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DISTRIBUTIONAL SOLUTIONS OF WILSON'S FUNCTIONAL EQUATIONS WITH INVOLUTION AND THEIR ERDÖS' PROBLEM

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Abstract. We find the distributional solutions of the Wilson's functional equations

$$u \circ T + u \circ T^{\sigma} - 2u \otimes v = 0,$$
$$u \circ T + u \circ T^{\sigma} - 2v \otimes u = 0,$$

where $u, v \in \mathcal{D}'(\mathbb{R}^n)$, the space of Schwartz distributions, $T(x, y) = x + y,\ T^{\sigma}(x, y) = x + \sigma y,\ x, y \in \mathbb{R}^n$, σ an involution, and $\circ$, $\otimes$ are pullback and tensor product of distributions, respectively. As a consequence, we solve the Erdös' problem for the Wilson's functional equations in the class of locally integrable functions. We also consider the Ulam-Hyers stability of the classical Wilson's functional equations

$$f(x + y) + f(x + \sigma y) = 2f(x)g(y),$$
$$f(x + y) + f(x + \sigma y) = 2g(x)f(y)$$

in the class of Lebesgue measurable functions.

1. Introduction

Throughout this paper we denote by G a commutative group, $\mathbb{R}^n$ the n-dimensional Euclidean space, $\mathbb{C}$ the set of complex numbers, and $f, g : G \to \mathbb{C}$ or $f, g : \mathbb{R}^n \to \mathbb{C}$. A function $\sigma : G \to G$ is said to be *an involution* if $\sigma(x + y) = \sigma(x) + \sigma(y)$ for all $x, y \in G$ and $\sigma(\sigma(x)) = x$ for all $x \in G$. For simplicity we write σx instead of $\sigma(x)$.

The functional equation

$$f(x + y) + f(x - y) = 2f(x)f(y), \quad \forall x, y \in G \tag{1.1}$$

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is known as the *d'Alembert's functional equation* [16, 17]. As the name suggests this functional equation was introduced by d'Alembert in connection with the composition of forces and plays a central role in determining the sum of two vectors in Euclidean and non-Euclidean geometries [25]. Wilson's functional equations

$$f(x+y)+f(x+\sigma y)=2f(x)g(y), \quad \forall x,y\in G, \tag{1.2}$$

$$f(x+y)+f(x+\sigma y)=2g(x)f(y), \quad \forall x,y\in G \tag{1.3}$$

are generalizations of d'Alembert's functional equation. Among others Wilson's functional equation was studied by Wilson [31, 32], Kaczmarz [24], van der Lyn [29], Fenyö [19], Angheluta [3], Aczél Chung, and Ng [2], Chung, Ebanks, Ng and Sahoo [10], Aczél [1] and Stetkær in [28]. Recently, Chung and Sahoo [8] solve the equation (1.2) and (1.3) for arbitrary commutative semigroup.

In 1950, Laurent Schwartz introduced the theory of distributions in his monograph *Théorie des distributions* [26]. In this book Schwartz systematizes the theory of generalized functions, basing it on the theory of linear topological spaces, relates all the earlier approaches, and obtains many important results. After his elegant theory appeared, many important concepts and results on the classical spaces of functions have been generalized to the space of distributions. In this paper, as distributional version of the equations (1.2) and (1.3) we first consider the equations

$$u\circ T+u\circ T^{\sigma}-2u\otimes v=0, \tag{1.4}$$

$$u\circ T+u\circ T^{\sigma}-2v\otimes u=0, \tag{1.5}$$

where $u\in\mathcal{D}'(\mathbb{R}^n)$, the space of Schwartz distribution, $T(x,y)=x+y$, $T^{\sigma}(x,y)=x+\sigma y$, $x,y\in\mathbb{R}^n$, and $\circ$, $\otimes$ are pullback and tensor product of distributions, respectively. As consequences of the results in distributions, we obtain the Erdös' problem (see [5, 18, 23]) for Wilson's functional equation, namely, we solve the equations

$$f(x+y)+f(x+\sigma y)-2f(x)g(y)=0, \tag{1.6}$$

$$f(x+y)+f(x+\sigma y)-2g(x)f(y)=0 \tag{1.7}$$

for all $(x,y)\in(\mathbb{R}^n\times\mathbb{R}^n)\setminus\Omega$, where Ω is a subset of $\mathbb{R}^{2n}$ with $2n$-dimensional Lebesgue measure zero and f,g are locally integrable functions.

Secondly, we consider the Ulam-Hyers stability of the Wilson's functional equations (1.6) and (1.7) in the class of Lebesgue measurable functions with exponential perturbation, i.e., we consider the functional inequalities

$$|f(x+y)+f(x+\sigma y)-2f(x)g(y)|\le e^{\gamma\cdot y}, \tag{1.8}$$

$$|f(x+y)+f(x+\sigma y)-2g(x)f(y)|\le e^{\gamma\cdot y}, \tag{1.9}$$

where $f,g:\mathbb{R}^n\to\mathbb{C}$ are Lebesgue measurable functions and $\gamma\in\mathbb{R}^n$. For more known results for d'Alembert's functional equation and Wilson's functional equations we refer the reader to [4, 6, 7, 11, 12, 13, 14, 15, 25].

