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CERTAIN FRACTIONAL INTEGRAL INEQUALITIES ASSOCIATED WITH PATHWAY FRACTIONAL INTEGRAL OPERATORS

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Abstract. During the past two decades or so, fractional integral inequalities have proved to be one of the most powerful and far-reaching tools for the development of many branches of pure and applied mathematics. Very recently, many authors have presented some generalized inequalities involving the fractional integral operators. Here, using the pathway fractional integral operator, we give some presumably new and potentially useful fractional integral inequalities whose special cases are shown to yield corresponding inequalities associated with Riemann-Liouville type fractional integral operators. Relevant connections of the results presented here with those earlier ones are also pointed out.

1. Introduction and preliminaries

In recent years certain interesting and useful fractional integral inequalities involving functions of independent variables in applied sciences have been presented via fractional integral operators. During the last two decades or so, several interesting and useful extensions of many of the fractional integral inequalities have been considered by many authors (see, *e.g.*, [1, 7, 20, 21, 25, 26]; see also the very recent work [3] and [4]). Recently many authors have presented a number of interesting integral inequalities of Pólya and Szegö type by using the Riemann-Liouville fractional integral operator (see [4, 23]). Nair [22] introduced and investigated a new fractional integral operator through the idea of pathway model given by Mathai [17] (and further studied by Mathai and Haubold [18, 19]). Here, motivated essentially by the above works, we aim at establishing certain (presumably) new Pólya-Szegö type inequalities associated with the pathway fractional integral operator. Relevant connections of the results presented here with those involving Riemann-Liouville fractional integrals are also indicated.

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Throughout this paper, let $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{C}$ be the sets of real, positive real and complex numbers, respectively.

We begin by recalling the well-known celebrated functional introduced and defined by Chebyshev [6]:

(1.1)
$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right),$$

where f and g are two real-valued integrable functions which are synchronous on $[a,\, b]$: That is,

$$(f(x)-f(y))\,(g(x)-g(y)) \geq 0 \tag{1.2}$$

for any x, $y \in [a,b]$. The functional (1.1) has attracted many researchers' attention due mainly to its demonstrated applications in numerical quadrature, transform theory, probability and statistical problems. Among those applications, the functional (1.1) has also been employed to yield a number of integral inequalities (see, *e.g.*, [2, 5, 8, 9, 11, 16, 24, 30]; for a very recent work, see also [31]).

In 1935, Grüss [13] proved the inequality

$$|T(f,\, g)| \leq \frac{(M-m)(N-n)}{4}, \tag{1.3}$$

where f and g are two integrable and synchronous functions on $[a,\, b]$ which are also bounded on $[a,\, b]$: That is,

$$m \leq f(x) \leq M \quad \text{and} \quad n \leq g(x) \leq N \tag{1.4}$$

for all x, $y \in [a,b]$ and for some m, M, n, and $N \in \mathbb{R}$.

In the sequel, under the same assumptions as in (1.3), Pólya and Szegö [27] introduced the following inequality:

$$\frac{\int_a^b f^2(x)dx\ \int_a^b g^2(x)dx}{\left(\int_a^b f(x)dx\ \ \int_a^b g(x)dx\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{MN}{mn}} + \sqrt{\frac{mn}{MN}}\right)^2. \tag{1.5}$$

Dragomir and Diamond [10] proved that

$$|T(f,\, g)| \leq \frac{(M-m)\ (N-n)}{4(b-a)^2\sqrt{mMnN}}\int_a^b f(x)dx\ \int_a^b g(x)dx, \tag{1.6}$$

where f and g are two integrable and synchronous functions on $[a,\, b]$ which are positive and bounded on $[a,\, b]$: That is,

$$0 < m \leq f(x) \leq M \quad \text{and} \quad 0 < n \leq g(x) \leq N \tag{1.7}$$

for all x, $y \in [a,b]$ and for some m, M, n, and $N \in \mathbb{R}$.

The following definitions and some related properties will be required for our purpose.

Definition 1. A real-valued function $f(t)$ $(t > 0)$ is said to be in the space C^n_μ $(n,\, \mu \in \mathbb{R})$, if there exists a real number $p > \mu$ such that $f^{(n)}(t) = t^p\, \phi(t)$, where $\phi(t) \in C(0,\, \infty)$. Here, for the case $n = 1$, we use a simpler notation $C^1_\mu = C_\mu$.

Definition 2. Let $L[a, b]$ denote the set of Lebesgue measurable real or complex valued functions f defined on $[a,\, b]$ such that

$$\int_a^b |f(t)|\, dt < \infty.$$

Let $f \in L[a, b]$, $a > 0$, $\eta \in \mathbb{C}$ with $\Re(\eta) > 0$ and let us take a pathway parameter $\alpha < 1$. Then the pathway fractional integration operator $P^{(\eta,\alpha,a)}_{0^+} f$ for the function f is defined as follows (see [22, p. 239]):

$$P^{(\eta,\alpha,a)}_{0^+}\{f(\tau)\}(t) := t^\eta \int_0^{\frac{t}{a(1-\alpha)}} \left[1 - \frac{a\,(1-\alpha)\,\tau}{t}\right]^{\frac{\eta}{1-\alpha}} f(\tau)\, d\tau, \tag{1.8}$$

