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CERTAIN SEMISYMMETRY PROPERTIES OF (κ, μ)-CONTACT METRIC MANIFOLDS

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Abstract. The object of the present paper is to characterize (κ, μ)-contact metric manifolds whose concircular curvature tensor satisfies certain semisymmetry conditions. We also verify that the result holds by a concrete example.

1. Introduction

In [3], Blair, Koufogiorgos and Papantoniou introduced (κ, μ)-contact metric manifolds. A class of contact metric manifolds with contact metric structure (φ, ξ, η, g) in which the curvature tensor R satisfies the condition

$$R(X,Y)\xi = (\kappa I + \mu h)\{\eta(Y)X - \eta(X)Y\}$$

for all X and $Y \in TM$, where $(\kappa, \mu) \in \mathbb{R}^2$ is called (κ, μ)-metric manifolds.

A transformation of an $(2n+1)$-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle is called a concircular transformation ([8], [12]). Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. A concircular transformation is always a conformal transformation [8].

Concircular curvature tensor is defined by [12]

$$Z(X,Y)W = R(X,Y)W - \frac{r}{2n(2n+1)}\{g(Y,W)X - g(X,W)Y\}, \tag{1.1}$$

where $X, Y, W \in TM$ and R and r is the curvature tensor and the scalar curvature respectively.

Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

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In [2], D. E. Blair et al. started a study of concircular curvature tensor of contact metric manifolds. Also concircular curvature tensor in (κ, μ)-contact metric manifolds has been studied by U. C. De and Sujit Ghosh [6].

A Riemannian manifold is said to be semisymmetric if its curvature tensor R satisfies $R(X, Y) \cdot R = 0$, $X, Y \in TM$, where $R(X, Y)$ acts on R.

Recently, in [13] Yildiz and De studied φ-projectively semisymmetric and h-projectively semisymmetric (κ, μ)-contact metric manifolds.

Motivated by the above studies, we study in this paper certain semisymmetry properties of the concircular curvature tensor in (κ, μ)-contact metric manifolds.

The paper is organized as follows:

In Section 2, we give necessary details about (κ, μ)-contact metric manifolds. Section 3 deals with φ-concircularly semisymmetric (κ, μ)-contact metric manifolds. In Section 4, h-concircularly semisymmetric (κ, μ)-contact metric manifolds have been studied. Finally, we construct an example of a (κ, μ)-contact metric manifold which verifies Theorem 5.1.

2. Preliminaries

An $(2n+1)$-dimensional differentiable manifold M is called an almost contact manifold if there is an almost contact structure (φ, ξ, η) consisting of a $(1,1)$-tensor field φ, a vector field ξ, a 1-form η satisfying

$$\varphi^2(X) = -X + \eta(X)\xi, \ \eta(\xi) = 1, \ \varphi\xi = 0, \ \eta \circ \varphi = 0. \tag{2.1}$$

An almost contact structure is said to be normal if the induced almost complex structure J on the product manifold $M^{2n+1} \times \mathbb{R}$ defined by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$ is integrable, where X is tangent to M, t is the coordinate of R and f is a smooth function on $M^{2n+1} \times \mathbb{R}$.

The condition for being normal is equivalent to vanishing of the torsion tensor $[\varphi, \varphi] + 2d\eta \otimes \xi$, where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ.

Let g be a compatible Riemannian metric with structure (φ,ξ,η), that is,

$$g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y), \tag{2.2}$$

or equivalently,

$$g(X, \xi) = \eta(X), \ g(\varphi X, Y) = -g(X, \varphi Y) \tag{2.3}$$

for all $X, Y \in TM$.

An almost contact metric structure becomes a contact metric structure if

$$g(X, \varphi Y) = d\eta(X, Y) \tag{2.4}$$

for all $X, Y \in TM$.

Given a contact metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$, we define a $(1,1)$-tensor field h by $h = \frac{1}{2}L_\xi\varphi$, where L denotes the Lie differentiation. Then h is symmetric and satisfies

$$h\xi = 0, \ h\varphi + \varphi h = 0, \tag{2.5}$$

$$\nabla\xi = -\varphi - \varphi h,\ trace(h) = trace(\varphi h) = 0, \tag{2.6}$$

where ∇ is the Levi-Civita connection.

