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VALUE DISTRIBUTION OF SOME q-DIFFERENCE POLYNOMIALS

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Abstract. For a transcendental entire function $f(z)$ with zero order, the purpose of this article is to study the value distributions of q-difference polynomial $f(qz)-a(f(z))^n$ and $f(q_1z)f(q_2z)\cdots f(q_mz)-a(f(z))^n$. The property of entire solution of a certain q-difference equation is also considered.

1. Introduction and main results

A meromorphic function $f(z)$ means meromorphic in the complex plane $\mathbb{C}$. If no poles occur, then $f(z)$ reduces to an entire function. For every real number $x \geq 0$, we define $\log^+ x := \max\{0, \log x\}$. Assume that $n(r,f)$ counts the number of the poles of f in $|z| \leq r$, each pole is counted according to its multiplicity, and that $\overline{n}(r,f)$ counts the number of the distinct poles of f in $|z| \leq r$, ignoring the multiplicity. The characteristic function of f is defined by

$$T(r,f) := m(r,f) + N(r,f),$$

where

$$N(r,f) := \int_0^r \frac{n(t,f)-n(0,f)}{t}dt + n(0,f)\log r$$

and

$$m(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f(re^{i\theta})\right| d\theta.$$

The notation $\overline{N}(r,f)$ is similarly defined with $\overline{n}(r,f)$ instead of $n(r,f)$. For more notations and definitions of the Nevanlinna's value distribution theory of meromorphic functions, we refer to [10, 17].

A meromorphic function $\alpha(z)$ is called a small function with respect to $f(z)$, if $T(r,\alpha) = S(r,f)$, where $S(r,f)$ denotes any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside a possible exceptional set E of logarithmic density

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0. The order and the exponent of convergence of zeros of meromorphic function $f(z)$ is respectively defined as

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r},$$

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log N(r,\frac{1}{f})}{\log r}.$$

The difference operators for a meromorphic function f are defined as

$$\triangle_c f(z) = f(z+c) - f(z) \quad (c \neq 0),$$

$$\nabla_q f(z) = f(qz) - f(z) \quad (q \neq 0, 1).$$

A Borel exceptional value of $f(z)$ is any value a satisfying $\lambda(f-a) < \sigma(f)$.

The zero distribution of differential polynomials is a classical topic in the theory of meromorphic functions. In [9], Hayman discussed Picard values of a meromorphic function and its derivatives. In particular, he proved the following result.

Theorem A ([9])**.** *Let $f(z)$ be a transcendental entire function. Then*

(a) *for $n \geq 3$ and $a \neq 0$, $\psi(z) = f^{'}(z) - a(f(z))^n$ assumes all finite values infinitely often.*

(b) *For $n \geq 2$, $\phi(z) = f^{'}(z)(f(z))^n$ assumes all finite values except possibly zero infinitely often.*

Recently, the difference variant of Nevanlinna theory has been established independently in [2, 6, 7, 8]. With the development of difference analogue of Nevanlinna theory, many authors paid their attentions to the difference version of Hayman conjecture. For example, Laine and Yang [12] proved that if $f(z)$ is a transcendental entire function of finite order, c is a nonzero complex constant and $n \geq 2$, then $f^n(z)f(z+c)$ takes every nonzero value infinitely often.

Liu and Qi [14] proved the following theorem by considering q-difference polynomials, which can be seen as a q-difference counterpart of Theorem A(b).

Theorem B ([14, Theorems 1.1 and 1.2])**.** *If $f(z)$ is a transcendental meromorphic function of zero order, a, q are nonzero complex constants. If $n \geq 6$, then $f^n(z)f(qz+c)$ assumes every nonzero value $b \in \mathbb{C}$ infinitely often. If $n \geq 8$, then $f^n(z) + a[f(qz+c) - f(z)]$ assumes every nonzero value $b \in \mathbb{C}$ infinitely often.*

In [13], Liu-Liu-Cao extended this result by considering zeros distribution of q-difference products $f^n(z)(f^m(z)-a)f(qz+c)$ and $f^n(z)(f^m(z)-a)[f(qz+c) - f(z)]$ for the meromorphic function f of zero order.

