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THE BERGMAN KERNEL FOR INTERSECTION OF TWO COMPLEX ELLIPSOIDS

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Abstract. In this paper we obtain the closed forms of some hypergeometric functions. As an application, we obtain the explicit forms of the Bergman kernel functions for intersection of two complex ellipsoids $\{z \in \mathbb{C}^3 \colon |z_1|^p + |z_2|^q < 1,\ |z_1|^p + |z_3|^r < 1\}$. We consider cases $p = 6$, $q = r = 2$ and $p = q = r = 2$. We also investigate the Lu Qi-Keng problem for $p = q = r = 2$.

1. Introduction

In 1921, S. Bergman introduced a kernel function, which is now known as the Bergman kernel function. It is well known that there exists a unique Bergman kernel function for each bounded domain in $\mathbb{C}^n$. Computation of the Bergman kernel function by explicit formulas is an important research direction in several complex variables. Let D be a bounded domain in $\mathbb{C}^n$. The Bergman space $L^2_a(D)$ is the space of all square integrable holomorphic functions on D. Then the Bergman kernel $K_D(z, w)$ is defined [1] by

$$K_D(z, w) = \sum_{j=0}^{\infty} \phi_j(z)\overline{\phi_j(w)}, \quad (z, w) \in D \times D,$$

where $\{\phi_j() \colon j = 0, 1, 2, \ldots\}$ is a complete orthonormal basis for $L^2_a(D)$. If D is the Hermitian unit ball B_n defined by

$$B_n = \{z \in \mathbb{C}^n \colon |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 < 1\},$$

it is easy to see that z^α, $\alpha \in (\mathbb{Z}_+)^n$ form an orthogonal basis of $L^2_a(B_n)$. A direct computation shows that $\|z^\alpha\| = \sqrt{\frac{\alpha!\pi^n}{(n+|\alpha|)!}}$. So the functions $\varphi_\alpha = \sqrt{\frac{(n+|\alpha|)!}{\alpha!\pi^n}} z^\alpha$, $\alpha \in (\mathbb{Z}_+)^n$, form an orthonormal basis of $L^2_a(D)$. An easy computation gives:

$$K_{B_n}(z, w) = \frac{n!}{\pi^n} \frac{1}{(1 - \langle z, w\rangle)^{n+1}},$$

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where $\langle z,w\rangle := z_1\overline{w}_1+\cdots+z_n\overline{w}_n$. J.-D. Park in [18] compute Bergman kernel for nonhomogeneous domain

$$D_{q_1,q_2}=\{(z_1,z_2)\in\mathbb{C}^2 : |z_1|^{4/q_1}+|z_2|^{4/q_2}<1\}$$

for any positive integers q_1 and q_2. The goal of this paper is to give Bergman kernel for $\{z\in\mathbb{C}^3\colon |z_1|^p+|z_2|^q<1,\ |z_1|^p+|z_3|^r<1\}$ in cases when $p=6$, $q=r=2$ or $p=q=r=2$.

2. Main results

The following are the main theorems of this paper.

Theorem 2.1. *The Bergman kernel for* $D_1=\{(z_1,z_2,z_3)\in\mathbb{C}^3 : |z_1|^2+|z_2|^2<1,\ |z_1|^2+|z_3|^2<1\}$ *is given by*

$$K_{D_1}((z_1,z_2,z_3),(w_1,w_2,w_3))=\frac{3-6\nu_1+3\nu_1^2+\nu_1(\nu_2+\nu_3)-\nu_2-\nu_3-\nu_2\nu_3}{\pi^3(1-\nu_1-\nu_2)^3(1-\nu_1-\nu_3)^3},$$

where $\nu_i=z_i\overline{w_i}$ *for* $i=1,2,3$.