which, when the integral variable τ does not matter, is denoted by

$$P^{(\eta,\alpha,a)}_{0^+}\{f\}(t). \tag{1.9}$$

Let $[a,\, b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The left-sided and right-sided Riemann-Liouville fractional integrals $I^\eta_{a+} f$ and $I^\eta_{b-} f$ of order $\eta \in \mathbb{C}$ $(\Re(\eta) > 0)$ are defined, respectively, by

$$\left(I^\eta_{a+} f\right)(x) := \frac{1}{\Gamma(\eta)} \int_a^x \frac{f(t)\, dt}{(x-t)^{1-\eta}} \quad (x > a;\ \Re(\eta) > 0) \tag{1.10}$$

and

$$\left(I^\eta_{b-} f\right)(x) := \frac{1}{\Gamma(\eta)} \int_x^b \frac{f(t)\, dt}{(t-x)^{1-\eta}} \quad (x < b;\ \Re(\eta) > 0), \tag{1.11}$$

where $f \in C_\mu$ $(\mu \geq -1)$ (see, *e.g.*, [15, p. 69]) and Γ is the familiar Gamma function (see, *e.g.*, [28, Section 1.1] and [29, Section 1.1]).

Remark 1. The special case of the pathway fractional integration operator in (1.8) when $\alpha = 0$, $a = 1$, and $\eta \to \eta - 1$ reduces immediately to the left-sided Riemann-Liouville fractional integrals as follows:

$$\left(P^{(\eta-1,0,1)}_{0^+} f\right)(t) = \int_0^t (t-\tau)^{\eta-1}\, f(\tau)\, d\tau = \Gamma(\eta)\, \left(I^\eta_{0+} f\right)(t) \quad (\Re(\eta) > 0). \tag{1.12}$$

Further one of the Erdélyi-Kober type fractional integrals (see [15, p. 105, Eq. (2.6.1)]) defined by

$$\left(I^\eta_{a+;\sigma,\alpha} f\right)(t) := \frac{\sigma\, t^{-\sigma(\eta+\alpha)}}{\Gamma(\eta)} \int_a^t \frac{\tau^{\sigma\,\alpha+\sigma-1}\, f(\tau)\, d\tau}{\left(t^\sigma - \tau^\sigma\right)^{1-\eta}} \tag{1.13}$$

$$(0 \leqq a < t < b \leqq \infty;\ \Re(\eta) > 0;\ \sigma > 0;\ \alpha \in \mathbb{C})$$

appears to be closely related to the pathway fractional integration operator (1.8). It is found that one of the two integral operators (1.8) and (1.13) cannot contain the other one as a purely special case. Yet it is easy to see that some special cases of the two integrals have, for example, the following relationship:

$$\left(P_{0^+}^{(\eta-1,0,1)} f\right)(t) = \Gamma(\eta)\, t^{\eta} \left(I_{0+;1,0}^{\eta} f\right)(t). \tag{1.14}$$

The case $f(t) = t^{\beta-1}$ of (1.8) is known to give the following formula (see [22, Eq. (12)]):

$$P_{0^+}^{(\eta,\alpha,a)}\left(t^{\beta-1}\right) = \frac{t^{\eta+\beta}}{[a(1-\alpha)]^{\beta}}\, \frac{\Gamma(\beta)\, \Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\Gamma\left(\frac{\eta}{1-\alpha}+\beta+1\right)} \tag{1.15}$$

$$(\alpha < 1;\ \Re(\eta) > 0;\ \Re(\beta) > 0).$$

Indeed, setting $f(t) = t^{\beta-1}$ in (1.8) and then putting $u = \frac{a(1-\alpha)\tau}{t}$, some algebra gives that

$$P_{0^+}^{(\eta,\alpha,a)}\left(t^{\beta-1}\right) = \frac{t^{\eta+\beta}}{[a(1-\alpha)]^{\beta}}\, B\left(\frac{\eta}{1-\alpha}+1,\, \beta\right),$$

where $B(\alpha, \beta)$ is the well-known Beta function which is closely related to the Gamma function as follows:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\, \Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{1.16}$$

where α and β are complex numbers which are neither 0 nor negative integers (see, *e.g.*, [28, pp. 9–11] and [29, pp. 7–10]).

2. Certain Pólya-Szegö type fractional integral inequalities associated with pathway operator

In this section, we establish certain Pólya-Szegö type integral inequalities for the synchronous functions involving the pathway fractional integral operator (1.8), which can be established by similar arguments used in the proof of Theorems 1, 2 and 3 in [14]. So the proof details are omitted.

Lemma 1. *Let u, v, m, M, n and N be continuous and positive integrable functions on* $[0, \infty)$ *with*

$$0 < m(\tau) \le u(\tau) \le M(\tau),\ 0 < n(\tau) \le v(\tau) \le N(\tau)\ (\tau \in [0, t],\ t > 0). \tag{2.1}$$

Then the following inequality holds true:

$$\frac{P_{0^+}^{(\eta,\alpha,a)}\left\{n\, N\, u^2\right\}(t)\, P_{0^+}^{(\eta,\alpha,a)}\left\{m\, M\, v^2\right\}(t)}{\left(P_{0^+}^{(\eta,\alpha,a)}\left\{(n\, m + M\, N)\, uv\right\}(t)\right)^2} \le \frac{1}{4} \tag{2.2}$$

for all $a > 0$, $\alpha < 1$, $t > 0$, and $\eta > 0$.