2. Solutions of Eq. (1.4), (1.5), (1.6) and (1.7)

A function $m : G \to \mathbb{C}$ is called *an exponential function* provided that $m(x+y) = m(x)m(y)$ for all $x, y \in G$ and $a : G \to \mathbb{C}$ is called *an additive function* provided that $a(x+y) = a(x) + a(y)$ for all $x, y \in G$. Now, we briefly introduce the space $\mathcal{D}'(\mathbb{R}^n)$ of distributions. We denote by $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of non-negative integers, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $\partial_j = \frac{\partial}{\partial x_j}$, $j = 1, 2, \ldots, n$.

Definition 2.1. Let $C_c^\infty(\mathbb{R}^n)$ the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports. A distribution u is a linear form on $C_c^\infty(\mathbb{R}^n)$ such that for every compact set $K \subset \mathbb{R}^n$ there exist constants $C > 0$ and $k \in \mathbb{N}_0$ for which

$$|\langle u, \varphi \rangle| \le C \sum_{|\alpha| \le k} \sup |\partial^\alpha \varphi|$$

holds for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ with supports contained in K. The set of all distributions on $\mathbb{R}^n$ is denoted by $\mathcal{D}'(\mathbb{R}^n)$.

Definition 2.2. Let $u \in \mathcal{D}'(\mathbb{R}^{n_2})$ and $\lambda : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ with $n_1 \ge n_2$ a smooth function such that for each $x \in \mathbb{R}^{n_1}$ the derivative $\lambda'(x)$ is surjective, that is, the Jacobian matrix $\nabla\lambda$ of λ has rank n_2. Then there exists a unique continuous linear map $\lambda^* : \mathcal{D}'(\mathbb{R}^{n_2}) \to \mathcal{D}'(\mathbb{R}^{n_1})$ such that $\lambda^* u = u \circ \lambda$ when u is a continuous function. We call $\lambda^* u$ the pullback of u by λ and usually denoted by $u \circ \lambda$.

We refer to ([22], chapter VI) for pullbacks of distributions. As a matter of fact, the pullbacks $u \circ T$ and $u \circ T^\sigma$ in the following (2.1) and (2.2) can be written in a transparent way:

$$\langle u \circ T,\ \varphi(x,y) \rangle = \langle u_x,\ \int \varphi(x-y, y)\,dy \rangle,$$

$$\langle u \circ T^\sigma,\ \varphi(x,y) \rangle = \langle u_x,\ \int \varphi(x-\sigma y, y)\,dy \rangle$$

for all $\varphi \in C_c^\infty(\mathbb{R}^{2n})$.

Definition 2.3. Let $u_j \in \mathcal{D}'(\mathbb{R}^{n_j})$ for $j = 1, 2$. Then the tensor product $u_1 \otimes u_2$ of u_1 and u_2, defined by

$$\langle u_1 \otimes u_2, \varphi(x_1, x_2) \rangle = \langle u_1\,,\ \langle u_2\,,\ \varphi(x_1, x_2) \rangle\ \rangle$$

for $\varphi(x_1, x_2) \in C_c^\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, belongs to $\mathcal{D}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

Let $u \in \mathcal{D}'(\mathbb{R}^n)$, $\phi \in C_c^\infty(\mathbb{R}^n)$. Then the convolution $u * \phi$ of u and ϕ is defined by

$$(u * \phi)(x) = \langle u_y, \phi(x-y) \rangle.$$

It is well known that $(u * \phi)(x)$ is a smooth function on $\mathbb{R}^n$.

As main results in this section we first consider the functional equations

$$u \circ T + u \circ T^\sigma - 2u \otimes v = 0, \tag{2.1}$$

$$u \circ T + u \circ T^{\sigma} - 2v \otimes u = 0, \tag{2.2}$$

where $u \in \mathcal{D}'(\mathbb{R}^n)$, σ is an involution on $\mathbb{R}^n$ and T, $T^{\sigma} : \mathbb{R}^{2n} \to \mathbb{R}^n$ are given by

$$T(x, y) = x + y, \quad T^{\sigma}(x, y) = x + \sigma y, \quad x, y \in \mathbb{R}^n.$$

As we see in Definition 2.2, pullback $u \circ T^{\sigma}$ makes sense only when σ is a smooth function. Since every smooth involution σ on $\mathbb{R}^n$ is given by a linear transformation, we denotes σ an $n \times n$ matrix such that $\sigma^2 = I$, where I is the identity matrix.

For the proof of our main result we need the following two lemmas.

Lemma 2.4 ([8])**.** *Let $f, g : G \to \mathbb{C}$ satisfy the functional equation*

$$f(x + y) + f(x + \sigma y) = 2f(x)g(y)$$

for all $x, y \in G$. Then either (g, f) has the form

$$g(x) = \frac{m(x) + m(\sigma x)}{2}, \quad f(x) = \alpha_1\, m(x) + \alpha_2\, m(\sigma x)$$

for all $x \in G$, where $m : G \to \mathbb{C}$ is an exponential function satisfying $m \neq m \circ \sigma$ and $\alpha_1, \alpha_2 \in \mathbb{C}$, or (g, f) has the form

$$g(x) = m(x), \quad f(x) = m(x)\big(\beta + a(x)\big)$$

for all $x \in G$, where $m : G \to \mathbb{C}$ is an exponential function satisfying $m = m \circ \sigma$ and $a : G \to \mathbb{C}$ is an additive function satisfying $a = -a \circ \sigma$, and $\beta \in \mathbb{C}$.