A contact metric manifold is said to be an η-Einstein manifold if

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.7}$$

where a, b are smooth functions on M and S is the Ricci tensor.

A normal contact metric manifold is called a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X. \tag{2.8}$$

On a Sasakian manifold, the following relation holds

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.9}$$

for all $X, Y \in TM$.

Blair, Koufogiorgos and Papantoniou [3] considered the $(\kappa,\ \mu)$-nullity condition and gave several reasons for studying it. The $(\kappa,\ \mu)$-nullity distribution $N(\kappa,\ \mu)$ ([3], [10]) of a contact metric manifold M is defined by

$$N(\kappa,\mu) : p \mapsto N_p(\kappa,\mu).$$

Here $N_p(\kappa,\mu) = [W \in T_pM \mid R(X,Y)W = (\kappa I + \mu h)(g(Y,W)X - g(X,W)Y)]$ for all $X, Y \in TM$, where $(\kappa,\mu) \in \mathbb{R}^2$.

A contact metric manifold M^{2n+1} with $\xi \in N(\kappa,\mu)$ is called a $(\kappa,\ \mu)$-contact metric manifold. Then we have

$$\begin{aligned} R(X,Y)\xi &= \kappa[\eta(Y)X - \eta(X)Y] \\ &\quad + \mu[\eta(Y)hX - \eta(X)hY] \ \text{ for all } X,Y \in TM. \end{aligned} \tag{2.10}$$

For $(\kappa,\ \mu)$-metric manifolds, it follows that $h^2 = (\kappa - 1)\varphi^2$. This class contains Sasakian manifolds for $\kappa = 1$ and $h = 0$. In fact, for a $(\kappa,\ \mu)$-metric manifold, the condition of being Sasakian manifold, κ-contact manifold, $\kappa = 1$ and $h = 0$ are equivalent. If $\mu = 0$, then the $(\kappa,\ \mu)$-nullity distribution $N(\kappa,\ \mu)$ is reduced to κ-nullity distribution $N(\kappa)$ [11]. If $\xi \in N(\kappa)$, then we call contact metric manifold M an $N(\kappa)$-contact metric manifold.

$(\kappa,\ \mu)$-contact metric manifolds have been studied by several authors ([1], [4], [5], [6], [7], [9]) and many other authors.

In a $(\kappa,\ \mu)$-contact metric manifold, the following relations hold [3]:

$$h^2 = (\kappa - 1)\varphi^2, \tag{2.11}$$

$$(\nabla_X\varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.12}$$

$$R(\xi,X)Y = \kappa\{g(X,Y)\xi - \eta(Y)X\} + \mu\{g(hX,Y)\xi - \eta(Y)hX\}, \tag{2.13}$$

$$S(X,\xi) = 2n\kappa\eta(X), \tag{2.14}$$

$$\begin{aligned} S(X,Y) &= \{(2n-2) - n\mu\}g(X,Y) + \{(2n-2) + \mu\}g(hX,Y) \\ &\quad + \{(2-2n) + n + (2\kappa + \mu)\}\eta(X)\eta(Y), \end{aligned} \tag{2.15}$$

$$r = 2n\big(2n - 2 + \kappa - n\mu\big), \tag{2.16}$$

$$\begin{aligned} S(X, hY) = &\{(2n-2) - n\mu\}g(X, hY) \\ &- (\kappa - 1)\{(2n-2) + \mu\}g(X, Y) \\ &+ (\kappa - 1)\{(2n-2) + \mu\}\eta(X)\eta(Y), \end{aligned} \tag{2.17}$$

$$Q\varphi - \varphi Q = 2\{(2n-2) + \mu\}h\varphi, \tag{2.18}$$

where Q is the Ricci operator defined by $g(QX, Y) = S(X, Y)$.