We present another inequality of Pólya-Szegö type involving the pathway fractional integral operator in (1.8) asserted by the following lemma.

Lemma 2. *Let u, v, m, M, n and N be continuous and positive integrable functions on* $[0, \infty)$ *satisfying the inequalities* (2.1). *Then the following inequality holds true*:

$$\frac{\left(P_{0+}^{(\eta,\alpha,a)}\{m\,M\}(t)\right)\left(P_{0+}^{(\zeta,\alpha,a)}\{n\,N\}(t)\right)-\left(P_{0+}^{(\eta,\alpha,a)}\{u^2\}(t)\right)\left(P_{0+}^{(\zeta,\alpha,a)}\{v^2\}(t)\right)}{\left(P_{0+}^{(\eta,\alpha,a)}\{u\,m\}(t)\,P_{0+}^{(\zeta,\alpha,a)}\{v\,n\}(t)+P_{0+}^{(\eta,\alpha,a)}\{u\,M\}(t)\,P_{0+}^{(\zeta,\alpha,a)}\{v\,N\}(t)\right)^2} \leq \frac{1}{4} \tag{2.3}$$

for all $a > 0$, $\alpha < 1$, $t > 0$, $\zeta > 0$, and $\eta > 0$.

Lemma 3. *Let u, v, m, M, n and N be continuous and positive integrable functions on* $[0, \infty)$ *satisfying the inequalities* (2.1). *Then the following inequality holds true*:

$$\begin{aligned}&\left(P_{0+}^{(\eta,\alpha,a)}\left\{u^2\right\}(t)\right)\left(P_{0+}^{(\zeta,\alpha,a)}\left\{v^2\right\}(t)\right)\\ \leq\ &\left(P_{0+}^{(\eta,\alpha,a)}\left\{\frac{M\,u\,v}{n}\right\}(t)\right)\left(P_{0+}^{(\zeta,\alpha,a)}\left\{\frac{N\,u\,v}{m}\right\}(t)\right)\end{aligned} \tag{2.4}$$

for all $a > 0$, $\alpha < 1$, $t > 0$, $\zeta > 0$, and $\eta > 0$.

3. Chebyshev type fractional integral inequalities associated with the pathway operator

In this section, we establish certain Chebyshev type fractional integral inequalities associated with the pathway operator with the help of Pólya-Szegö type fractional integral inequalities given by Lemmas 1 and 2.

Theorem 1. *Let u, v, m, M, n and N be continuous and positive integrable functions on* $[0, \infty)$ *satisfying the inequalities* (2.1). *Then the following inequality holds true*:

$$\begin{aligned}&\left|\frac{t^{\eta+1}}{a(1-\alpha+\eta)}\,P_{0+}^{(\eta,\alpha,a)}\{f\,g\}(t)-P_{0+}^{(\eta,\alpha,a)}\{f\}(t)\,P_{0+}^{(\eta,\alpha,a)}\{g\}(t)\right|\\ \leq\ &\left|\mathcal{H}(f,m,M)\,\mathcal{H}(g,n,N)\right|^{1/2}, \quad (a>0;\ \alpha<1;\ t>0;\ \eta>0),\end{aligned} \tag{3.1}$$

where

$$\mathcal{H}(f,m,M) := \frac{t^{\eta+1}}{4\,a(1-\alpha+\eta)}\frac{\left(P_{0+}^{(\eta,\alpha,a)}\{(m+M)f\}(t)\right)^2}{P_{0+}^{(\eta,\alpha,a)}\{m\,M\}(t)}-\left(P_{0+}^{(\eta,\alpha,a)}\{f\}(t)\right)^2 \tag{3.2}$$

and

$$\mathcal{H}(g,n,N) := \frac{t^{\eta+1}}{4\,a(1-\alpha+\eta)}\frac{\left(P_{0+}^{(\eta,\alpha,a)}\{(n+N)g\}(t)\right)^2}{P_{0+}^{(\eta,\alpha,a)}\{n\,N\}(t)}-\left(P_{0+}^{(\eta,\alpha,a)}\{g\}(t)\right)^2. \tag{3.3}$$

Proof. For all τ, $\rho \in (0, t]$ with $t > 0$, let

$$\begin{aligned}A(\tau,\rho) :&= (f(\tau)-f(\rho))\,(g(\tau)-g(\rho))\\ &= f(\tau)g(\tau)+f(\rho)g(\rho)-f(\tau)g(\rho)-f(\rho)g(\tau).\end{aligned} \tag{3.4}$$

