

## SOME CURIOSITIES OF THE ALGEBRA OF BOUNDED DIRICHLET SERIES

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ABSTRACT. It is shown that the algebra  $\mathcal{H}^\infty$  of bounded Dirichlet series is not a coherent ring, and has infinite Bass stable rank. As corollaries of the latter result, it is derived that  $\mathcal{H}^\infty$  has infinite topological stable rank and infinite Krull dimension.

### 1. Introduction

The aim of this short note is to make explicit two observations about algebraic properties of the ring  $\mathcal{H}^\infty$  of bounded Dirichlet series. In particular we will show that

- (1)  $\mathcal{H}^\infty$  is not a coherent ring. (This is essentially an immediate consequence of Eric Amar's proof of the noncoherence of the Hardy algebra  $H^\infty(\mathbb{D}^n)$  of the polydisk  $\mathbb{D}^n$  for  $n \geq 3$  [1].)
- (2)  $\mathcal{H}^\infty$  has infinite Bass stable rank. (This is a straightforward adaptation of the first author's proof of the fact that the stable rank of the infinite polydisk algebra is infinite [12]). As corollaries, we obtain that  $\mathcal{H}^\infty$  has infinite topological stable rank, and infinite Krull dimension.

Before giving the relevant definitions, we briefly mention that  $\mathcal{H}^\infty$  is a closed Banach subalgebra of the classical Hardy algebra  $H^\infty(\mathbb{C}_{>0})$  consisting of all bounded and holomorphic functions in the open right half plane

$$\mathbb{C}_{>0} := \{s \in \mathbb{C} : \operatorname{Re}(s) > 0\},$$

and it is striking to compare our findings with the corresponding results for  $H^\infty(\mathbb{C}_{>0})$ :

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	$H^\infty(\mathbb{C}_{>0})$	$\mathcal{H}^\infty$
Coherent?	Yes (See [11])	No
Bass stable rank	1 (See [17])	$\infty$
Topological stable rank	2 (See [16])	$\infty$
Krull dimension	$\infty$ (See [13])	$\infty$

Nevertheless the above results for  $\mathcal{H}^\infty$  lend support to Harald Bohr's idea of interpreting Dirichlet series as functions of infinitely many complex variables, a key theme used in the proofs of the main results in this note.

We recall the pertinent definitions below.

### 1.1. The algebra $\mathcal{H}^\infty$ of bounded Dirichlet series

$\mathcal{H}^\infty$  denotes the set of Dirichlet series

$$(1.1) \quad f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where  $(a_n)_{n \in \mathbb{N}}$  is a sequence of complex numbers, such that  $f$  is holomorphic and bounded in  $\mathbb{C}_{>0}$ . Equipped with pointwise operations and the supremum norm,

$$\|f\|_\infty := \sup_{s \in \mathbb{C}_{>0}} |f(s)|, \quad f \in \mathcal{H}^\infty,$$

$\mathcal{H}^\infty$  is a unital commutative Banach algebra. In [8, Theorem 3.1], it was shown that the Banach algebra  $\mathcal{H}^\infty$  is precisely the multiplier space of the Hilbert space  $\mathcal{H}$  of Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

for which

$$\|f\|_{\mathcal{H}}^2 := \sum_{n=1}^{\infty} |a_n|^2 < \infty.$$

The importance of the Hilbert space  $\mathcal{H}$  stems from the fact that its kernel function  $K_{\mathcal{H}}(z, w)$  is related to the Riemann zeta function  $\zeta$ :

$$K_{\mathcal{H}}(z, w) = \zeta(z + \bar{w}).$$

For  $m \in \mathbb{N}$ , let  $\mathcal{H}_m^\infty$  be the closed subalgebra of  $\mathcal{H}^\infty$  consisting of Dirichlet series of the form (1.1) involving only integers  $n$  generated by the first  $m$  primes  $2, 3, \dots, p_m$ .

### 1.2. $\mathcal{H}^\infty = H^\infty(\mathbb{D}^\infty)$

In [8, Lemma 2.3 and the proof of Theorem 3.1], it was established that  $\mathcal{H}^\infty$  is isometrically (Banach algebra) isomorphic to  $H^\infty(\mathbb{D}^\infty)$ , a certain algebra of functions analytic in the infinite dimensional polydisk, defined below. As this plays a central role in what follows, we give an outline of this based on [8], [10] and [15].