Lemma 2.5 ([8])**.** *Let $f, g : G \to \mathbb{C}$ satisfy*

$$f(x + y) + f(x + \sigma y) = 2\, g(x)\, f(y)$$

for all $x, y \in G$. Then (g, f) has the form

$$f(x) = \frac{m(x) + m(\sigma x)}{2\lambda}, \quad g(x) = \frac{m(x) + m(\sigma x)}{2}$$

for all $x \in G$, where $m : G \to \mathbb{C}$ is an exponential function satisfying $m = m \circ \sigma$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$.

As the first step of solving (2.1) we construct a σ-symmetric δ-sequence δ_t, $t > 0$. Define ρ on $\mathbb{R}^n$ by

$$\rho(x) = \begin{cases} q\, e^{-(1-|x|^2)^{-1}}, & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1, \end{cases}$$

where $q = \left(\int_{|x|<1} e^{-(1-|x|^2)^{-1}} dx\right)^{-1}$. It is easy to see that ρ is an infinitely differentiable function with support $\{x : |x| \leq 1\}$. Now, we employ

$$\delta_t(x) = \frac{\rho_t(x) + \rho_t(\sigma x)}{2}$$

for all $x \in \mathbb{R}^n$, where $\rho_t(x) := t^{-n}\rho(x/t)$, $t > 0$. Let $u \in \mathcal{D}'(\mathbb{R}^n)$. Then for each $t > 0$,

$$(u * \delta_t)(x) = \langle u_y, \delta_t(x - y)\rangle \to u \text{ as } t \to 0^+$$

in $\mathcal{D}'(\mathbb{R}^n)$, i.e.,

$$\lim_{t\to 0^+} \int (u * \delta_t)(x)\varphi(x)\, dx = \langle u, \varphi\rangle,\ \forall \varphi \in C_c^\infty(\mathbb{R}^n).$$

Furthermore, δ_t is σ-symmetric, i.e., $\delta_t = \delta_t \circ \sigma$ for all $t > 0$.

In the following, we exclude the case when $u = 0$ or $v = 0$. We denote by $c \cdot x$ the inner product of $c = (c_1, c_2, \ldots, c_n) \in \mathbb{C}^n$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ which defined as $c \cdot x = \sum_{j=1}^n c_j x_j$.

Theorem 2.6. *Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$ satisfy* (2.1). *Then either (v, u) is given by*

$$v = \frac{e^{c\cdot x} + e^{c\sigma\cdot x}}{2},\ u = \alpha_1 e^{c\cdot x} + \alpha_2 e^{c\sigma\cdot x}, \tag{2.3}$$

where $c \in \mathbb{C}^n$ with $c \neq c\sigma$, $\alpha_1, \alpha_2 \in \mathbb{C}$ and $c\sigma$ denotes matrix multiplication, or else

$$v = e^{(c+c\sigma)\cdot x},\quad u = e^{(c+c\sigma)\cdot x}\big(\beta + (d - d\sigma)\cdot x\big) \tag{2.4}$$

for all $x \in \mathbb{R}^n$, where $\beta \in \mathbb{C}$, $c, d \in \mathbb{C}^n$.

Proof. Convolving $(\delta_t \otimes \delta_s)(x, y) := \delta_t(x)\delta_s(y)$ in $u \circ T^\sigma$ and using $\delta_t \circ \sigma = \delta_t$, $(\delta_t * \delta_s) \circ \sigma = \delta_t * \delta_s$ we have

$$\begin{aligned} [(u \circ T^\sigma) * (\delta_t \otimes \delta_s)](x, y) &= \langle (u \circ T^\sigma)_{\xi,\eta}, \delta_t(x - \xi)\delta_s(y - \eta)\rangle \\ &= \langle u_z, \int_{\mathbb{R}^n} \delta_t(x - z + \sigma\eta)\delta_s(y - \eta)\, d\eta\rangle \\ &= \langle u_z, \int_{\mathbb{R}^n} \delta_t(\sigma x - \sigma z + \eta)\delta_s(y - \eta)\, d\eta\rangle \\ &= \langle u_z, \int_{\mathbb{R}^n} \delta_t(\eta)\delta_s(y + \sigma x - \sigma z - \eta)\, d\eta\rangle \\ &= \langle u_z, (\delta_t * \delta_s)(y + \sigma x - \sigma z)\rangle \\ &= \langle (u_z, (\delta_t * \delta_s)(x + \sigma y - z)\rangle \\ &= (u * \delta_t * \delta_s)(x + \sigma y) \end{aligned} \tag{2.5}$$