Multiplying both sides of (3.4) by $t^{2\eta}\left[1-\frac{a(1-\alpha)\tau}{t}\right]^{\frac{\eta}{1-\alpha}}$ $[1-\frac{a(1-\alpha)\rho}{t}]^{\frac{\eta}{1-\alpha}}$ and integrating each side of the resulting equality with respect to τ and ρ from 0 to $\frac{t}{a(1-\alpha)}$, respectively, and using (1.8), we get

$$\begin{aligned}&\mathcal{F}_{f,g}^{(\eta,\alpha,a)}(t)\\ &=2\frac{t^{\eta+1}}{a(1-\alpha+\eta)}P_{0^+}^{(\eta,\alpha,a)}\{fg\}(t)-2\left(P_{0^+}^{(\eta,\alpha,a)}f(t)\right)\left(P_{0^+}^{(\eta,\alpha,a)}g(t)\right),\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}\mathcal{F}_{f,g}^{(\eta,\alpha,a)}(t):=t^{2\eta}\int_0^{\frac{t}{a(1-\alpha)}}\int_0^{\frac{t}{a(1-\alpha)}}&\left[1-\frac{a\,(1-\alpha)\,\tau}{t}\right]^{\frac{\eta}{1-\alpha}}\\ \times\left[1-\frac{a\,(1-\alpha)\,\rho}{t}\right]^{\frac{\eta}{1-\alpha}}&A(\tau,\rho)\,d\tau\,d\rho.\end{aligned}\tag{3.6}$$

On the other hand, by using the Cauchy's inequality for integrals (see, *e.g.*, [12, p. 243]), we have

$$\left|\mathcal{F}_{f,g}^{(\eta,\alpha,a)}(t)\right|^2\le\mathcal{H}_f^{(\eta,\alpha,a)}(t)\,\mathcal{H}_g^{(\eta,\alpha,a)}(t),\tag{3.7}$$

where

$$\begin{aligned}\mathcal{H}_f^{(\eta,\alpha,a)}(t):=t^{2\eta}\int_0^{\frac{t}{a(1-\alpha)}}\int_0^{\frac{t}{a(1-\alpha)}}&\left[1-\frac{a\,(1-\alpha)\,\tau}{t}\right]^{\frac{\eta}{1-\alpha}}\\ \times\left[1-\frac{a\,(1-\alpha)\,\rho}{t}\right]^{\frac{\eta}{1-\alpha}}&(f(\tau)-f(\rho))^2\,d\tau\,d\rho\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}_g^{(\eta,\alpha,a)}(t):=t^{2\eta}\int_0^{\frac{t}{a(1-\alpha)}}\int_0^{\frac{t}{a(1-\alpha)}}&\left[1-\frac{a\,(1-\alpha)\,\tau}{t}\right]^{\frac{\eta}{1-\alpha}}\\ \times\left[1-\frac{a\,(1-\alpha)\,\rho}{t}\right]^{\frac{\eta}{1-\alpha}}&(g(\tau)-g(\rho))^2\,d\tau\,d\rho.\end{aligned}$$

It is easy to see the following:

$$\mathcal{H}_f^{(\eta,\alpha,a)}(t)=2\left[\frac{t^{\eta+1}}{a(1-\alpha+\eta)}P_{0^+}^{(\eta,\alpha,a)}\left\{f^2\right\}(t)-\left(P_{0^+}^{(\eta,\alpha,a)}\left\{f\right\}(t)\right)^2\right]\tag{3.8}$$

and

$$\mathcal{H}_g^{(\eta,\alpha,a)}(t)=2\left[\frac{t^{\eta+1}}{a(1-\alpha+\eta)}P_{0^+}^{(\eta,\alpha,a)}\left\{g^2\right\}(t)-\left(P_{0^+}^{(\eta,\alpha,a)}\left\{g\right\}(t)\right)^2\right].\tag{3.9}$$

Setting $u=f$ and $n=N=v=1$ in Lemma 1 yields

$$P_{0^+}^{(\eta,\alpha,a)}\left\{f^2\right\}(t)\le\frac{1}{4}\frac{\left(P_{0^+}^{(\eta,\alpha,a)}\left\{(m+M)\,f\right\}(t)\right)^2}{P_{0^+}^{(\eta,\alpha,a)}\left\{m\,M\right\}(t)}.\tag{3.10}$$

Then applying (3.10) to (3.8) gives

$$\begin{aligned}&\frac{1}{2}\mathcal{H}_f^{(\eta,\alpha,a)}(t)\\ &\leq \frac{t^{\eta+1}}{4a(1-\alpha+\eta)}\frac{\left(P_{0^+}^{(\eta,\alpha,a)}\left\{(m+M)\,f\right\}(t)\right)^2}{P_{0^+}^{(\eta,\alpha,a)}\left\{m\,M\right\}(t)}-\left(P_{0^+}^{(\eta,\alpha,a)}\left\{f\right\}(t)\right)^2.\end{aligned}\tag{3.11}$$

Similarly, we get

$$\begin{aligned}&\frac{1}{2}\mathcal{H}_f^{(\eta,\alpha,a)}(t)\\ &\leq \frac{t^{\eta+1}}{4a(1-\alpha+\eta)}\frac{\left(P_{0^+}^{(\eta,\alpha,a)}\left\{(m+M)\,g\right\}(t)\right)^2}{P_{0^+}^{(\eta,\alpha,a)}\left\{m\,M\right\}(t)}-\left(P_{0^+}^{(\eta,\alpha,a)}\left\{g\right\}(t)\right)^2.\end{aligned}\tag{3.12}$$