A seminal observation made by H. Bohr [3], is that if we put

$$z_1 = \frac{1}{2^s}, \quad z_2 = \frac{1}{3^s}, \quad z_3 = \frac{1}{5^s}, \quad \dots, \quad z_n = \frac{1}{p_n^s}, \quad \dots,$$

where  $p_n$  denotes the  $n$ th prime, then, in view of the Fundamental Theorem of Arithmetic, formally a Dirichlet series in  $\mathcal{H}_n^\infty$  or  $\mathcal{H}^\infty$  can be considered as a power series of infinitely many variables. Indeed, each  $n$  has a unique expansion

$$n = p_1^{\alpha_1(n)} \cdots p_{r(n)}^{\alpha_{r(n)}(n)},$$

with nonnegative  $\alpha_j(n)$ s, and so, from (1.1), we obtain the formal power series

$$(1.2) \quad F(\mathbf{z}) = \sum_{n=1}^{\infty} a_n z_1^{\alpha_1(n)} \cdots z_{r(n)}^{\alpha_{r(n)}(n)},$$

where  $\mathbf{z} = (z_1, \dots, z_m)$  or  $\mathbf{z} = (z_1, z_2, z_3, \dots)$  depending on whether  $f$  is a function in  $\mathcal{H}_m^\infty$  or in  $\mathcal{H}^\infty$ . Let us recall Kronecker's Theorem on diophantine approximation [7, Chapter XXIII]:

**Proposition 1.1.** *For each  $m \in \mathbb{N}$ , the map*

$$t \mapsto (2^{-it}, 3^{-it}, \dots, p_m^{-it}) : (0, \infty) \rightarrow \mathbb{T}^m$$

*has dense range in  $\mathbb{T}^m$ , where  $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ .*

Using the above and the Maximum Principle, it can be shown that for  $f \in \mathcal{H}_m^\infty$ ,

$$(1.3) \quad \|f\|_\infty = \|F\|_\infty,$$

where the norm on the right hand side is the  $H^\infty(\mathbb{D}^m)$  norm. Here  $H^\infty(\mathbb{D}^m)$  denotes the usual Hardy algebra of bounded holomorphic functions on the polydisk  $\mathbb{D}^m$ , endowed with the supremum norm:

$$\|F\|_\infty := \sup_{\mathbf{z} \in \mathbb{D}^m} |F(\mathbf{z})|, \quad F \in H^\infty(\mathbb{D}^m).$$

In [8], it was shown that this result also holds in the infinite dimensional case. In order to describe this result, we introduce some notation. Let  $c_0$  be the Banach space of complex sequences tending to 0 at infinity, with the induced norm from  $\ell^\infty$ , and let  $B$  be the open unit ball of that Banach space. Thus with  $\mathbb{N} := \{1, 2, 3, \dots\}$  and  $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ ,

$$B = c_0 \cap \mathbb{D}^{\mathbb{N}}.$$

For a point  $\mathbf{z} = (z_1, \dots, z_m, \dots) \in B$ , we set  $\mathbf{z}^{(m)} := (z_1, \dots, z_m, 0, \dots)$ , that is,  $z_k = 0$  for  $k > m$ . Substituting  $\mathbf{z}^{(m)}$  in the argument of  $F$  given formally by (1.2), we obtain a function

$$(z_1, \dots, z_m) \mapsto F(\mathbf{z}^{(m)}),$$

which we call the  $m$ th-section  $F_m$  (after Bohr's terminology "mte abschnitt").  $F$  is said to be in  $H^\infty(\mathbb{D}^\infty)$  if the  $H^\infty$  norm of these functions  $F_m$  are uniformly bounded, and denote the supremum of these norms to be  $\|F\|_\infty$ . Using Schwarz's Lemma for the polydisk, it can be seen that for  $m < \ell$ ,

$$|F(\mathbf{z}^{(m)}) - F(\mathbf{z}^{(\ell)})| \leq 2\|f\|_\infty \cdot \max\{|z_j| : m < j \leq \ell\},$$

and so we may define

$$F(\mathbf{z}) = \lim_{m \rightarrow \infty} F(\mathbf{z}^{(m)}).$$

It was shown in [8] that (1.3) remains true in the infinite dimensional case, and so we may associate  $\mathcal{H}^\infty$  with  $H^\infty(\mathbb{D}^\infty)$ .

**Proposition 1.2** ([8]). *There exists a Banach algebra isometric isomorphism  $\iota : \mathcal{H}^\infty \rightarrow H^\infty(\mathbb{D}^\infty)$ .*

### 1.3. Coherence

**Definition 1.3.** Let  $R$  be a unital commutative ring, and for  $n \in \mathbb{N}$ , let  $R^n = R \times \cdots \times R$  ( $n$  times).