Theorem C ([13, Theorems 1.1 and 1.3])**.** *If $f(z)$ is a transcendental meromorphic function of zero order, a, q are nonzero complex constants, $\alpha(z)$ is a nonzero small function with respect to f. If $n \geq 6$, then $f^n(z)(f^m(z)-a)f(qz+c) - \alpha(z)$ has infinitely many zeros. If $n \geq 7$, then $f^n(z)(f^m(z)-a)[f(qz+c) - f(z)] - \alpha(z)$ has infinitely many zeros.*

In this paper, we obtain a q-difference counterpart of Theorem A(a) and generalize it to more general cases.

Theorem 1.1. *Let $f(z)$ be a transcendental entire function of zero order, a be a nonzero complex constant, $q \in \mathbb{C} \setminus \{0,1\}$, $n \in \mathbb{N}^+$. Considering q-difference polynomial*

$$H(z) = f(qz) - a(f(z))^n,$$

(1) *if $n \geq 3$, then $H(z) - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a nonzero small function with respect to $f(z)$.*

(2) *In particular, if $\alpha(z)$ is a nonzero rational function, then the condition $n \geq 3$ can be reduced to $n > 1$.*

From the proof of Theorem 1.1(2), one can immediately get the following corollary.

Corollary 1.2. *The q-difference equation $f(qz) - a(f(z))^n - R(z) = 0$ has no transcendental entire solution of zero order when $n > 1$, where $R(z)$ is a nonzero rational function.*

In the following, we obtain more general results by considering the value distribution of q-difference polynomial

$$F(z) = f(q_1 z)f(q_2 z)\cdots f(q_m z) - a(f(z))^n.$$

Theorem 1.3. *Let $f(z)$ be a transcendental entire function of zero order, $q_1, q_2, \ldots, q_m$ be nonzero complex constants such that at least one of them is not equal to 1, $a \in \mathbb{C} \setminus \{0\}$, $m, n \in \mathbb{N}^+$. Considering q-difference polynomial*

$$F(z) = f(q_1 z)f(q_2 z)\cdots f(q_m z) - a(f(z))^n,$$

(1) *if $m < \frac{n-1}{2-\frac{1}{n}}$, then $F(z) - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a nonzero small function with respect to $f(z)$.*

(2) *In particular, if $\alpha(z)$ is a nonzero rational function, then the condition $m < \frac{n-1}{2-\frac{1}{n}}$ can be reduced to $n > m$.*

Remark. Theorem 1.1 is a special case of Theorem 1.3, for $m = 1$. Thus, we need only give the proof of Theorem 1.3.

Corollary 1.4. *The q-difference equation $f(q_1 z)f(q_2 z)\cdots f(q_m z) - a(f(z))^n - R(z) = 0$ has no transcendental entire solution of zero order when $n > m$, where $R(z)$ is a nonzero rational function.*

However, by another way of proving, we have the following more general result.

Theorem 1.5. *Let $f(z)$ be a transcendental entire function of zero order, $q_1, q_2, \ldots, q_m$ be nonzero complex constants such that at least one of them is not equal to 1, $a \in \mathbb{C} \setminus \{0\}$, $m, n \in \mathbb{N}^+$. If $m \neq n$, then $F(z) - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a nonzero small function with respect to $f(z)$.*

Note that all of the above theorems discuss the case when $f(z)$ is a transcendental entire function of zero order. It is natural to ask how about value distribution of q-difference polynomial $F(z)$ for the transcendental entire function $f(z)$ with positive order? We have the following theorem.

