Y$, the following conditions are equivalent*:

(1) *f is quasi-sT-continuous*;
(2) *For each $x \in X$ and each $V \in SO(Y, f(x))$, there exists $U \in TO(X, x)$ such that $f(tCl(U)) \subseteq sCl(V)$*;
(3) *$f^{-1}(V) \in TR(X, x)$ for every $V \in SR(Y, f(x))$.*

Proof. (1) ⇒ (2): Let $x \in X$ and $V \in SO(Y, f(x))$. Then, by Propositions 5.1 and 5.2 $sCl(V)$ is both semi-θ-open and semi-θ-closed. Put $U = f^{-1}(sCl(V))$. Then it follows from Theorem 5.4 that $U \in TR(X)$. Thus we obtain $U \in TO(X)$, $U = tCl(U)$ and $f(tCl(U)) \subseteq sCl(V)$.

(2) ⇒ (1): It is obvious.

(1) ⇒ (3): Let $V \in SR(Y)$. By Proposition 5.2 V is semi-θ-open and semi-θ-closed in Y. It follows from Theorem 5.4 that $f^{-1}(V) \in TR(X, x)$.

(3) ⇒ (1): Let $x \in X$ and $V \in SO(Y, f(x))$. By Proposition 5.1 $sCl(V) \in SR(Y, f(x))$ and $f^{-1}(sCl(V)) \in TR(X, x)$. Put $U = f^{-1}(sCl(V))$, then $U \in TO(X, x)$ and $f(U) \subseteq sCl(V)$. This shows that f is quasi-sT-continuous. □

Definition 5.7 ([16])**.** A topological space X is said to be s-closed if for every semi-open cover $\{U_\alpha : \alpha \in \Lambda\}$ of X there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $X = \cup\{sCl(U_\alpha) : \alpha \in \Lambda_0\}$.

A subset S of X is said to be s-closed relative to X [6] if for every cover $\{U_\alpha : \alpha \in \Lambda\}$ of S by semi-open sets in X, there exists a finite subfamily Λ_0 of Λ such that $S \subseteq \cup\{sCl(U_\alpha) : \alpha \in \Lambda_0\}$.

Theorem 5.8. *If $f : X \to Y$ is quasi-sT-continuous and A is semi-compact relative to X, then $f(A)$ is s-closed relative to Y.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be any cover of $f(A)$ by semi-open sets of Y. For each $x \in A$, there exists an $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since f is quasi-sT-continuous, there exists $U_x \in TO(X)$ containing x such that $f(U_x) \subseteq sCl(V_{\alpha(x)})$. Since $\{U_x : x \in A\}$ is a cover of A by T-open sets of X, by Theorem 3.25 there exist finite points, say, $x_1, x_2, \ldots, x_n$ of A such that $A \subseteq \cup\{U_{x_i} : i = 1, 2, \ldots, n\}$. Therefore, we obtain

$$f(A) \subseteq \bigcup_{i=1}^{n} f(U_{x_i}) \subseteq \bigcup_{i=1}^{n} sCl(V_{\alpha(x_i)}).$$

This shows that $f(A)$ is s-closed relative to Y. □

Corollary 5.9. *If $f : X \to Y$ is a quasi-sT-continuous surjection and X is semi-compact, then Y is s-closed.*

6. Strongly-sT-continuous functions

Definition 6.1. A function $f : X \to Y$ is said to be strongly-sT-continuous if for each $x \in X$ and each semi-open set V of Y containing $f(x)$, there exists $U \in TO(X, x)$ such that $f(Cl(U)) \subseteq V$.

Theorem 6.2. *If $f : X \to Y$ is a strongly-sT-continuous surjection and X is S-closed, then Y is semi-compact.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be any semi-open cover of Y. For each $x \in X$, there exists an $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since f is strongly-sT-continuous, there exists a $U_x \in TO(X)$ containing x such that $f(Cl(U_x)) \subseteq V_{\alpha(x)}$. Since $\{U_x : x \in X\}$ is a T-open cover of X, by Corollary 3.8 there exist finite points, say, $x_1, x_2, \ldots, x_n$ of X such that $X = \cup\{Cl(U_{x_i}) : i = 1, 2, \ldots, n\}$. Since f is surjective,

$$Y = \bigcup_{i=1}^{n} f(Cl(U_{x_i})) \subseteq \bigcup_{i=1}^{n} V_{\alpha(x_i)}.$$

This shows that Y is semi-compact. □

Definition 6.3. A function $f : X \to Y$ is said to be θ-sT-continuous if for each $x \in X$ and each semi-open set V of Y containing $f(x)$, there exists $U \in TO(X, x)$ such that $f(Cl(U)) \subseteq sCl(V)$.

Theorem 6.4. *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function, given by $g(x) = (x, f(x))$ for every $x \in X$. If g is θ-sT-continuous, then f is θ-sT-continuous.*

Proof. Let $x \in X$ and $V \in SO(Y, f(x))$. Then $X \times V$ is a semi-open set of $X \times Y$ containing $g(x)$ and hence there exists $U \in TO(X, x)$ such that $g(Cl(U)) \subseteq sCl(X \times V) \subseteq X \times sCl(V)$. By the definition of g, we have $f(Cl(U)) \subseteq sCl(V)$. Therefore, f is θ-sT-continuous. □

Theorem 6.5. *If $f : X \to Y$ is a θ-sT-continuous surjection and X is S-closed, then Y is s-closed.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be any semi-open cover of Y. For each $x \in X$, there exists an $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since f is θ-sT-continuous, there exists $U_x \in TO(X)$ containing x such that $f(Cl(U_x)) \subseteq sCl(V_{\alpha(x)})$. Since $\{U_x : x \in X\}$ is a T-open cover of X, by Corollary 3.8 there exist finite points, say, $x_1, x_2, \ldots, x_n$ of X such that $X = \cup\{Cl(U_{x_i}) : i = 1, 2, \ldots, n\}$. Since f is surjective,

$$Y = \bigcup_{i=1}^{n} f(Cl(U_{x_i})) \subseteq \bigcup_{i=1}^{n} sCl(V_{\alpha(x_i)}).$$

This shows that Y is s-closed. □

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