For  $\mathbf{f} = (f_1, \dots, f_n) \in R^n$ , a relation  $\mathbf{g}$  on  $\mathbf{f}$  is an  $n$ -tuple  $\mathbf{g} = (g_1, \dots, g_n)$  in  $R^n$  such that

$$g_1 f_1 + \cdots + g_n f_n = 0.$$

The set of all relations on  $\mathbf{f}$  is denoted by  $\mathbf{f}^\perp$ .

The ring  $R$  is said to be *coherent* if for each  $n$  and each  $\mathbf{f} \in R^n$ , the  $R$ -module  $\mathbf{f}^\perp$  is finitely generated.

A property which is equivalent to coherence is that the intersection of any two finitely generated ideals in  $R$  is finitely generated, and the annihilator of any element is finitely generated [4]. We refer the reader to the article [6] and the monograph [5] for the relevance of the property of coherence in commutative algebra. All Noetherian rings are coherent, but not all coherent rings are Noetherian. (For example, the polynomial ring  $\mathbb{C}[x_1, x_2, x_3, \dots]$  is not Noetherian because the sequence of ideals  $\langle x_1 \rangle \subset \langle x_1, x_2 \rangle \subset \langle x_1, x_2, x_3 \rangle \subset \cdots$  is ascending and not stationary, but  $\mathbb{C}[x_1, x_2, x_3, \dots]$  is coherent [5, Corollary 2.3.4].)

In the context of algebras of holomorphic functions in the unit disk  $\mathbb{D}$ , we mention [11], where it was shown that the Hardy algebra  $H^\infty(\mathbb{D})$  is coherent, while the disk algebra  $A(\mathbb{D})$  isn't. For  $n \geq 3$ , Amar [1] showed that the Hardy algebra  $H^\infty(\mathbb{D}^n)$  is not coherent. (It is worth mentioning that whether the Hardy algebra  $H^\infty(\mathbb{D}^2)$  of the bidisk is coherent or not seems to be an open problem.) Using Amar's result, we will prove the following result:

**Theorem 1.4.**  *$\mathcal{H}^\infty$  is not coherent.*

### 1.4. Stable rank

In algebraic  $K$ -theory, the notion of (Bass) stable rank of a ring was introduced in order to facilitate  $K$ -theoretic computations [2].

**Definition 1.5.** Let  $R$  be a commutative ring with an identity element (denoted by 1).

An element  $(a_1, \dots, a_n) \in R^n$  is called *unimodular* if there exist elements  $b_1, \dots, b_n$  in  $R$  such that

$$b_1 a_1 + \dots + b_n a_n = 1.$$

The set of all unimodular elements of  $R^n$  is denoted by  $U_n(R)$ .

We say that  $a = (a_1, \dots, a_{n+1}) \in U_{n+1}(R)$  is *reducible* if there exists an element  $(x_1, \dots, x_n) \in R^n$  such that

$$(a_1 + x_1 a_{n+1}, \dots, a_n + x_n a_{n+1}) \in U_n(R).$$

The *Bass stable rank* of  $R$  is the least integer  $n \in \mathbb{N}$  for which every  $a \in U_{n+1}(R)$  is reducible. If there is no such integer  $n$ , we say that  $R$  has *infinite stable rank*.

Using the same idea as in [12, Proposition 1] (that the infinite polydisk algebra  $A(\mathbb{D}^\infty)$  has infinite Bass stable rank), we show the following.

**Theorem 1.6.** *The Bass stable rank of  $\mathcal{H}^\infty$  is infinite.*

For Banach algebras, an analogue of the Bass stable rank, called the topological stable rank, was introduced by Marc Rieffel in [14].

**Definition 1.7.** Let  $R$  be a commutative complex Banach algebra with unit element 1. The least integer  $n$  for which  $U_n(R)$  is dense in  $R^n$  is called the *topological stable rank* of  $R$ . We say  $R$  has *infinite topological stable rank* if no such integer  $n$  exists.

**Corollary 1.8.** *The topological stable rank of  $\mathcal{H}^\infty$  is infinite.*

*Proof.* This follows from the inequality that the Bass stable rank of a commutative unital semisimple complex Banach algebra is at most equal to its topological stable rank; see [14, Corollary 2.4].  $\square$

**Definition 1.9.** The Krull dimension of a commutative ring  $R$  is the supremum of the lengths of chains of distinct proper prime ideals of  $R$ .