Theorem 2.2. *The Bergman kernel for* $D_2=\{(z_1,z_2,z_3)\in\mathbb{C}^3 : |z_1|^6+|z_2|^2<1,\ |z_1|^6+|z_3|^2<1\}$ *is given by*

$$\begin{aligned}
&K_{D_2}((z_1,z_2,z_3),(w_1,w_2,w_3))\\
&=\frac{3}{2\pi^3}\frac{\partial^2}{\partial\nu_2\partial\nu_3}\left\{\frac{\nu_1+2}{2(1-\nu_1^3-\nu_3)^2}+\frac{\nu_2\nu_3(V_1(\nu_1,\nu_2,\nu_3)+\nu_1V_2(\nu_1,\nu_2,\nu_3))}{(1-\nu_1^3)^2(1-\nu_1^3-\nu_2)^2(1-\nu_1^3-\nu_3)^2}\right.\\
&\quad+\frac{\frac{2\nu_3}{\sqrt[3]{1-\nu_2}}-\frac{2\nu_2}{\sqrt[3]{1-\nu_3}}+(2+\nu_1)(\nu_2-\nu_3)-\frac{\nu_2\nu_1}{(1-\nu_3)^{2/3}}+\frac{\nu_3\nu_1}{(1-\nu_2)^{2/3}}}{3\left(1-\nu_1^3-\nu_3\right)\left(\nu_3-\nu_2\right)}\\
&\quad+\frac{2\nu_2(1-\nu_3)^{2/3}-2\nu_3(1-\nu_2)^{2/3}-\nu_3\sqrt[3]{1-\nu_2}+\nu_2\sqrt[3]{1-\nu_3}}{2\left(1-\nu_1^3-\nu_3\right)^2(\nu_3-\nu_2)}\\
&\quad+\frac{\nu_3\left(1+\nu_1-\nu_1\sqrt[3]{1-\nu_2}-(1-\nu_2)^{2/3}\right)\left(2\nu_1^3+\nu_2+\nu_3-2\right)}{(1-\nu_1^3-\nu_2)^2(1-\nu_1^3-\nu_3)^2}\\
&\quad\left.+\frac{\nu_3((3\nu_1+2)(1-\nu_2)-2(1-\nu_2)^{2/3}-3\nu_1\sqrt[3]{1-\nu_2})}{3(1-\nu_2)(1-\nu_1^3-\nu_2)(1-\nu_1^3-\nu_3)}\right\},
\end{aligned}$$

where $\nu_i=z_i\overline{w_i}$ *for* $i=1,2,3$, *and*

$$\begin{aligned}
V_1(\nu_1,\nu_2,\nu_3)&=6\nu_1^9+\nu_1^6(6\nu_2+6\nu_3-7)+\nu_1^3(\nu_2(6\nu_3-2)-2(\nu_3+2))\\
&\quad+\nu_2(3\nu_3-4)-4\nu_3+5+\nu_1(\nu_2(6\nu_3-7)-7\nu_3+8),\\
V_2(\nu_1,\nu_2,\nu_3)&=3\nu_1^9+\nu_1^6(3\nu_2+3\nu_3+2)+\nu_1^3(\nu_2(3\nu_3+4)+4\nu_3-13)\\
&\quad+3\nu_1^7+2\nu_1^4(\nu_2+\nu_3-3)+\nu_1\nu_2(\nu_3-2)-2\nu_1\nu_3+3\nu_1.
\end{aligned}$$

Theorem 2.3. *The Bergman kernel for* $D_3=\{(z_1,z_2,z_3)\in\mathbb{C}^3 : |z_1|^2+|z_2|^2<1,\ |z_1|^4+|z_3|^2<|z_1|^2\}$ *is given by*

$$K_{D_3}(z,w)=\frac{\nu_1(\nu_3(\nu_1-\nu_2-1)+(\nu_1-1)\nu_1(3\nu_1+\nu_3-3))}{\pi^3(\nu_1+\nu_2-1)^3((\nu_1-1)\nu_1+\nu_3)^3},$$

where $z = (z_1, z_2, z_3)$, $w = (w_1, w_2, w_3)$ *and* $\nu_i = z_i\overline{w_i}$ *for* $i = 1, 2, 3$.