Finally, combining (3.5), (3.11) and (3.12) into (3.7) produces the desired inequality (3.1). This completes the proof. □

Theorem 2. *Let u, v, m, M, n and N be continuous and positive integrable functions on* $[0,\infty)$ *satisfying the inequalities* (2.1)*. Then, for all $a>0$, $\alpha<1$, $t>0$, $\eta>0$ and $\zeta>0$, we have*

$$\begin{aligned}&\Big|\frac{t^{\zeta+1}}{a(1-\alpha)+a\zeta}P_{0^+}^{(\eta,\alpha,a)}\left\{fg\right\}(t)+\frac{t^{\eta+1}}{a(1-\alpha)+a\eta}P_{0^+}^{(\zeta,\alpha,a)}\left\{fg\right\}(t)\\ &\quad-P_{0^+}^{(\eta,\alpha,a)}\left\{f\right\}(t)\,P_{0^+}^{(\zeta,\alpha,a)}\left\{g\right\}(t)-P_{0^+}^{(\eta,\alpha,a)}\left\{g\right\}(t)\,P_{0^+}^{(\zeta,\alpha,a)}\left\{f\right\}(t)\Big|\\ &\leq \left|\mathcal{H}_\zeta(f,m,M)(t)+\mathcal{H}_\eta(f,m,M)(t)\right|^{1/2}\left|\mathcal{H}_\zeta(g,n,N)(t)+\mathcal{H}_\eta(g,n,N)(t)\right|^{1/2}.\end{aligned}\tag{3.13}$$

where

$$\begin{aligned}\mathcal{H}_\zeta(u,v,w)(t):=&\frac{t^{\zeta+1}}{4(a(1-\alpha)+a\zeta)}\frac{\left(P_{0^+}^{(\eta,\alpha,a)}\left\{(v+w)u\right\}(t)\right)^2}{P_{0^+}^{(\eta,\alpha,a)}\left\{vw\right\}(t)}\\ &-P_{0^+}^{(\eta,\alpha,a)}\left\{u\right\}(t)\,P_{0^+}^{(\zeta,\alpha,a)}\left\{u\right\}(t)\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}\mathcal{H}_\eta(u,v,w)(t):=&\frac{t^{\eta+1}}{4(a(1-\alpha)+a\eta)}\frac{\left(P_{0^+}^{(\zeta,\alpha,a)}\left\{(v+w)u\right\}(t)\right)^2}{P_{0^+}^{(\zeta,\alpha,a)}\left\{vw\right\}(t)}\\ &-P_{0^+}^{(\zeta,\alpha,a)}\left\{u\right\}(t)\,P_{0^+}^{(\eta,\alpha,a)}\left\{u\right\}(t).\end{aligned}\tag{3.15}$$

Proof. Multiplying both sides of (3.4) by

$$t^{\eta+\zeta}\left[1-\frac{a\,(1-\alpha)\,\tau}{t}\right]^{\frac{\eta}{1-\alpha}}\;[1-\frac{a\,(1-\alpha)\,\rho}{t}]^{\frac{\zeta}{1-\alpha}}$$

and integrating each side of the resulting inequality with respect to τ and ρ, respectively, from 0 to $\frac{t}{a(1-\alpha)}$, and using (1.8), we get

$$\begin{aligned}
&\mathcal{F}_{f,g}^{(\zeta,\eta,\alpha,a)}(t)\\
&= \frac{t^{\eta+1}}{a(1-\alpha+\eta)} P_{0^+}^{(\zeta,\alpha,a)}\left\{f\,g\right\}(t) + \frac{t^{\zeta+1}}{a(1-\alpha+\zeta)} P_{0^+}^{(\eta,\alpha,a)}\left\{f\,g\right\}(t)\\
&\quad - P_{0^+}^{(\eta,\alpha,a)}\left\{f\right\}(t)\, P_{0^+}^{(\zeta,\alpha,a)}\left\{g\right\}(t) - P_{0^+}^{(\eta,\alpha,a)}\left\{g\right\}(t)\, P_{0^+}^{(\zeta,\alpha,a)}\left\{f\right\}(t),
\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}
\mathcal{F}_{f,g}^{(\zeta,\eta,\alpha,a)}(t) := t^{\zeta}\, t^{\eta} \int_0^{\frac{t}{a(1-\alpha)}} \int_0^{\frac{t}{a(1-\alpha)}} \left[1 - \frac{a\,(1-\alpha)\,\tau}{t}\right]^{\frac{\eta}{1-\alpha}}\\
\times \left[1 - \frac{a\,(1-\alpha)\,\rho}{t}\right]^{\frac{\eta}{1-\alpha}} A(\tau,\rho)\, d\tau\, d\rho.
\end{aligned} \tag{3.17}$$