**Corollary 1.10.** *The Krull dimension of  $\mathcal{H}^\infty$  is infinite.*

*Proof.* This follows from the fact that if a ring has Krull dimension  $d$ , then its Bass stable rank is at most  $d + 2$ ; see [9].  $\square$

## 2. Noncoherence of $\mathcal{H}^\infty$

We will use the following fact due to Amar [1, Proof of Theorem 1.(ii)].

**Proposition 2.1.**  $(z_1 - z_2, z_2 - z_3)^\perp$  is not a finitely generated  $H^\infty(\mathbb{D}^3)$ -module.

*Proof of Theorem 1.4.* The main idea of the proof is that, using the isomorphism  $\iota$ , essentially we boil the problem down to working with  $H^\infty(\mathbb{D}^\infty)$ . Let

$$\begin{aligned} f_1 &:= \frac{1}{2^s} - \frac{1}{3^s}, \\ f_2 &:= \frac{1}{3^s} - \frac{1}{5^s}. \end{aligned}$$

Then  $\iota(f_1) = z_1 - z_2$  and  $\iota(f_2) = z_2 - z_3$ . Suppose that  $(f_1, f_2)^\perp$  is a finitely generated  $\mathcal{H}^\infty$ -module, say by

$$\begin{bmatrix} g_1^{(1)} \\ g_1^{(2)} \end{bmatrix}, \dots, \begin{bmatrix} g_r^{(1)} \\ g_r^{(2)} \end{bmatrix} \in (\mathcal{H}^\infty)^2.$$

We will show that the 3rd section of the image under  $\iota$  of the above elements generate  $(z_1 - z_2, z_2 - z_3)^\perp$  in  $H^\infty(\mathbb{D}^3)$ , contradicting Proposition 2.1. If

$$\begin{bmatrix} G^{(1)} \\ G^{(2)} \end{bmatrix} \in (H^\infty(\mathbb{D}^3))^2 \cap (F_1, F_2)^\perp,$$

then  $F_1 G^{(1)} + F_2 G^{(2)} = 0$ , and by applying  $\iota^{-1}$ , we see that

$$\begin{bmatrix} \iota^{-1} G^{(1)} \\ \iota^{-1} G^{(2)} \end{bmatrix} \in (f_1, f_2)^\perp.$$

So there exist  $\alpha^{(1)}, \dots, \alpha^{(r)} \in \mathcal{H}^\infty$  such that

$$\begin{bmatrix} \iota^{-1} G^{(1)} \\ \iota^{-1} G^{(2)} \end{bmatrix} = \alpha^{(1)} \begin{bmatrix} g_1^{(1)} \\ g_1^{(2)} \end{bmatrix} + \dots + \alpha^{(r)} \begin{bmatrix} g_r^{(1)} \\ g_r^{(2)} \end{bmatrix}.$$

Applying  $\iota$ , we obtain

$$\begin{bmatrix} G^{(1)} \\ G^{(2)} \end{bmatrix} = \iota(\alpha^{(1)}) \begin{bmatrix} \iota(g_1^{(1)}) \\ \iota(g_1^{(2)}) \end{bmatrix} + \dots + \iota(\alpha^{(r)}) \begin{bmatrix} \iota(g_r^{(1)}) \\ \iota(g_r^{(2)}) \end{bmatrix}.$$

Finally taking the 3rd section, we obtain

$$\begin{bmatrix} G^{(1)}(z_1, z_2, z_3) \\ G^{(2)}(z_1, z_2, z_3) \end{bmatrix} = \sum_{j=1}^r (\iota(\alpha^{(j)}))(\mathbf{z}^{(3)}) \begin{bmatrix} (\iota(g_j^{(1)}))(\mathbf{z}^{(3)}) \\ (\iota(g_j^{(2)}))(\mathbf{z}^{(3)}) \end{bmatrix}.$$

So it follows that

$$\begin{bmatrix} (\iota(g_1^{(1)}))(\mathbf{z}^{(3)}) \\ (\iota(g_1^{(2)}))(\mathbf{z}^{(3)}) \end{bmatrix}, \dots, \begin{bmatrix} (\iota(g_r^{(1)}))(\mathbf{z}^{(3)}) \\ (\iota(g_r^{(2)}))(\mathbf{z}^{(3)}) \end{bmatrix}$$

generate  $(z_1 - z_2, z_2 - z_3)^\perp$ , a contradiction to Amar's result, Proposition 2.1.  $\square$

### 3. Stable rank of $\mathcal{H}^\infty$

The proof of Theorem 1.6 is a straightforward adaptation of the first author's proof of the fact that the Bass stable rank of the infinite polydisk algebra is infinite [12, Proposition 1]. In [12], the infinite polydisk algebra  $A(\mathbb{D}^\infty)$  is the uniform closure of the algebra generated by the coordinate functions  $z_1, z_2, z_3, \dots$  on the countably infinite polydisk  $\overline{\mathbb{D}} \times \overline{\mathbb{D}} \times \overline{\mathbb{D}} \times \dots$ .