Theorem 1.6. *Let $f(z)$ be a transcendental entire function of finite and positive order $\sigma(f)$, $q_1, q_2, \ldots, q_m$ be nonzero complex constants such that at least one of them is not equal to 1 and $q_1^{\sigma(f)} + q_2^{\sigma(f)} + \cdots + q_m^{\sigma(f)} \neq n$, $a \in \mathbb{C} \setminus \{0\}$, $m, n \in \mathbb{N}^+$. If $f(z)$ has finitely many zeros, then $F(z) - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a nonzero small function with respect to $f(z)$.*

Next, we will consider a special q-difference equation and obtain the following result.

Theorem 1.7. *Let $G(z)$ be an entire function with order less than one, $q_1, q_2, \ldots, q_m$ be nonzero complex constants such that at least one of them is not equal to 1 and $q_1^{\sigma(f)} + q_2^{\sigma(f)} + \cdots + q_m^{\sigma(f)} \neq n$, $a \in \mathbb{C} \setminus \{0\}$, $m, n \in \mathbb{N}^+$. Suppose that $f(z)$ is a finite and positive order transcendental entire solution of the q-difference equation*

$$f(q_1 z) f(q_2 z) \cdots f(q_m z) - a(f(z))^n = G(z). \tag{1.1}$$

Then $f(z)$ has infinitely many zeros.

2. Lemmas

To prove our results, we need some lemmas. The first one is the characteristic function relationship between $f(z)$ and $f(qz)$, provided that $f(z)$ is a nonconstant meromorphic function of zero order.

Lemma 2.1 ([19])**.** *If $f(z)$ is a nonconstant meromorphic function of zero order, and $q \in \mathbb{C} \setminus \{0\}$, then*

$$T(r, f(qz)) = (1 + o(1)) T(r, f) \tag{2.1}$$

on a set of lower logarithmic density 1.

Lemma 2.2 ([2])**.** *Let $f(z)$ be a nonconstant meromorphic function of zero order, and let $q \in \mathbb{C} \setminus \{0\}$, then*

$$m(r, \frac{f(qz)}{f(z)}) = S(r, f) \tag{2.2}$$

on a set of logarithmic density 1.

The following Lemma 2.3 is the well-known Weierstrass factorization theorem and Hadamard factorization theorem.

Lemma 2.3 ([1])**.** *If an entire function f has a finite exponent of convergence $\lambda(f)$ for its zero-sequence, then f has a representation in the form*

$$f(z) = Q(z) e^{g(z)},$$

satisfying $\lambda(Q) = \sigma(Q) = \lambda(f)$. *Further, if* f *is of finite order, then* g *in the above form is a polynomial of degree less or equal to the order of* f.

Lemma 2.4 ([18])**.** *Suppose that* $f_1(z), f_2(z), \dots, f_n(z), (n \geq 2)$ *are meromorphic functions and* $g_1(z), g_2(z), \dots, g_n(z)$ *are entire functions satisfying the following conditions*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;

(2) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$;

(3) *for* $1 \leq j \leq n, 1 \leq h < k \leq n$, $T(r, f_j) = o(T(r, e^{g_h - g_k}))$ $(r \to \infty, r \notin E)$.

Then $f_j(z) \equiv 0$ $(j = 1, 2, \dots, n)$.

3. The proofs

3.1. Proof of Theorem 1.3

(1) Denote

$$\psi(z) = \frac{f(q_1 z) \cdots f(q_m z) - \alpha(z)}{a(f(z))^n}.$$

The condition $m < \frac{n-1}{2-\frac{1}{n}}$ implies $n > m$. Since $f(z)$ is a transcendental entire function of zero order, by Lemma 2.1, we obtain

$$\begin{aligned} nT(r, f) &= T(r, \frac{f(q_1 z) \cdots f(q_m z) - \alpha(z)}{a\psi(z)}) \\ &\leq T(r, f(q_1 z) \cdots f(q_m z)) + T(r, \psi(z)) + O(1) \\ &\leq mT(r, f) + T(r, \psi) + S(r, f), \end{aligned}$$