Now, by using the Cauchy's inequality for integrals, we have

$$\left|\mathcal{F}_{f,g}^{(\zeta,\eta,\alpha,a)}(t)\right|^2 \leq \mathcal{H}_{f}^{(\zeta,\eta,\alpha,a)}(t)\, \mathcal{H}_{g}^{(\zeta,\eta,\alpha,a)}(t), \tag{3.18}$$

where

$$\begin{aligned}
\mathcal{H}_{f}^{(\zeta,\eta,\alpha,a)}(t) := t^{\zeta}\, t^{\eta} \int_0^{\frac{t}{a(1-\alpha)}} \int_0^{\frac{t}{a(1-\alpha)}} \left[1 - \frac{a\,(1-\alpha)\,\tau}{t}\right]^{\frac{\eta}{1-\alpha}}\\
\times \left[1 - \frac{a\,(1-\alpha)\,\rho}{t}\right]^{\frac{\eta}{1-\alpha}} (f(\tau)-f(\rho))^2\, d\tau\, d\rho
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{H}_{g}^{(\zeta,\eta,\alpha,a)}(t) := t^{\zeta}\, t^{\eta} \int_0^{\frac{t}{a(1-\alpha)}} \int_0^{\frac{t}{a(1-\alpha)}} \left[1 - \frac{a\,(1-\alpha)\,\tau}{t}\right]^{\frac{\eta}{1-\alpha}}\\
\times \left[1 - \frac{a\,(1-\alpha)\,\rho}{t}\right]^{\frac{\eta}{1-\alpha}} (g(\tau)-g(\rho))^2\, d\tau\, d\rho.
\end{aligned}$$

Then, it is easy to find the following:

$$\begin{aligned}
\mathcal{H}_{f}^{(\zeta,\eta,\alpha,a)}(t) &= \frac{t^{\eta+1}}{a(1-\alpha+\eta)} P_{0^+}^{(\zeta,\alpha,a)}\left\{f^2\right\}(t) + \frac{t^{\zeta+1}}{a(1-\alpha+\zeta)} P_{0^+}^{(\eta,\alpha,a)}\left\{f^2\right\}(t)\\
&\quad - 2P_{0^+}^{(\eta,\alpha,a)}\left\{f\right\}(t)\, P_{0^+}^{(\zeta,\alpha,a)}\left\{f\right\}(t)
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
\mathcal{H}_{g}^{(\zeta,\eta,\alpha,a)}(t) &= \frac{t^{\eta+1}}{a(1-\alpha+\eta)} P_{0^+}^{(\zeta,\alpha,a)}\left\{g^2\right\}(t) + \frac{t^{\zeta+1}}{a(1-\alpha+\zeta)} P_{0^+}^{(\eta,\alpha,a)}\left\{g^2\right\}(t)\\
&\quad - 2P_{0^+}^{(\eta,\alpha,a)}\left\{g\right\}(t)\, P_{0^+}^{(\zeta,\alpha,a)}\left\{g\right\}(t).
\end{aligned} \tag{3.20}$$

Setting $n = N = v = 1$ in Lemma 1 gives

$$P_{0^+}^{(\eta,\alpha,a)}\left\{u^2\right\}(t) \le \frac{1}{4}\frac{\left(P_{0^+}^{(\eta,\alpha,a)}\left\{(m+M)\,u\right\}(t)\right)^2}{P_{0^+}^{(\eta,\alpha,a)}\left\{m\,M\right\}(t)}. \tag{3.21}$$

Then applying (3.21) to (3.19) and (3.20), respectively, yields the following inequalities:

$$\begin{aligned}\mathcal{H}_f^{(\zeta,\eta,\alpha,a)}(t) \le{}& \frac{t^{\zeta+1}}{4a(1-\alpha+\zeta)}\frac{\left(P_{0^+}^{(\eta,\alpha,a)}\left\{(m+M)\,f\right\}(t)\right)^2}{P_{0^+}^{(\eta,\alpha,a)}\left\{m\,M\right\}(t)}\\ &+\frac{t^{\eta+1}}{4a(1-\alpha+\eta)}\frac{\left(P_{0^+}^{(\zeta,\alpha,a)}\left\{(m+M)\,f\right\}(t)\right)^2}{P_{0^+}^{(\zeta,\alpha,a)}\left\{m\,M\right\}(t)}\\ &-2\,P_{0^+}^{(\eta,\alpha,a)}\left\{f\right\}(t)\,P_{0^+}^{(\zeta,\alpha,a)}\left\{f\right\}(t)\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}\mathcal{H}_g^{(\zeta,\eta,\alpha,a)}(t) \le{}& \frac{t^{\zeta+1}}{4a(1-\alpha+\zeta)}\frac{\left(P_{0^+}^{(\eta,\alpha,a)}\left\{(m+M)\,g\right\}(t)\right)^2}{P_{0^+}^{(\eta,\alpha,a)}\left\{m\,M\right\}(t)}\\ &+\frac{t^{\eta+1}}{4a(1-\alpha+\eta)}\frac{\left(P_{0^+}^{(\zeta,\alpha,a)}\left\{(m+M)\,g\right\}(t)\right)^2}{P_{0^+}^{(\zeta,\alpha,a)}\left\{m\,M\right\}(t)}\\ &-2\,P_{0^+}^{(\eta,\alpha,a)}\left\{f\right\}(t)\,P_{0^+}^{(\zeta,\alpha,a)}\left\{g\right\}(t).\end{aligned} \tag{3.23}$$

Finally using (3.16), (3.22) and (3.23) for the inequality (3.18) is immediately seen to yield the desired inequality (3.13). The proof is complete. □

Remark 2. It may be noted that the inequality in (3.13) when $\zeta = \eta$ reduces immediately to that in (3.1). As noted previously in (1.13), since a special case of the pathway fractional integral operator when the parameters are suitably chosen reduces to the left-sided Riemann-Liouville fractional integral operator, the results in Theorems 1 and 2 yield some known ones, for example, see [23].