*Proof of Theorem 1.6.* Fix  $n \in \mathbb{N}$ . Let  $g \in \mathcal{H}^\infty$  be given by

$$(3.1) \quad g(s) := \prod_{j=1}^n \left( 1 - \frac{1}{(p_j p_{n+j})^s} \right) \in \mathcal{H}^\infty.$$

Set

$$\mathbf{f} := \left( \frac{1}{2^s}, \dots, \frac{1}{p_n^s}, g \right) \in (\mathcal{H}^\infty)^{n+1}.$$

We will show that  $\mathbf{f} \in U_{n+1}(\mathcal{H}^\infty)$  is not reducible. First let us note that  $\mathbf{f}$  is unimodular. Indeed, by expanding the product on the right hand side of (3.1), we obtain

$$g = 1 + \frac{1}{2^s} \cdot g_1 + \dots + \frac{1}{p_n^s} \cdot g_n$$

for some appropriate  $g_1, \dots, g_n \in \mathcal{H}^\infty$ . Now suppose that  $\mathbf{f}$  is reducible, and that there exist  $h_1, \dots, h_n \in \mathcal{H}^\infty$  such that

$$\left( \frac{1}{2^s} + gh_1, \dots, \frac{1}{p_n^s} + gh_n \right) \in U_n(\mathcal{H}^\infty).$$

Let  $y_1, \dots, y_n \in \mathcal{H}^\infty$  be such that

$$\left( \frac{1}{2^s} + gh_1 \right) y_1 + \dots + \left( \frac{1}{p_n^s} + gh_n \right) y_n = 1.$$

Applying  $\iota$ , we obtain

$$(3.2) \quad (z_1 + \iota(g)\iota(h_1))\iota(y_1) + \dots + (z_n + \iota(g)\iota(h_n))\iota(y_n) = 1.$$

Let  $\mathbf{h} := (\iota(h_1), \dots, \iota(h_n))$ . For  $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ , we define

$$\Phi(\mathbf{z}) = \begin{cases} -\mathbf{h}(z_1, \dots, z_n, \overline{z_1}, \dots, \overline{z_n}, 0, \dots) \prod_{j=1}^n (1 - |z_j|^2) & \text{for } |z_j| < 1, j = 1, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Then  $\Phi$  is a continuous map from  $\mathbb{C}^n$  into  $\mathbb{C}^n$ . But  $\Phi$  vanishes outside  $\mathbb{D}^n$ , and so

$$\max_{\mathbf{z} \in \mathbb{D}^n} \|\Phi(\mathbf{z})\|_2 = \sup_{\mathbf{z} \in \mathbb{C}^n} \|\Phi(\mathbf{z})\|_2.$$

This implies that there must exist an  $r \geq 1$  such that  $\Phi$  maps  $K := r\overline{\mathbb{D}}^n$  into  $K$ . As  $K$  is compact and convex, by Brouwer's Fixed Point Theorem it follows that there exists a  $\mathbf{z}_* \in K$  such that

$$\Phi(\mathbf{z}_*) = \mathbf{z}_*.$$

Since  $\Phi$  is zero outside  $\mathbb{D}^n$ , we see that  $\mathbf{z}_* \in \mathbb{D}^n$ . Let  $\mathbf{z}_* = (\zeta_1, \dots, \zeta_n)$ . Then for each  $j \in \{1, \dots, n\}$ , we obtain

$$\begin{aligned} 0 &= \zeta_j + (\iota(h_j))(\zeta_1, \dots, \zeta_n, \overline{\zeta_1}, \dots, \overline{\zeta_n}, 0, \dots) \prod_{k=1}^n (1 - |\zeta_k|^2) \\ (3.3) \quad &= \zeta_j + (\iota(h_j)\iota(g))(\zeta_1, \dots, \zeta_n, \overline{\zeta_1}, \dots, \overline{\zeta_n}, 0, \dots). \end{aligned}$$

But from (3.2), we know that

$$\sum_{j=1}^n (z_j + \iota(h_j)\iota(g))\iota(y_j) = 1,$$

and this contradicts (3.3). As the choice of  $n \in \mathbb{N}$  was arbitrary, it follows that the Bass stable rank of  $\mathcal{H}^\infty$  is infinite.  $\square$

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