Theorem 2.4. *The Bergman kernel for domain* D_4 *defined*

$$\left\{z \in \mathbb{C}^4 : |z_1|^2 + |z_2|^2 + |z_3|^2 < 1,\ \left(|z_1|^2 + |z_2|^2\right)^2 + |z_4|^2 < |z_1|^2 + |z_2|^2\right\}$$

is given by

$$\begin{aligned}
&K_{D_4}((z_1, z_2, z_3, z_4), (w_1, w_2, w_3, w_4))\\
&= \frac{1}{\pi^4}\frac{\partial^2}{\partial\nu_3\partial\nu_4}\left\{\frac{\nu_3}{\nu_3 - \nu_3^2 - \nu_4}\right.\\
&\quad\left(\frac{1 + \nu_1 + \nu_2 - 2\nu_3}{(1 - \nu_1 - \nu_2)^3} + \frac{\nu_4\nu_3(\nu_4 - \nu_3)}{(\nu_3 - \nu_4 - 1)(1 - \nu_1 - \nu_2)^2}\right.\\
&\quad + \frac{\nu_4\nu_3(2 - 2\nu_1 - 2\nu_2 - \nu_3)}{(\nu_3 - \nu_4 - 1)(1 - \nu_1 - \nu_2)^2(1 - \nu_1 - \nu_2 - \nu_3)^2}\\
&\quad + \frac{\nu_4\left(6\nu_3(\nu_1 + \nu_2 - 1) + 6(\nu_1 + \nu_2 - 1)^2 + 2\nu_3^2\right)}{(\nu_1 + \nu_2 - 1)^3(\nu_1 + \nu_2 + \nu_3 - 1)^3}\\
&\quad + \frac{8\left(W_1(\nu_1, \nu_2, \nu_3, \nu_4) + \sqrt{1 - 4\nu_4}W_2(\nu_1, \nu_2, \nu_3, \nu_4)\right)}{(1 - 4\nu_4)^{3/2}\nu_4\left(\sqrt{1 - 4\nu_4} - 2\nu_1 + 1\right)\left(\sqrt{1 - 4\nu_4} - 2\nu_1 - 2\nu_2 + 1\right)^3}\\
&\quad \left.\left. + \frac{\nu_4\nu_3\left(\nu_3 - \nu_4\right)\left(\sqrt{1 - 4\nu_4} + 1\right)^2 + 4\nu_4}{(\nu_3 - \nu_4 - 1)\sqrt{1 - 4\nu_4}\left(\sqrt{1 - 4\nu_4} - 2\nu_1 - 2\nu_2 + 1\right)^2}\right)\right\},
\end{aligned}$$

where $\nu_i = z_i\overline{w_i}$ *for* $i = 1, 2, 3, 4$, *and*

$$\begin{aligned}
W_1(\nu_1, \nu_2, \nu_3, \nu_4) &= \nu_4^2(8\nu_1 + 8\nu_2 - 4\nu_3 + 2) - (\nu_1 - 1)(\nu_1 + \nu_2 + 2\nu_3 + 1)\\
&\quad + \nu_4\left(\nu_1^2(4\nu_3 + 6) + \nu_1(4\nu_2\nu_3 + 6\nu_2 + 4\nu_3 - 4)\right.\\
&\quad \left.-2\nu_2\nu_3 - 7\nu_2 - 8\nu_3 - 4\right),\\
W_2(\nu_1, \nu_2, \nu_3, \nu_4) &= w(4\nu_1(\nu_1 + \nu_2 - 1) - 2\nu_2\nu_3 - 5\nu_2 - 4\nu_3 - 2)\\
&\quad - (\nu_1 - 1)(\nu_1 + \nu_2 + 2\nu_3 + 1).
\end{aligned}$$