for all $x, y \in \mathbb{R}^n$. Letting $\sigma = I$ in (2.5) we have

$$[(u \circ T) * (\delta_t \otimes \delta_s)](x, y) = (u * \delta_t * \delta_s)(x + y) \tag{2.6}$$

for all $x, y \in \mathbb{R}^n$. Similarly, we have

$$[(u \otimes v) * (\delta_t \otimes \delta_s)](x, y) = (u * \delta_t(x)(v * \delta_s)(y) \tag{2.7}$$

for all $x, y \in \mathbb{R}^n$. Convolving $(\delta_t \otimes \delta_s)(x, y)$ in (2.1), from (2.5), (2.6) and (2.7) we have the functional equation

$$(u * \delta_t * \delta_s)(x + y) + (u * \delta_t * \delta_s)(x + \sigma y) - 2(u * \delta_t)(x)(v * \delta_s)(y) = 0 \tag{2.8}$$

for all $x, y \in \mathbb{R}^n$. Since $u * \delta_s$ is a smooth function, it is well known that

$$u * \delta_t * \delta_s \to u * \delta_s \tag{2.9}$$

uniformly on all compact subsets $K \subset \mathbb{R}^n$ as $t \to 0^+$. It follows from (2.8) and (2.9) that

$$f(x) := \lim_{t\to 0^+} (u * \delta_t)(x) \tag{2.10}$$

exists for all $x \in \mathbb{R}$ and the convergence is uniform on all compact subsets $K \subset \mathbb{R}^n$, which implies

$$\begin{aligned} \langle u, \varphi \rangle &= \lim_{t\to 0^+} \int_{\mathbb{R}^n} (u * \delta_t)(x)\varphi(x)\, dx \\ &= \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx \end{aligned}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$, i.e., $u = f$ in $\mathcal{D}'(\mathbb{R}^n)$. Similarly, it follows from (2.8) that

$$g(y) := \lim_{s\to 0^+} (u * \delta_s)(y) \tag{2.11}$$

exists for all $y \in \mathbb{R}$ and the convergence is uniform on all compact subsets $K \subset \mathbb{R}^n$, which implies $v = g$ in $\mathcal{D}'(\mathbb{R}^n)$. Letting $t \to 0^+$ and then $s \to 0^+$ in (2.8) we have

$$f(x + y) + f(x + \sigma y) - 2f(x)g(y) = 0 \tag{2.12}$$

for all $x, y \in \mathbb{R}^n$. By Lemma 2.4, we have

$$g(x) = \frac{m(x) + m(\sigma x)}{2} \tag{2.13}$$

for all $x \in \mathbb{R}^n$, where m is an exponential function. In view of the proof in [27], m is given by $m(x) = g(x) - \alpha(g(x+y_0) - g(x+\sigma y_0))$ for some $\alpha \in \mathbb{C}$, $y_0 \in \mathbb{R}^n$, which implies that m is a measurable function since g is a measurable function. It is well known that every measurable exponential function $m : \mathbb{R}^n \to \mathbb{C}$ is given by $m(x) = e^{c\cdot x}$ for some $c \in \mathbb{C}^n$ and every measurable additive function $a : \mathbb{R}^n \to \mathbb{C}$ is given by $a(x) = d \cdot x$ for some $d \in \mathbb{C}^n$. Thus, from (2.13) we have

$$g(x) = \frac{e^{c\cdot x} + e^{c\cdot \sigma x}}{2} = \frac{e^{c\cdot x} + e^{c\sigma\cdot x}}{2} \tag{2.14}$$

for all $x \in \mathbb{R}^n$, where $c\sigma$ denotes matrix multiplication. By Lemma 2.4, if $c \neq c\sigma$, then f is given by

$$f(x) = \alpha_1 e^{c\cdot x} + \alpha_2 e^{c\sigma\cdot x} \tag{2.15}$$

for all $x \in \mathbb{R}^n$ and for some $\alpha_1, \alpha_2 \in \mathbb{C}$, and if $c = c\sigma$ we have

$$f(x) = e^{c\cdot x}\big(\beta + d \cdot x\big) \tag{2.16}$$

for all $x \in \mathbb{R}^n$, where $d \in \mathbb{C}$ satisfies $d = -d\sigma$ and $\beta \in \mathbb{C}$. From the equalities $c = c\sigma$ and $d = -d\sigma$, replacing c by $2c$ and d by $2d$ we can write

$$g(x) = e^{(c+c\sigma)\cdot x}, \quad f(x) = e^{(c+c\sigma)\cdot x}\big(\beta + (d - d\sigma) \cdot x\big) \tag{2.17}$$

for all $x \in \mathbb{R}^n$. This completes the proof. □

We denote by $L^1_{loc}(\mathbb{R}^n)$ the set of all $f : \mathbb{R}^n \to \mathbb{C}$ such that $\int_K |f(x)|dx < \infty$ for every bounded measurable set $K \subset \mathbb{R}^n$. Every $f \in L^1_{loc}(\mathbb{R}^n)$ is viewed as a distribution via the correspondence

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. Thus, as a direct consequence of Theorem 2.6 we solve an Erdos' type problem [18] for Wilson's functional equation.