which implies

$$(n - m)T(r, f) + S(r, f) \leq T(r, \psi) \tag{3.1}$$

on a set of lower logarithmic density 1. The above inequality implies that $\psi(z)$ is transcendental since $f(z)$ is a transcendental entire function and $n > m$. On the other hand,

$$\begin{aligned} T(r, \psi) &= T(r, \frac{f(q_1 z) \dots f(q_m z) - \alpha(z)}{a(f(z))^n}) \\ &\leq T(r, f(q_1 z) \dots f(q_m z)) + nT(r, f(z)) + O(1), \end{aligned}$$

thus by Lemma 2.1, we get

$$T(r, \psi) \leq (n + m)T(r, f) + S(r, f). \tag{3.2}$$

By (3.1), (3.2), and $n > m$, we obtain

$$T(r, \psi) = O(T(r, f)).$$

Suppose $F(z) - \alpha(z)$ has finitely many zeros, then $\psi(z)$ has only finite 1-points, that is

$$N(r, \frac{1}{\psi - 1}) = S(r, \psi) = S(r, f).$$

Thus we get from the Second Main Theorem that

$$\begin{aligned}T(r,\psi) &\leq \overline{N}(r,\psi)+\overline{N}(r,\frac{1}{\psi})+\overline{N}(r,\frac{1}{\psi-1})+S(r,\psi)\\ &\leq \frac{1}{n}N(r,\psi)+\overline{N}(r,\frac{1}{f(q_1z)\cdots f(q_mz)-\alpha(z)})+S(r,\psi)\\ &\leq \frac{1}{n}T(r,\psi)+mT(r,f)+S(r,\psi)+S(r,f),\end{aligned}$$

which implies that

$$(1-\frac{1}{n})T(r,\psi)\leq mT(r,f)+S(r,f). \tag{3.3}$$

By (3.1) and (3.3), we obtain

$$(1-\frac{1}{n})T(r,\psi)\leq \frac{m}{n-m}T(r,\psi)+S(r,\psi),$$

that is

$$(1-\frac{1}{n}-\frac{m}{n-m})T(r,\psi)\leq S(r,\psi). \tag{3.4}$$

Since $m<\frac{n-1}{2-\frac{1}{n}}$, we have $1-\frac{1}{n}-\frac{m}{n-m}>0$, it is clearly that (3.4) is a contradiction.

Hence, $F(z)-\alpha(z)$ has infinitely many zeros.

(2) By Lemma 2.1, we get

$$\begin{aligned}T(r,F) &\leq T(r,f(q_1z)\cdots f(q_mz))+nT(r,f)\\ &\leq (n+m)T(r,f)+S(r,f).\end{aligned}$$

On the other hand,

$$\begin{aligned}nT(r,f) &= T(r,f(q_1z)\cdots f(q_mz)-F(z))\\ &\leq T(r,f(q_1z)\cdots f(q_mz))+T(r,F(z))\\ &\leq mT(r,f)+T(r,F)+S(r,f).\end{aligned}$$

Thus by the above inequalities we have

$$(n-m)T(r,f)+S(r,f)\leq T(r,F)\leq (n+m)T(r,f)+S(r,f). \tag{3.5}$$

From (3.5), we obtain $T(r,F)=O(T(r,f))$, $F(z)$ is transcendental as $f(z)$ is a transcendental entire function and $n>m$. Since $\sigma(f)=0$, clearly, $F(z)$ is also of zero order.

Suppose $F(z)-\alpha(z)$ has finitely many zeros, since $\alpha(z)$ is a nonzero rational function and $F(z)$ is a function of zero order, then we get

$$F(z)-\alpha(z)=R(z),$$

where $R(z)$ is a rational function. Thus $T(r,F)=S(r,F)$, which is a contradiction.

Hence, $F(z)-\alpha(z)$ has infinitely many zeros.