From [3] we can state the following results:

Lemma 2.1. *Let M be an $(2n+1)$-dimensional contact metric manifold with ξ belonging to the (κ, μ)-nullity distribution. Then we have*

$$\begin{aligned} &R(X, Y)\varphi W - \varphi R(X, Y)W \\ &= \{(1-\kappa)[g(\varphi Y, W)\eta(X) - g(\varphi X, W)\eta(Y)] \\ &\quad + (1-\mu)[g(\varphi hY, W)\eta(X) - g(\varphi hX, W)\eta(Y)]\}\xi \\ &\quad - g(Y + hY, W)(\varphi X + \varphi hX) + g(X + hX, W)(\varphi Y + \varphi hY) \\ &\quad - g(\varphi Y + \varphi hY, W)(X + hX) + g(\varphi X + \varphi hX, W)(Y + hY) \\ &\quad - \eta(W)\{(1-\kappa)[\eta(X)\varphi Y - \eta(Y)\varphi X] \\ &\quad + (1-\mu)[\eta(X)\varphi hY - \eta(Y)\varphi hX]\} \end{aligned} \tag{2.19}$$

for any vector fields X, Y, W.

Lemma 2.2. *Let M be an $(2n+1)$-dimensional contact metric manifold with ξ belonging to the (κ, μ)-nullity distribution. Then we have*

$$\begin{aligned} &R(X, Y)hW - hR(X, Y)W \\ &= \{\kappa[g(hY, W)\eta(X) - g(hX, W)\eta(Y)] \\ &\quad + \mu(1-\kappa)[g(X, W)\eta(Y) - g(Y, W)\eta(X)]\}\xi \\ &\quad + \kappa\{g(Y, \varphi W)\varphi hX - g(X, \varphi W)\varphi hY \\ &\quad + g(W, \varphi hY)\varphi X - g(W, \varphi hX)\varphi Y \\ &\quad + \eta(W)[\eta(X)hY - \eta(Y)hX]\} \\ &\quad - \mu\{\eta(Y)[(1-\kappa)\eta(W)X + \mu\eta(X)hW] \\ &\quad - \eta(X)[(1-\kappa)\eta(W)Y + \mu\eta(Y)hW] + 2g(X, \varphi Y)\varphi hW\} \end{aligned} \tag{2.20}$$

for any vector fields X, Y, W.

3. η-Einstein $(\kappa,\ \mu)$-contact metric manifolds

In general, in a $(\kappa,\ \mu)$-contact metric manifold, the Ricci operator Q does not commute with φ. However, Yildiz and De [13] proved the following:

Proposition 3.1. *In a non-Sasakian $(\kappa,\ \mu)$-contact metric manifold, the following conditions are equivalent*:
(a) η*-Einstein manifold,*
(b) $Q\varphi = \varphi Q$.

For $n = 1$, from (2.18) and Proposition 3.1 we can state the following:

Corollary 3.1. *A 3-dimensional non-Sasakian η-Einstein $(\kappa,\ \mu)$-contact metric manifold is an $N(\kappa)$-contact metric manifold.*

4. φ-concircularly semisymmetric $(\kappa,\ \mu)$-contact metric manifolds

Definition 4.1. A (κ, μ)-contact metric manifold is said to be φ-concircularly semisymmetric if $Z(X, Y) \cdot \varphi = 0$ for all $X, Y \in TM$.

Suppose M be an $(2n+1)$-dimensional φ-concircularly semisymmetric (κ, μ)-contact metric manifold. Then we get

$$Z(X, Y)\varphi W - \varphi(Z(X, Y)W) = 0. \tag{4.1}$$

Using (1.1) and (2.19) in (4.1) we have

$$\begin{aligned}
&\{(1-\kappa)[g(\varphi Y, W)\eta(X) - g(\varphi X, W)\eta(Y)] \\
&+(1-\mu)[g(\varphi hY, W)\eta(X) - g(\varphi hX, W)\eta(Y)]\}\xi \\
&-g(Y + hY, W)(\varphi X + \varphi hX) + g(X + hX, W)(\varphi Y + \varphi hY) \\
&-g(\varphi Y + \varphi hY, W)(X + hX) + g(\varphi X + \varphi hX, W)(Y + hY) \\
&-\eta(W)\{(1-\kappa)[\eta(X)\varphi Y - \eta(Y)\varphi X] + (1-\mu)[\eta(X)\varphi hY - \eta(Y)\varphi hX]\} \\
&-\frac{r}{2n(2n+1)}[g(Y, \varphi W)X - g(X, \varphi W)Y - g(Y, W)\varphi X + g(X, W)\varphi Y] \\
&= 0.
\end{aligned} \tag{4.2}$$