Corollary 2.7. *Let Ω be a subset of $\mathbb{R}^{2n}$ with $2n$-dimensional Lebesgue measure zero. Suppose that $f, g \in L^1_{loc}(\mathbb{R}^n)$ satisfy*

$$f(x+y) + f(x+\sigma y) - 2f(x)g(y) = 0 \tag{2.18}$$

for all $(x, y) \in (\mathbb{R}^n \times \mathbb{R}^n) \setminus \Omega$. Then either there exist a set $U \subset \mathbb{R}^n$ of n-dimensional Lebesgue measure zero, $\alpha_1, \alpha_2 \in \mathbb{C}$ and $c \in \mathbb{C}^n$ with $c \neq c\sigma$ such that

$$g(x) = \frac{e^{c\cdot x} + e^{c\sigma\cdot x}}{2}, \; f(x) = \alpha_1 e^{c\cdot x} + \alpha_2 e^{c\sigma\cdot x} \tag{2.19}$$

for all $x \in \mathbb{R}^n \setminus U$, or else there exist a set $V \subset \mathbb{R}^n$ of n-dimensional Lebesgue measure zero and $c, d \in \mathbb{C}^n$, $\beta \in \mathbb{C}$ such that

$$g(x) = e^{(c+c\sigma)\cdot x}, \quad f(x) = e^{(c+c\sigma)\cdot x}\big(\beta + (d - d\sigma)\cdot x\big) \tag{2.20}$$

for all $x \in \mathbb{R}^n \setminus V$.

Proof. By Theorem 2.6, equalities (2.19) and (2.20) hold in the sense of distributions, which implies the equalities hold for almost every $x \in \mathbb{R}^n$. Let $U_1 = \{x \in \mathbb{R}^n : g(x) \neq \frac{e^{c\cdot x}+e^{c\sigma\cdot x}}{2}\}$, $U_2 = \{x \in \mathbb{R}^n : f(x) \neq \alpha_1 e^{c\cdot x} + \alpha_2 e^{c\sigma\cdot x}\}$, $V_1 = \{x \in \mathbb{R}^n : g(x) \neq e^{(c+c\sigma)\cdot x}\}$, $V_2 = \{x \in \mathbb{R}^n : f(x) \neq e^{(c+c\sigma)\cdot x}(\beta + (d - d\sigma)\cdot x)\}$. Then we get (2.19) with $U = U_1 \cup U_2$ and get (2.20) with $V = V_1 \cup V_2$. This completes the proof. □

Using the same method as in the proof of Theorem 2.6 we obtain the following.

Theorem 2.8. *Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$ satisfy* (2.2). *Then (v, u) has the form*

$$v = \frac{e^{c\cdot x} + e^{c\sigma\cdot x}}{2}, \; u = \frac{e^{c\cdot x} + e^{c\sigma\cdot x}}{2\lambda}, \tag{2.21}$$

where $c \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$.

Finally, we consider the functional equations (2.1) and (2.2) in the space $\mathcal{G}'(\mathbb{R}^n)$ of Gelfand generalized functions. Generalizing the Schwartz tempered distribution [22], Gelfand and Shilov [20, 21] introduced the following space of generalized functions.

Definition 2.9 ([20, 21])**.** We denote by $\mathcal{G}(\mathbb{R}^n)$ the Gelfand–Shilov space of all infinitely differentiable functions φ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{A,B} = \sup_{x\in\mathbb{R}^n,\, \alpha,\, \beta\in\mathbb{N}_0^n} \frac{|x^\alpha \partial^\beta \varphi(x)|}{A^{|\alpha|}B^{|\beta|}\alpha!^{1/2}\beta!^{1/2}} < \infty$$

for some $A > 0$, $B > 0$. We say that $\varphi_j \to 0$ as $j \to \infty$ if $||\varphi_j||_{A,B} \to 0$ as $j \to \infty$ for some A, $B > 0$, and denote by $\mathcal{G}'(\mathbb{R}^n)$ the dual space of $\mathcal{G}(\mathbb{R}^n)$ and call its elements Gelfand–Shilov generalized functions.

It is known that the space $\mathcal{G}(\mathbb{R}^n)$ consists of all infinitely differentiable functions $\varphi(x)$ on $\mathbb{R}^n$ which can be extended to an entire function on $\mathbb{C}^n$ satisfying

$$|\varphi(x+iy)| \le C\exp(-a|x|^2 + b|y|^2), \quad x, y \in \mathbb{R}^n \tag{2.22}$$

for some a, b, $C > 0$ (see [20]).

Remark. The space $\mathcal{G}'(\mathbb{R}^n)$ contains the space of Schwartz tempered distributions [22] and is a partial extension of $\mathcal{D}'(\mathbb{R}^n)$. As a brief example, any infinite sum $u = \sum_{k=1}^{\infty} a_k\delta^{(k)}$ does not belong to $\mathcal{D}'(\mathbb{R}^n)$, but belongs to $\mathcal{G}'(\mathbb{R}^n)$ under some growth conditions on the sequence a_k, $k = 1, 2, 3, \ldots$.