3. Explicit formulas of hypergeometric functions

A great interest in the theory of hypergeometric functions (that is, hypergeometric functions of several variables) is motivated essentially by the fact that the solutions of many applied problems involving (for example) partial differential equations are obtainable with the help of such hypergeometric function (see, for details, ([22], p. 47); see also other works ([8, 9], [17]) and the references cited therein). For instance, the energy absorbed by some non-ferromagnetic conductor sphere included in an internal magnetic field can be calculated with the help of such functions [11], [14]. Hypergeometric functions of several variables are used in physical and quantum chemical applications as well [13], [21]. Especially, many problems in gas dynamics lead to solutions of degenerate second-order partial differential equations, which are then solvable in terms of multiple hypergeometric functions. Among examples, we can cite the problem of adiabatic flat-parallel gas flow without whirlwind, the flow problem of supersonic current from vessel with flat walls, and a number of other problems connected with gas flow [2], [7]. Multiple hypergeometric functions (that is, hypergeometric functions in several variables) occur naturally in a wide variety of problems. In particular, one of the Lauricella functions

$$F_8(a,b_1,b_2,b_3;c_1,c_2;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}(b_1)_m(b_2)_n(b_3)_p}{(c_1)_m(c_2)_{n+p}m!n!p!} x^m y^n z^p,$$

Appell's functions F, F_1 and F_2 defined by

$$F(a,b;c;x) = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m m!} x^m,$$

$$F_1(a,b_1,b_2;c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}m!n!} x^m y^n,$$

$$F_2(a,b_1,b_2;c_1,c_2;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n m!n!} x^m y^n,$$

and Horn's function H_3

$$H_3(a,b;c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(b)_n}{(c)_{m+n}m!n!} x^m y^n.$$

For function F_1 we have the following integral representation (see [6])

$$F_1(a,b_1,b_2;c;x,y) = \frac{\Gamma(c)}{\Gamma(b_1)\Gamma(b_2)\Gamma(c-b_1-b_2)} \iint\limits_{\substack{u\geq 0, v\geq 0\\ u+v\leq 1}} u^{b_1-1}v^{b_2-1}(1-u-v)^{c-b_1-b_2-1}(1-ux-vy)^{-a}\,dudv,$$

where $\Re(b_1) > 0, \Re(b_2) > 0, \Re(c-b_1-b_2) > 0$.

Picard has pointed out that F_1 can be represented by a single integral in the form

$$F_1(a,b_1,b_2;c;x,y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux)^{-b_1}(1-uy)^{-b_2}\,du,$$

where $\Re(a) > 0, \Re(c-a) > 0$.

In [5] presented certain interesting integral representation for Horn's H_3 function

$$H_3(a,b;c;x,y) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-uy)^{-a}$$

$$\cdot F\left(\frac{a}{2}, \frac{a+1}{2}; c-b; \frac{4x(1-u)}{(1-uy)^2}\right) du,$$

where $\Re(c) > \Re(b) > 0$.

In this section we prove following recursion formula for

$$F_8(a,1;c_1,c_2;x,y,z) := F_8(a,1,1,1;c_1,c_2;x,y,z).$$

Proposition 3.1. *For any $a > 1$ and $|x|+|y| < 1$, $|x|+|z| < 1$, we have*

$$\begin{aligned} F_8(a,1;c_1,c_2;x,y,z) &= \frac{a-c_1-c_2+1}{(a-1)(1-x-z)} F_8(a-1,1;c_1,c_2;x,y,z) \\ &+ \frac{c_1-1}{a-1}\frac{1}{1-x-z} F_1(a-1,1,1;c_2;y,z) \\ &+ \frac{c_2-1}{a-1}\frac{1}{1-x-z} F_2(a-1,1,1;c_1,c_2-1;x,y). \end{aligned}$$

Proof. Using well know fact $(1)_k = k!$ we have

$$F_8(a,1;c_1,c_2;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}}{(c_1)_m (c_2)_{n+p}} x^m y^n z^p.$$

Next from $(s)_k = \frac{(s-1+k)(s-1)_k}{s-1}$ we have

$$F_8(a,1;c_1,c_2;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a-1+m+n+p)(a-1)_{m+n+p}}{(a-1)(c_1)_m (c_2)_{n+p}} x^m y^n z^p.$$