Taking inner product with Z of (4.2) and contracting Y, Z we obtain

$$g(\varphi X, W)\{\kappa - 3 - \frac{r(2n-1)}{2n(2n+1)} + (2n+1)\} + g(\varphi X, hW)(2 - \mu - 2n) = 0. \tag{4.3}$$

Putting $X = \varphi X$ and using (2.1) we have

$$g(X, hW) = ag(X, W) + b\eta(X)\eta(W), \tag{4.4}$$

where

$$a = -\frac{\kappa - 3 - \frac{r(2n-1)}{2n(2n+1)} + (2n+1)}{2 - \mu - 2n}$$

and

$$b = \frac{\kappa - 3 - \frac{r(2n-1)}{2n(2n+1)} + (2n+1)}{2 - \mu - 2n}.$$

Putting (4.4) in (2.15) we have

$$S(X, W) = a_1 g(X, W) + b_1\eta(X)\eta(W), \tag{4.5}$$

where

$$a_1 = \{(2n-2) - n\mu\} - \{(2n-2) + \mu\}\frac{\kappa - 3 - \frac{r(2n-1)}{2n(2n+1)} + (2n+1)}{2 - \mu - 2n}$$

and

$$b_1 = \{(2-2n) + n(2\kappa + \mu)\} + \{(2n-2) + \mu\}\frac{\kappa - 3 - \frac{r(2n-1)}{2n(2n+1)} + (2n+1)}{2 - \mu - 2n}.$$

From (4.5) we can conclude the following:

Theorem 4.1. *An* $(2n+1)$*-dimensional* φ*-concircularly semisymmetric* $(\kappa,\ \mu)$*-contact metric manifold reduces to an* η*-Einstein manifold.*

From Proposition 3.1 and Theorem 4.1 we can state that:

Corollary 4.1. *Let* M *be an* $(2n+1)$*-dimensional* φ*-concircularly semisymmetric* $(\kappa,\ \mu)$*-contact metric manifold. Then the Ricci operator* Q *commutes with* φ*. That is,* $Q\varphi = \varphi Q$*.*

5. 3-dimensional φ-concircularly semisymmetric $(\kappa,\ \mu)$-contact metric manifolds

Suppose M is a 3-dimensional φ-concircularly semisymmetric (κ, μ)-contact metric manifold.

Putting $n = 1$ in equation (4.3) we have

$$g(\varphi X, W)(\kappa - \frac{r}{6}) + g(\varphi X, hW)\mu = 0. \tag{5.1}$$

Substituting $W = hW$ in (5.1) we obtain

$$(\kappa - \frac{r}{6})hW + \mu h^2 W = 0. \tag{5.2}$$

Applying trace in both side of the equation (5.2) and using $traceh = 0$, we get

$$\mu = 0. \tag{5.3}$$

From (5.3) we can state the following:

Theorem 5.1. *A* 3*-dimensional* φ*-concircularly semisymmetric* $(\kappa,\ \mu)$*-contact metric manifold reduces to an* $N(\kappa)$*-contact metric manifold.*

6. h-concircularly semisymmetric $(\kappa,\ \mu)$-contact metric manifolds

Definition 6.1. A (κ, μ)-contact metric manifold is said to be h-concircularly semisymmetric if $Z(X, Y) \cdot h = 0$ for all $X, Y \in TM$.

Suppose M is an $(2n+1)$-dimensional h-concircularly semisymmetric (κ,μ)-contact metric manifold. Then we get

$$Z(X,Y)hW - h(Z(X,Y)W) = 0. \tag{6.1}$$

Using (1.1) and (2.20) in (6.1) we have

$$\begin{aligned}
&\{\kappa[g(hY,W)\eta(X) - g(hX,W)\eta(Y)]\\
&+\mu(1-\kappa)[g(X,W)\eta(Y) - g(Y,W)\eta(X)]\}\xi\\
&+\kappa\{g(Y,\varphi W)\varphi hX - g(X,\varphi W)\varphi hY\\
&+g(W,\varphi hY)\varphi X - g(W,\varphi hX)\varphi Y + \eta(W)[\eta(X)hY - \eta(Y)hX]\}\\
&-\mu\{\eta(Y)[(1-\kappa)\eta(W)X + \mu\eta(X)hW]\\
&-\eta(X)[(1-\kappa)\eta(W)Y + \mu\eta(Y)hW] + 2g(X,\varphi Y)\varphi hW\}\\
&-\frac{r}{2n(2n+1)}[g(Y,hW)X - g(X,hW)Y - g(Y,W)hX + g(X,W)hY]\\
&= 0.
\end{aligned} \tag{6.2}$$