In view of (2.22) it is easy to see that the n-dimensional heat kernel (see [30])

$$E_t(x) = (4\pi t)^{-n/2}\exp(-|x|^2/4t), \quad t > 0,$$

belongs to the Gelfand-Shilov space $\mathcal{G}(\mathbb{R}^n)$ for each $t > 0$. Thus, the convolution $(u * E_t)(x) := \langle u_y, E_t(x-y)\rangle$ is well defined for all $u \in \mathcal{G}'(\mathbb{R}^n)$. Instead of δ_t employed in the proof of Theorem 3.4, using

$$\gamma_t = \frac{E_t + E_t \circ \sigma}{2}$$

and following the same approach as in the proof of Theorem 2.6 we obtain the following.

Theorem 2.10. *Let $u, v \in \mathcal{G}'(\mathbb{R}^n)$ satisfy* (2.1)*. Then either (v, u) is given by*

$$v = \frac{e^{c\cdot x} + e^{c\sigma\cdot x}}{2}, \ u = \alpha_1 e^{c\cdot x} + \alpha_2 e^{c\sigma\cdot x}, \tag{2.23}$$

where $c \in \mathbb{C}^n$ with $c \ne c\sigma$, $\alpha_1, \alpha_2 \in \mathbb{C}$ and $c\,\sigma$ denotes matrix multiplication, or

$$v = e^{(c+c\sigma)\cdot x}, \quad u = e^{(c+c\sigma)\cdot x}\big(\beta + (d - d\sigma)\cdot x\big) \tag{2.24}$$

for all $x \in \mathbb{R}^n$, where $\beta \in \mathbb{C}$, $c, d \in \mathbb{C}^n$.

Theorem 2.11. *Let $u, v \in \mathcal{G}'(\mathbb{R}^n)$ satisfy* (2.2)*. Then (v, u) has the form*

$$v = \frac{e^{c\cdot x} + e^{c\sigma\cdot x}}{2}, \ u = \frac{e^{c\cdot x} + e^{c\sigma\cdot x}}{2\lambda}, \tag{2.25}$$

where $c \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ with $\lambda \ne 0$.

3. Ulam-Hyers stabilities of Eq. (1.6) and (1.7)

In this section, based on the results in [9] we consider the stability of functional equations (1.6) and (1.7) for all $x, y \in \mathbb{R}^n$, i.e., we deal with the functional inequalities (1.8) and (1.9).

Theorem 3.1. *Let $f, g : \mathbb{R}^n \to \mathbb{C}$ be unbounded Lebesgue measurable functions satisfying the functional inequality* (1.8) *for all $x, y \in \mathbb{R}^n$ and for some $\gamma \in \mathbb{R}^n$. Then, g has the form*

$$g(x) = \frac{e^{c\cdot x} + e^{c\cdot \sigma x}}{2} \tag{3.1}$$

for all $x \in \mathbb{R}^n$ and for some $c \in \mathbb{C}^n$. Assume that there exists $z_0 \in \mathbb{R}^n$ such that

$$\max\{\Re c \cdot z_0,\ \Re c \cdot \sigma z_0\} > \max\{0,\ \gamma \cdot z_0\}. \tag{3.2}$$

Then if $c \cdot x \neq c \cdot \sigma x$ for some $x \in \mathbb{R}^n$, f has the form

$$f(x) = \alpha_1\, e^{c\cdot x} + \alpha_2\, e^{c\cdot \sigma x} \tag{3.3}$$

for all $x \in \mathbb{R}^n$ and for some $\alpha_1, \alpha_2 \in \mathbb{C}$, and if $c \cdot x = c \cdot \sigma x$ for all $x \in \mathbb{R}^n$, f has the form

$$f(x) = \big(\beta + b \cdot (x - \sigma x)\big)\, e^{c\cdot x} \tag{3.4}$$

for all $x \in \mathbb{R}^n$ and for some $\beta \in \mathbb{C}, b \in \mathbb{C}^n$.

Proof. By the result in [9, Theorem 2.2] we get

$$g(x) = \frac{m(x) + m(\sigma x)}{2} \tag{3.5}$$

for all $x \in \mathbb{R}^n$. In view of the proof in [27], m is given by $m(x) = g(x) - \alpha(g(x + z_0) - g(x + \sigma z_0))$ for some $\alpha \in \mathbb{C}$, $z_0 \in \mathbb{R}^n$, which implies that m is Lebesgue measurable. It is well known that every Lebesgue measurable solution of the exponential functional equation is given by $m(x) = e^{c\cdot x}$ for some $c \in \mathbb{C}^n$. Thus, from (3.5) we get (3.1). Now, we prove that if (3.2) is satisfied, then there exists a sequence $z_k \in \mathbb{R}^n$, $k = 1, 2, 3, \ldots$, such that $|g(z_k)| \to \infty$ and $|g(z_k)|e^{-\gamma\cdot z_k} \to \infty$ as $k \to \infty$. Let

$$q(x) = |g(x)|\, e^{-\gamma\cdot x} = \frac{1}{2} e^{-\gamma\cdot x}\, |\, e^{c\cdot x} + e^{c\cdot\sigma x}\, | \,. \tag{3.6}$$