4. Special cases and concluding remarks

We can present a large number of special cases of our inequalities in Lemmas 1, 2, and 3 and Theorems 1 and 2. For example, let $m(\tau) = m$, $M(\tau) = M$, $n(\tau) = n$ and $N(\tau) = N$ be constant functions in Lemmas 1, 2, and 3, we obtain the following inequalities as given in Corollaries 1, 2, and 3, respectively.

Corollary 1. *Let* u and v *be two continuous and positive integrable functions on* $[0, \infty)$ *and there exist constants* m, M, n, *and* N *such that*

$$0 < m \le u(\tau) \le M < \infty,\ \ 0 < n \le v(\tau) \le N < \infty \quad (\tau \in [0,t],\, t > 0)\,. \tag{4.1}$$

Then the following inequality holds true:

$$\frac{P_{0^+}^{(\eta,\alpha,a)}\left\{u^2\right\}(t)\,P_{0^+}^{(\eta,\alpha,a)}\left\{v^2\right\}(t)}{\left(P_{0^+}^{(\eta,\alpha,a)}\left\{uv\right\}(t)\right)^2} \le \frac{1}{4}\left(\frac{mn}{MN}+\frac{MN}{mn}\right) \tag{4.2}$$

for all $a>0$*,* $\alpha<1$*,* $t>0$*, and* $\eta>0$*.*

Corollary 2. *Let* u and v *be two positive integrable functions on* $[0,\infty)$ *and there exist four constants* m*,* n*,* M*, and* N *satisfying the inequalities* (4.1)*. Then the following inequality holds true*:

$$\begin{aligned}&\frac{t^{\eta+\zeta+2}}{a^2(1-\alpha+\eta)\,(1-\alpha+\zeta)}\frac{\left(P_{0^+}^{(\eta,\alpha,a)}\left\{u^2\right\}(t)\right)\,\left(P_{0^+}^{(\zeta,\alpha,a)}\left\{v^2\right\}(t)\right)}{\left(P_{0^+}^{(\eta,\alpha,a)}\left\{u\right\}(t)\,P_{0^+}^{(\zeta,\alpha,a)}\left\{v\right\}(t)\right)^2}\\&\quad\le\frac{mn}{MN}+\frac{MN}{mn}\end{aligned} \tag{4.3}$$

for all $a>0$*,* $\alpha<1$*,* $t>0$*,* $\zeta>0$*, and* $\eta>0$*.*

Corollary 3. *Let* u *and* v *be two positive integrable functions on* $[0,\infty)$ *and there exist four constants* m*,* n*,* M*, and* N *satisfying the inequalities* (4.1)*. Then the following inequality holds true*:

$$\frac{\left(P_{0^+}^{(\eta,\alpha,a)}\left\{u^2\right\}(t)\right)\,\left(P_{0^+}^{(\zeta,\alpha,a)}\left\{v^2\right\}(t)\right)}{\left(P_{0^+}^{(\eta,\alpha,a)}\left\{u\,v\right\}(t)\right)\,\left(P_{0^+}^{(\zeta,\alpha,a)}\left\{u\,v\right\}(t)\right)} \le \frac{M\,N}{m\,n} \tag{4.4}$$

for all $a>0$*,* $\alpha<1$*,* $t>0$*,* $\zeta>0$*, and* $\eta>0$*.*

Setting $f(\tau)=\tau^{\lambda}$ in Theorems 1 and 2 and using the formula (1.15), the inequalities (3.1) and (3.13) give two interesting inequalities asserted by Corollaries 4 and 5, respectively.

Corollary 4. *Let* g*,* m*,* M*,* n*, and* N *be positive integrable functions on* $[0,\infty)$ *satisfying the inequalities* (2.1)*. Then the following inequality holds true*: *For all* $a>0$*,* $\alpha<1$*,* $t>0$*,* $\lambda>-1$ and $\eta>0$*,*

$$\begin{aligned}&\left|\frac{t^{\eta+1}}{a(1-\alpha+\eta)}\,P_{0^+}^{(\eta,\alpha,a)}\left\{\tau^{\lambda}\,g(\tau)\right\}(t)\right.\\&\qquad\left.-\frac{t^{\eta+\lambda+1}}{[a(1-\alpha)]^{\lambda+1}}\,\frac{\Gamma(\lambda+1)\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\Gamma\left(\frac{\eta}{1-\alpha}+\lambda+2\right)}\,P_{0^+}^{(\eta,\alpha,a)}\left\{g\right\}(t)\right|\\&\quad\le\;\left|\mathcal{J}(m,M)\,\mathcal{H}(g,n,N)\right|^{1/2},\end{aligned} \tag{4.5}$$