Taking inner product with Z of (6.2) and contracting Y, Z we obtain

$$\begin{aligned}
&\{\kappa + 2\mu + \frac{r}{2n}\}g(hW,X) + \mu(\kappa-1)g(X,W)\\
&-(2n+1)\mu(\kappa-1)\eta(X)\eta(W) = 0,
\end{aligned} \tag{6.3}$$

which implies that

$$g(X,hW) = ag(X,W) + b\eta(X)\eta(W), \tag{6.4}$$

where

$$a = -\frac{\mu(\kappa-1)}{\kappa+2\mu+\frac{r}{2n}}$$

and

$$b = \frac{(2n+1)\mu(\kappa-1)}{\kappa+2\mu+\frac{r}{2n}}.$$

Putting (6.4) in (2.15) we have

$$S(X,W) = a_1 g(X,W) + b_1\eta(X)\eta(W), \tag{6.5}$$

where

$$a_1 = \{(2n-2) - n\mu\} - \{(2n-2)+\mu\}\frac{\mu(\kappa-1)}{\kappa+2\mu+\frac{r}{2n}}$$

and

$$b_1 = \{(2-2n) + n(2\kappa+\mu)\} + \{(2n-2)+\mu\}\frac{(2n+1)(\kappa-1)\mu}{\kappa+2\mu+\frac{r}{2n}}.$$

From (6.5) we can conclude the following:

Theorem 6.1. *An $(2n+1)$-dimensional h-concircularly semisymmetric $(\kappa,\ \mu)$-contact metric manifold is an η-Einstein manifold.*

From Proposition 3.1 and Theorem 6.1 we can state that:

Corollary 6.1. *Let M be an $(2n+1)$-dimensional h-concircularly semisymmetric $(\kappa,\ \mu)$-contact metric manifold. Then the Ricci operator Q commutes with φ. That is, $Q\varphi = \varphi Q$.*

7. An example

Let us consider a 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3 : (x,y,z) \neq (0,0,0)\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. The vector fields $e_1 = e^{z-x}\frac{\partial}{\partial x}$, $e_2 = e^{z-y}\frac{\partial}{\partial y}$, $e_3 = \frac{\partial}{\partial z}$ are linearly independent at each point of M. Let g be the metric defined by

$$g(e_i, e_j) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases} \tag{7.1}$$

Here i and j runs from 1 to 3.

Let η be the 1-form defined by $\eta(Z) = g(Z, e_1)$ for any vector field Z tangent to M. Let φ be the $(1,1)$-tensor field defined by $\varphi e_2 = -e_3$, $\varphi e_3 = e_2$, $\varphi e_1 = 0$. From the properties of φ and η we can state the following: $g(e_i, \varphi e_j) = d\eta(e_i, e_j)$, where i and j runs from 1 to 3. Using the linearity property of φ and g we have

$$\eta(e_1) = 1,$$
$$\varphi^2 Z = -Z + \eta(Z)e_1$$
$$g(\varphi Z, \varphi W) = g(Z,W) - \eta(Z)\eta(W)$$

for any vector field Z, W.

Then for $e_1 = \xi$, the structure (φ, ξ, η, g) defines a contact metric structure on M.

Let ∇ be the Levi-Civita connection on M with respect to the metric g. Then we have

$$\begin{aligned}[e_1, e_2] &= e^{z-x}\frac{\partial}{\partial x}(e^{z-y}\frac{\partial}{\partial y}) - e^{z-y}\frac{\partial}{\partial y}(e^{z-x}\frac{\partial}{\partial x}) \\ &= e^{z-x}e^{z-y}\frac{\partial^2}{\partial x \partial y} - e^{z-x}e^{z-y}\frac{\partial^2}{\partial x \partial y} \\ &= 0.\end{aligned}$$

Similarly,

$$[e_1, e_3] = -e_1, \quad [e_2, e_3] = -e_2, \quad [e_2, e_1] = 0,$$
$$[e_3, e_1] = e_1, \quad [e_3, e_2] = e_2.$$