First, we assume that $\Re c \cdot z_0 \neq \Re c \cdot \sigma z_0$. Without loss of generality we may assume that $\Re c \cdot z_0 > \Re c \cdot \sigma z_0$. Putting $x = kz_0$, $k = 1, 2, 3, \ldots$ in (3.6) and using the triangle inequality we have

$$\begin{aligned} q(kz_0) &= \frac{1}{2}\, e^{-k\gamma\cdot z_0}\, \big|\, e^{kc\cdot z_0} + e^{kc\cdot\sigma z_0} \big| \\ &\geq \frac{1}{2}\, e^{-k\gamma\cdot z_0}\, \big|\, e^{k\Re c\cdot z_0} - e^{k\Re c\cdot\sigma z_0} \big| \\ &= \frac{1}{2}\, e^{k(\Re c - \gamma)\cdot z_0}\, \Big| 1 - e^{k\Re c\cdot(\sigma z_0 - z_0)} \Big| \end{aligned} \tag{3.7}$$

$$= \frac{1}{2} R^k \left|1 - r^k\right|,$$

where $R = e^{(\Re c - \gamma)\cdot z_0}$ and $r = e^{\Re c\cdot(\sigma z_0 - z_0)}$. By the condition (3.2) we see that

$$R > 1, \quad e^{\gamma\cdot z_0} R = e^{\Re c\cdot z_0} > 1 \quad \text{and} \quad 0 < r < 1. \tag{3.8}$$

Letting $k \to \infty$ in (3.7) we have

$$q(kz_0) \geq R^k \left|1 - r^k\right| \to \infty$$

and hence

$$|g(kz_0)| = e^{k\gamma\cdot z_0}\, q(kz_0) \geq (e^{\gamma\cdot z_0} R)^k\, |1 - r^k| \to \infty$$

as $k \to \infty$. Now, we assume that $\Re c \cdot z_0 = \Re c \cdot \sigma z_0$. Putting $x = kz_0$, $k = 1, 2, 3, \ldots$ in (3.6) and letting $R = e^{(\Re c - a)\cdot z_0}$, $\theta = \Im c \cdot (\sigma z_0 - z_0)$ we have

$$\begin{aligned} q(kz_0) &= \frac{1}{2} e^{-k\gamma\cdot z_0} | \, e^{kc\cdot z_0} + e^{kc\cdot\sigma z_0} | \\ &= \frac{1}{2} e^{k(\Re c - \gamma)\cdot z_0} \left|e^{ki\Im c\cdot z_0} + e^{ki\Im c\cdot\sigma z_0}\right| \\ &= \frac{1}{2} e^{k(\Re c - \gamma)\cdot z_0} \left|1 + e^{ki\Im c\cdot(\sigma z_0 - z_0)}\right| \\ &= \frac{1}{2} R^k \, |1 + e^{i\theta k}|. \end{aligned} \tag{3.9}$$

Note that the set $\{e^{i\theta k} \mid k = 1, 2, 3, \ldots\}$ forms either vertices of a regular polygon (including $\{1\}$ and $\{1, -1\}$) when θ/π is rational, or a dense subset of the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ when θ/π is irrational. Using this fact and the condition (3.8), we can see that there exists a sequence

$$k_1 < k_2 < k_3 < \cdots < k_j < \cdots$$

of positive integers such that $R^{k_j}\, |1 + e^{i\theta k_j}| \to \infty$ and $(e^{\gamma\cdot z_0} R)^{k_j}\, |1 + e^{i\theta k_j}| \to \infty$ as $j \to \infty$. Thus, we have

$$q(k_j z_0) \to \infty, \quad |g(k_j z_0)| \to \infty \tag{3.10}$$

as $j \to \infty$. Now, we repeat the proof in [9, Theorem 2.2] for the reader. Replacing y by $k_j z_0$ in (1.8) and dividing the result by $2\,|g(k_j z_0)|$ we have

$$\left| f(x) - \frac{f(x + k_j z_0) + f(x + \sigma k_j z_0)}{2\, g(k_j z_0)} \right| \leq \frac{e^{\gamma\cdot k_j z_0}}{2\, |g(k_j z_0)|} \tag{3.11}$$

for all $x \in \mathbb{R}^n$. Letting $j \to \infty$ in (3.11) we have

$$f(x) = \lim_{n\to\infty} \frac{f(x + k_j z_0) + f(x + \sigma k_j z_0)}{2\, g(k_j z_0)} \tag{3.12}$$

for all $x \in \mathbb{R}^n$. Multiplying both sides of (3.12) by $2\, g(y)$ and using (1.8) and (3.10) we have

(3.13)

$$2\, f(x) g(y) = \lim_{j\to\infty} \frac{2\, f(x + k_j z_0) g(y) + 2\, f(x + \sigma k_j z_0) g(y)}{2\, g(k_j z_0)}$$

$$= \lim_{j\to\infty} \frac{f(x+k_jz_0+y)+f(x+k_jz_0+\sigma y)+f(x+\sigma k_jz_0+y)+f(x+\sigma k_jz_0+\sigma y)}{2\,g(k_jz_0)}$$
$$= \lim_{j\to\infty} \left(\frac{f(x+y+k_jz_0)+f(x+y+\sigma k_jz_0)}{2\,g(k_jz_0)} + \frac{f(x+\sigma y+k_jz_0)+f(x+\sigma y+\sigma k_jz_0)}{2\,g(k_jz_0)}\right)$$
$$= f(x+y) + f(x+\sigma y)$$

for all $x, y \in \mathbb{R}^n$. Thus, using Lemma 2.4 with (3.13) we get the result. This completes the proof. □