After little calculation we obtain

$$\begin{aligned} F_8(a,1;c_1,c_2;x,y,z) &= \frac{a+1-c_1-c_2}{a-1} F_8(a-1,1;c_1,c_2;x,y,z) \\ &+ \frac{c_1-1}{a-1} \sum_{m,n,p=0}^{\infty} \frac{(a-1)_{m+n+p}}{(c_1-1)_m (c_2)_{n+p}} x^m y^n z^p \\ &+ \frac{c_2-1}{a-1} \sum_{m,n,p=0}^{\infty} \frac{(a-1)_{m+n+p}}{(c_1)_m (c_2-1)_{n+p}} x^m y^n z^p. \end{aligned}$$

Now summing the m and p respectively in the second and third lines, we have

$$\begin{aligned} &F_8(a,1;c_1,c_2;x,y,z) = \frac{a+1-c_1-c_2}{a-1} F_8(a-1,1;c_1,c_2;x,y,z) \\ &+ \frac{c_1-1}{a-1}\left(\sum_{n,p=0}^{\infty} \frac{(a-1)_{n+p}}{(c_2)_{n+p}} y^n z^p + \sum_{m=1,n,p=0}^{\infty} \frac{(a-1)_{m+n+p}}{(c_1-1)_m (c_2)_{n+p}} x^m y^n z^p\right) \\ &+ \frac{c_2-1}{a-1}\left(\sum_{m,n=0}^{\infty} \frac{(a-1)_{m+n}}{(c_1)_m (c_2-1)_n} x^m y^n + \sum_{m,n=0,p=1}^{\infty} \frac{(a-1)_{m+n+p}}{(c_1)_m (c_2-1)_{n+p}} x^m y^n z^p\right). \end{aligned}$$

Reversing the summation indexes, we can write

$$F_8(a,1;c_1,c_2;x,y,z) = \frac{a+1-c_1-c_2}{a-1}F_8(a-1,1;c_1,c_2;x,y,z) + \frac{c_1-1}{a-1}\sum_{n,p=0}^{\infty}\frac{(a-1)_{n+p}}{(c_2)_{n+p}}y^n z^p + xF_8(a,1;c_1,c_2;x,y,z) + \frac{c_2-1}{a-1}\sum_{m,n=0}^{\infty}\frac{(a-1)_{m+n}}{(c_1)_m(c_2-1)_n}x^m y^n + zF_8(a,1;c_1,c_2;x,y,z)$$

which complete proof. □

Now we calculate formulas in special cases.

Lemma 3.2. *For $|x|+|y|<1$, $|x|+|z|<1$ and $y\neq 0$, $z\neq 0$ $z\neq y$, we have*

$$F_8\left(\frac{10}{3},1;\frac{1}{3},3;x^3,y,z\right) = \frac{-\frac{9y}{\sqrt[3]{1-z}}+\frac{9z}{\sqrt[3]{1-y}}-9(z-y)}{7yz(z-y)M_{zx}} + \frac{9\left(\sqrt[3]{1-y}-1\right)}{7y\sqrt[3]{1-y}M_{yx}M_{zx}} + \frac{9P(x,y,z)}{14\left(1-x^3\right)^2 M_{yx}^2M_{zx}^2} + \frac{27\left((1-y)^{2/3}-1\right)\left(M_{yx}+M_{zx}\right)}{14yM_{yx}^2M_{zx}^2} - \frac{27\left(-z(1-y)^{2/3}+y(1-z)^{2/3}-y+z\right)}{14yz(y-z)M_{zx}^2},$$

$$F_8\left(\frac{11}{3},1;\frac{2}{3},3;x^3,y,z\right) = \frac{-\frac{9y}{(1-z)^{2/3}}+\frac{9z}{(1-y)^{2/3}}-9(z-y)}{40yz(z-y)M_{zx}} + \frac{9G(x,y,z)}{40\left(1-x^3\right)^2 M_{yx}^2M_{zx}^2} + \frac{27\left(\sqrt[3]{1-y}-1\right)\left(M_{yx}+M_{zx}\right)}{40yM_{yx}^2M_{zx}^2} - \frac{27\left(-z\sqrt[3]{1-y}+y\sqrt[3]{1-z}-y+z\right)}{80yz(y-z)M_{zx}^2} + \frac{27\left((1-y)^{2/3}-1\right)}{40y(1-y)^{2/3}M_{yx}M_{zx}},$$