3.2. Proof of Theorem 1.5

Suppose $F(z)-\alpha(z)$ has finitely many zeros, by Lemma 2.1, we have

$$\begin{aligned} T(r,F-\alpha) &\le \sum_{j=1}^{m} T(r,f(q_jz)) + nT(r,f) + T(r,\alpha) \\ &\le (m+n)T(r,f) + S(r,f). \end{aligned}$$

Thus

$$\sigma(F-\alpha)=0.$$

According to the Hadamard factorization theorem, we get

$$F(z)-\alpha(z) = f(q_1z)f(q_2z)\cdots f(q_mz) - a(f(z))^n - \alpha(z) = p(z), \tag{3.6}$$

where $p(z)$ is a polynomial.

Rewrite (3.6) as

$$f(q_1z)f(q_2z)\cdots f(q_mz) = a(f(z))^n + p(z) + \alpha(z). \tag{3.7}$$

When $n>m$, by (3.7) and Lemma 2.1, we have

$$\begin{aligned} nT(r,f) = T(r,af^n) &= T(r,\prod_{j=1}^{m} f(q_jz) - p - \alpha) \\ &\le \sum_{j=1}^{m} T(r,f(q_jz)) + S(r,f) \\ &\le mT(r,f) + S(r,f), \end{aligned}$$

which is a contradiction.

When $n<m$, by (3.7) and Lemma 2.2, we have

$$\begin{aligned} T(r,\prod_{j=1}^{m} f(q_jz)) = m(r,\prod_{j=1}^{m} f(q_jz)) &= m(r,f^m\prod_{j=1}^{m}\frac{f(q_jz)}{f}) \\ &\ge m(r,f^m) - m(r,\prod_{j=1}^{m}\frac{f}{f(q_jz)}) \\ &= mm(r,f) - S(r,f) \\ &= mT(r,f) - S(r,f). \end{aligned}$$

On the other hand, by (3.7), we get

$$T(r,\prod_{j=1}^{m} f(q_jz)) = T(r,af^n+p+\alpha) \le nT(r,f) + S(r,f).$$

Thus we have

$$mT(r,f) \le nT(r,f) + S(r,f).$$

Which is a contradiction.

Hence, $F(z)-\alpha(z)$ has infinitely many zeros.

3.3. Proof of Theorem 1.6

Since $f(z)$ is a transcendental entire function of finite order and has finitely many zeros, by Lemma 2.3, $f(z)$ can be written as

$$f(z) = g(z)e^{h(z)},$$

where $g(z)(\not\equiv 0)$, $h(z)$ are polynomials. Set

$$h(z) = a_k z^k + \cdots + a_0,$$

where $a_k, \ldots, a_0$ are constants, $a_k \neq 0$. Since $\sigma(f) \neq 0$, it follows that $\sigma(f) = \deg(h(z)) = k \geq 1$. We obtain

$$f(q_1 z) \cdots f(q_m z) = p_1(z) e^{a_k(q_1^k + \cdots + q_m^k) z^k}, \tag{3.8}$$

where $p_1(z) = g(q_1 z) \cdots g(q_m z) e^{a_{k-1}(q_1^{k-1} + \cdots + q_m^{k-1}) z^{k-1} + \cdots + m a_0}$, $\sigma(p_1) \leq k - 1 < k$. On the other hand, we have

$$(f(z))^n = (g(z))^n e^{n a_k z^k + n a_{k-1} z^{k-1} + \cdots + n a_0} = p_2(z) e^{n a_k z^k}, \tag{3.9}$$

where $p_2(z) = (g(z))^n e^{n a_{k-1} z^{k-1} + \cdots + n a_0}$, $\sigma(p_2) \leq k - 1 < k$.