Remark 3.2. Let $a, b \in \mathbb{R}^n$ be two nonzero vectors that are not parallel, i.e., $b \neq ra$ for all $r \in \mathbb{R}$, $r \neq 1$. Then, the hyperplane $b \cdot x = 0$ is not parallel to $(b-a)\cdot x = 0$ and hence there exists $x_0 \in \mathbb{R}^n$ such that $b \cdot x_0 > 0$ and $(b-a)\cdot x_0 > 0$. If $b = ra$ for some $r \in \mathbb{R}$, $r \neq 1$, then there exists $x_0 \in \mathbb{R}^n$ such that $b \cdot x_0 > 0$ and $(b-a)\cdot x_0 > 0$ if and only if $r > 1$. Thus, if the involution $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ in Theorem 3.1 is given by a linear map, i.e., $\sigma = A$, an $n \times n$ matrix, then using the above fact with $a = \gamma$ and $b = \Re c$, $\Re c \cdot A$, it is easy to see that condition (3.4) is equivalent to

$$\Re c \neq r\gamma \ \text{ or } \ \Re c \cdot A \neq r\gamma$$

for all $r \leq 1$. Now, the following example gives a transparent description of the solutions of a functional inequality of the type (1.8).

Example 3.3. In Theorem 3.1, let $n = 2$, $\gamma = (2, 1)$ and $\sigma(u, v) = (2u + 3v, -u - 2v)$ for all $u, v \in \mathbb{R}$. Then the functional inequality (3.1) becomes

$$|f(t+u, s+v) + f(t+2u+3v, s-u-2v) - 2f(t,s)g(u,v)| \leq e^{2u+v} \tag{3.14}$$

for all $t, s, u, v \in \mathbb{R}$. Now, using Theorem 3.1 and Remark 3.2 we can exhibit regular solutions (continuous, Lebesgue measurable solutions, etc.) of the functional inequality (3.14) when f is unbounded. By Theorem 3.1 we have

$$g(t,s) = \frac{1}{2}\left(e^{c_1t+c_2s} + e^{(2c_1-c_2)t+(3c_1-2c_2)s}\right) \tag{3.15}$$

for all $t, s \in \mathbb{R}$ and for some $c_1, c_2 \in \mathbb{C}$. Since either $\Re(c_1, c_2)$ or $\Re(2c_1-c_2, 3c_1-2c_2)$ is not parallel to $\gamma = (2, 1)$, using Remark 3.2 we can see that condition (3.2) is satisfied. Thus, if $(c_1, c_2) \neq (2c_1 - c_2, 3c_1 - 2c_2)$, i.e., $c_1 \neq c_2$, then f has the form

$$f(t,s) = \alpha_1\, e^{c_1t+c_2s} + \alpha_2\, e^{(2c_1-c_2)t+(3c_1-2c_2)s} \tag{3.16}$$

for all $t, s \in \mathbb{R}$ and for some $\alpha_1, \alpha_2 \in \mathbb{C}$, and if $c_1 = c_2$, then f has the form

$$f(t,s) = \big(\beta + d_2(t+3s)\big)\, e^{d_1(t+s)} \tag{3.17}$$

for all $t, s \in \mathbb{R}$ and for some $\beta, d_1, d_2 \in \mathbb{C}$.

Following the same methods as in the proof of Theorem 3.1 and using the result in [9, Theorem 2.4] we obtain the following.

Theorem 3.4. *Let $f, g : \mathbb{R}^n \to \mathbb{C}$ be unbounded Lebesgue measurable functions satisfying the functional inequality* (1.9) *for all $x, y \in \mathbb{R}^n$ and for some $\gamma \in \mathbb{R}^n$. Then, f has the form*

$$f(x) = \frac{e^{c\cdot x} + e^{c\cdot \sigma x}}{2\lambda} \tag{3.18}$$

for all $x \in \mathbb{R}^n$ and for some $c \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. In particular, the condition (3.4) *is satisfied. Then if $c \cdot x \neq c \cdot \sigma x$ for some $x \in \mathbb{R}^n$, f has the form*

$$f(x) = \alpha_1\, e^{c\cdot x} + \alpha_2\, e^{c\cdot \sigma x} \tag{3.19}$$

for all $x \in \mathbb{R}^n$ and for some α_1, $\alpha_2 \in \mathbb{C}$, and if $c \cdot x = c \cdot \sigma x$ for all $x \in \mathbb{R}^n$, f has the form

$$f(x) = \big(\beta + b \cdot (x - \sigma x)\big)\, e^{c\cdot x} \tag{3.20}$$

for all $x \in \mathbb{R}^n$ and for some $\beta \in \mathbb{C}, b \in \mathbb{C}^n$.

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