where

$$\mathcal{J}(m,M) := \frac{t^{\eta+1}}{4\,a(1-\alpha+\eta)} \frac{\left(P_{0^+}^{(\eta,\alpha,a)}\left\{(m+M)\tau^{\lambda}\right\}(t)\right)^2}{P_{0^+}^{(\eta,\alpha,a)}\{m\,M\}(t)} - \left(\frac{t^{\eta+\lambda+1}}{[a(1-\alpha)]^{\lambda+1}}\,\frac{\Gamma(\lambda+1)\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\Gamma\left(\frac{\eta}{1-\alpha}+\lambda+2\right)}\right)^2 \tag{4.6}$$

and $\mathcal{H}(g,n,N)$ *is given in Theorem 1.*

Corollary 5. *Let* g, m, M, n, *and* N *be positive integrable functions on* $[0,\,\infty)$ *satisfying the inequalities* (2.1). *Then the following inequality holds true*: *For all* $a>0$, $\alpha<1$, $t>0$, $\lambda>-1$ *and* $\eta>0$,

$$\begin{aligned}&\left|\frac{t^{\zeta+1}}{a(1-\alpha)+a\zeta} P_{0^+}^{(\eta,\alpha,a)}\left\{\tau^{\lambda}\,g(\tau)\right\}(t) + \frac{t^{\eta+1}}{a(1-\alpha)+a\eta} P_{0^+}^{(\zeta,\alpha,a)}\left\{\tau^{\lambda}\,g(\tau)\right\}(t)\right.\\ &\quad - \frac{t^{\eta+\lambda+1}}{[a(1-\alpha)]^{\lambda+1}}\,\frac{\Gamma(\lambda+1)\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\Gamma\left(\frac{\eta}{1-\alpha}+\lambda+2\right)} P_{0^+}^{(\zeta,\alpha,a)}\{g\}(t)\\ &\quad \left. - \frac{t^{\zeta+\lambda+1}}{[a(1-\alpha)]^{\lambda+1}}\,\frac{\Gamma(\lambda+1)\,\Gamma\left(1+\frac{\zeta}{1-\alpha}\right)}{\Gamma\left(\frac{\zeta}{1-\alpha}+\lambda+2\right)} P_{0^+}^{(\eta,\alpha,a)}\{g\}(t)\right|\\ &\leq \left|\mathcal{L}_1(m,M)(t)+\mathcal{L}_2(m,M)(t)\right|^{1/2}\,\left|\mathcal{H}_{\zeta}(g,n,N)(t)+\mathcal{H}_{\eta}(g,n,N)(t)\right|^{1/2},\end{aligned} \tag{4.7}$$

where

$$\mathcal{L}_1(m,M)(t) = \frac{t^{\zeta+1}}{4(a(1-\alpha)+a\zeta)} \frac{\left(P_{0^+}^{(\eta,\alpha,a)}\left\{(m+M)\tau^{\lambda}\right\}(t)\right)^2}{P_{0^+}^{(\eta,\alpha,a)}\{m\,M\}(t)} - \frac{t^{\eta+\zeta+2\lambda+2}}{[a(1-\alpha)]^{\lambda+\zeta+2}}\,\frac{\Gamma^2(\lambda+1)\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)\,\Gamma\left(1+\frac{\zeta}{1-\alpha}\right)}{\Gamma\left(\frac{\eta}{1-\alpha}+\lambda+2\right)\,\Gamma\left(\frac{\zeta}{1-\alpha}+\lambda+2\right)} \tag{4.8}$$

and

$$\mathcal{L}_2(m,M)(t) = \frac{t^{\eta+1}}{4(a(1-\alpha)+a\eta)} \frac{\left(P_{0^+}^{(\zeta,\alpha,a)}\left\{(m+M)\tau^{\lambda}\right\}(t)\right)^2}{P_{0^+}^{(\zeta,\alpha,a)}\{mM\}(t)} - \frac{t^{\eta+\zeta+2\lambda+2}}{[a(1-\alpha)]^{\lambda+\zeta+2}}\,\frac{\Gamma^2(\lambda+1)\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)\,\Gamma\left(1+\frac{\zeta}{1-\alpha}\right)}{\Gamma\left(\frac{\eta}{1-\alpha}+\lambda+2\right)\,\Gamma\left(\frac{\zeta}{1-\alpha}+\lambda+2\right)} \tag{4.9}$$

and $\mathcal{H}_{\zeta}(g,n,N)(t)$ *and* $\mathcal{H}_{\eta}(g,n,N)(t)$ *are given in Theorem 2.*

We conclude our present investigation by remarking further that the results obtained here are useful in deriving various fractional integral inequalities involving such relatively more familiar fractional integral operators as (for example) the Riemann- Liouville fractional integral operator $\left(I^{\eta}_{0+}f\right)(t)$ given by (1.10) and the Erdélyi-Kober fractional integral operators $\left(I^{\eta}_{0+;1,0}f\right)(t)$ given by (1.14), respectively.

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