By (3.8) and (3.9), we get

$$F(z) = p_1(z) e^{a_k(q_1^k + \cdots + q_m^k) z^k} - a p_2(z) e^{n a_k z^k}. \tag{3.10}$$

Since $p_1(z)(\not\equiv 0)$, $p_2(z)(\not\equiv 0)$, $\sigma(p_1) < k$, $\sigma(p_2) < k$, $q_1^k + q_2^k + \cdots + q_m^k \neq n$, it follows that $F(z)$ is a transcendental entire function and $\sigma(F) = \sigma(f) = k$.

Suppose $F(z) - \alpha(z)$ has finitely many zeros, then $\lambda(F - \alpha) < \sigma(F) = \sigma(f)$, $F(z) - \alpha(z)$ can be written as

$$F(z) - \alpha(z) = s(z) e^{t z^k}, \tag{3.11}$$

where $s(z)$ is an entire function with $\sigma(s) < k$, $t \neq 0$ is a constant. By (3.10) and (3.11), we obtain

$$p_1(z) e^{a_k(q_1^k + \cdots + q_m^k) z^k} - a p_2(z) e^{n a_k z^k} - s(z) e^{t z^k} - \alpha(z) = 0. \tag{3.12}$$

Since $q_1^k + q_2^k + \cdots + q_m^k \neq n$,

Case 1: $a_k(q_1^k + \cdots + q_m^k) \neq t, n a_k \neq t$. By Lemma 2.4, we obtain

$$p_1(z) \equiv 0, p_2(z) \equiv 0, s(z) \equiv 0, \alpha(z) \equiv 0.$$

This is a contradiction.

Case 2: $a_k(q_1^k + \cdots + q_m^k) = t$. Then (3.12) can be written as

$$(p_1(z) - s(z)) e^{a_k(q_1^k + \cdots + q_m^k) z^k} - a p_2(z) e^{n a_k z^k} - \alpha(z) = 0.$$

By Lemma 2.4, we obtain

$$p_1(z) - s(z) \equiv 0, p_2(z) \equiv 0, \alpha(z) \equiv 0,$$

which is a contradiction.

Case 3: $n a_k = t$. Then using the same method as above, we also obtain a contradiction.

Hence $F(z) - \alpha(z)$ has infinitely many zeros.

3.4. Proof of Theorem 1.7

Suppose $f(z)$ has finitely many zeros. Since $f(z)$ is a transcendental entire function of finite and positive order, by Lemma 2.3, $f(z)$ can be written as

$$f(z) = g(z)e^{h(z)}, \tag{3.13}$$

where $g(z)(\not\equiv 0)$, $h(z)$ are polynomials. Set

$$h(z) = a_k z^k + \cdots + a_0,$$

where $a_k, \ldots, a_0$ are constants, $a_k \neq 0$. Since $\sigma(f) \neq 0$, then $\sigma(f) = \deg(h(z)) = k \geq 1$. Substituting (3.13) into (1.1), we obtain

$$p_1(z)e^{a_k(q_1^k+\cdots+q_m^k)z^k} - ap_2(z)e^{na_k z^k} = G(z), \tag{3.14}$$

where

$$p_1(z) = g(q_1 z)\cdots g(q_m z)e^{a_{k-1}(q_1^{k-1}+\cdots+q_m^{k-1})z^{k-1}+\cdots+ma_0}, \ \ \sigma(p_1) \leq k-1 < k;$$

$$p_2(z) = (g(z))^n e^{na_{k-1}z^{k-1}+\cdots+na_0}, \ \ \sigma(p_2) \leq k-1 < k.$$

Since $p_1(z)(\not\equiv 0)$, $p_2(z)(\not\equiv 0)$, $\sigma(p_1) < k$, $\sigma(p_2) < k$, $q_1^k + q_2^k + \cdots + q_m^k \neq n$, $\sigma(G) < 1 < k$, by (3.14) and Lemma 2.4, we obtain

$$p_1(z) \equiv 0, p_2(z) \equiv 0, G(z) \equiv 0,$$

which is a contradiction.

Hence $f(z)$ has infinitely many zeros.

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