From Koszul's formula, the Riemannian connection ∇ of the metric g is given by

$$\begin{aligned}2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]).\end{aligned} \tag{7.2}$$

Using (7.2) we have

$$\nabla_{e_1}e_1 = e_3, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = -e_1,$$
$$\nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = e_3, \quad \nabla_{e_2}e_3 = -e_2,$$
$$\nabla_{e_3}e_1 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = 0.$$

We also know that

$$\nabla_{e_2}e_1 = -\varphi e_2 - \varphi h e_2.$$

Comparing the above two relations for $\nabla_{e_2}e_1$ and using $\varphi e_1 = 0$, $\varphi e_3 = e_2$ and $\varphi e_2 = -e_3$, we have

$$he_2 = -e_2.$$

Similarly, we obtain

$$he_3 = -e_3 \text{ and } he_1 = 0.$$

It is known that Riemannian curvature tensor

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{7.3}$$

Using (7.3) we obtain

$$R(e_2,e_1)e_1 = -e_2,$$
$$R(e_3,e_1)e_1 = -e_3,$$
$$R(e_2,e_3)e_1 = 0.$$

We conclude that e_1 belongs to the (κ,μ)-nullity distribution, where $\kappa = -1, \mu = 0$. Hence the manifold reduces to an $N(\kappa)$-contact metric manifold.

All nonzero components of the curvature tensor can be written as follows:

$$R(e_1,e_2)e_2 = -e_1, \quad R(e_1,e_3)e_3 = -e_1,$$
$$R(e_2,e_3)e_3 = -e_2, \quad R(e_2,e_3)e_2 = e_3,$$
$$R(e_1,e_3)e_3 = -e_1, \quad R(e_3,e_1)e_1 = -e_3, \quad R(e_2,e_1)e_1 = -e_2.$$

From the above results, we have the Ricci tensor

$$\begin{aligned} S(e_1,e_1) &= g(R(e_2,e_1)e_1,e_2) + g(R(e_3,e_1)e_1,e_3) \\ &= -2. \end{aligned}$$

Similarly, we obtain $S(e_2,e_2) = -2$, $S(e_3,e_3) = -2$ and the scalar curvature $r = S(e_1,e_1) + S(e_2,e_2) + S(e_3,e_3 = -6$.

From the above calculation we can conclude that $S(X,Y) = -2g(X,Y)$ for $X = a_1e_1 + a_2e_2 + a_3e_3$ and $Y = b_1e_1 + b_2e_2 + b_3e_3$.

For 3-dimensional (κ, μ)-contact metric manifolds, Riemannian curvature tensor can be written as follows:

$$\begin{aligned} R(X,Y)W = {} & [S(Y,W)X - S(X,W)Y + g(X,W)QX - g(X,W)QY] \\ & - \frac{r}{2}[g(Y,W)X - g(X,W)Y]. \end{aligned} \tag{7.4}$$

Using the values of Ricci tensors and the scalar curvature we obtain

$$R(X,Y)W = -[g(Y,W)X - g(X,W)Y]. \tag{7.5}$$

From the definition of φ-concircularly semisymmetric manifold we obtain

$$\begin{aligned}(Z(X,Y)\cdot\varphi)W &= Z(X,Y)\varphi W - \varphi Z(X,Y)W \\ &= R(X,Y)\varphi W - \varphi R(X,Y)W \\ &\quad - \frac{r}{6}[g(Y,\varphi W)X - g(X,\varphi W)Y \\ &\quad - g(Y,W)\varphi X + g(X,W)\varphi Y].\end{aligned} \tag{7.6}$$

Using (7.5) and the value of Ricci tensor, (7.6) yields

$$\begin{aligned}(Z(X,Y)\cdot\varphi)W = &-[g(Y,\varphi W)X - g(X,\varphi W)Y] \\ &+ [g(Y,W)\varphi X - g(X,W)\varphi Y] \\ &+ [g(Y,\varphi W)X - g(X,\varphi W)Y \\ &- g(Y,W)\varphi X + g(X,W)\varphi Y] \\ = &\ 0.\end{aligned} \tag{7.7}$$

Thus Theorem 5.1 is verified.

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