$$F_8\left(4,1;1,3;x^3,y,z\right) = \frac{3x^6+2x^3(y+z-3)+yz-2y-2z+3}{3\left(x^3-1\right)^2M_{yx}^2M_{zx}^2},$$

where $P(x,y,z) = 2x^9+x^6(2y+2z+3)+2x^3(yz+y+z-6)+yz-4y-4z+7$, $G(x,y,z) = x^9+x^6(y+z+6)+x^3(yz+4y+4z-15)+2yz-5y-5z+8$ *and* $M_{tx} = 1-x^3-t$.

If $y=0$, then we have

$$F_8\left(\frac{10}{3},1;\frac{1}{3},3;x^3,y,z\right) = F_2\left(\frac{10}{3},1,1;\frac{1}{3},3;x^3,z\right)$$

$$= \frac{x^3\left(2x^3\left(18-x^3-z\right)+16z-21\right)+4z-13}{14\left(x^3-1\right)^3 M_{zx}^2} + \frac{27(1-z)^{2/3}+18z-27}{14z^2M_{zx}^2} + \frac{3z\sqrt[3]{1-z}+9\sqrt[3]{1-z}-9}{7z^2\sqrt[3]{1-z}M_{zx}},$$

$$F_8\left(\frac{11}{3},1;\frac{2}{3},3;x^3,y,z\right) = F_2\left(\frac{11}{3},1,1;\frac{2}{3},3;x^3,z\right)$$

$$= \frac{-6x^9+x^6(45-6z)+x^3(30z+9)+6(5z-8)}{40\left(x^3-1\right)^3 M_{zx}^2} + \frac{27\sqrt[3]{1-z}+9z-27}{40z^2M_{zx}^2} + \frac{6z(1-z)^{2/3}+9(1-z)^{2/3}-9}{40z^2(1-z)^{2/3}M_{zx}},$$

where as before $M_{tx} = 1-x^3-t$*. In the case when* $z=y=0$*, we have*

$$F_8\left(\frac{10}{3},1;\frac{1}{3},3;x^3,y,z\right) = F\left(\frac{10}{3},1;\frac{1}{3};x^3\right) = \frac{4x^9-21x^6+84x^3+14}{14\left(1-x^3\right)^4},$$

$$F_8\left(\frac{11}{3},1;\frac{2}{3},3;x^3,y,z\right) = F\left(\frac{11}{3},1;\frac{2}{3};x^3\right) = \frac{5x^9-24x^6+60x^3+40}{40\left(1-x^3\right)^4}.$$

If $y=z$ *and* $z\neq 0$*, then we have*

$$F_8\left(\frac{10}{3},1;\frac{1}{3},3;x^3,z,z\right) = \lim_{y\to z} F_8\left(\frac{10}{3},1;\frac{1}{3},3;x^3,y,z\right),$$

$$F_8\left(\frac{11}{3},1;\frac{2}{3},3;x^3,z,z\right) = \lim_{y\to z} F_8\left(\frac{11}{3},1;\frac{2}{3},3;x^3,y,z\right).$$

Proof. Using twice recursion formula for F_8 function, we have

$$F_8\left(\frac{10}{3},1;\frac{1}{3},3;x^3,y,z\right) = \frac{-3F_1\left(\frac{4}{3},1,1;3;y,z\right)}{14(1-x^3-z)^2} + \frac{9F_2\left(\frac{4}{3},1,1;\frac{1}{3},2;x^3,y\right)}{14(1-x^3-z)^2} + \frac{-2F_1\left(\frac{7}{3},1,1;3;y,z\right)}{7(1-x^3-z)} + \frac{6F_2\left(\frac{7}{3},1,1;\frac{1}{3},2;x^3,y\right)}{7(1-x^3-z)}.$$

Now using recursion formula for F_2 (see [18] or more general version [19]), we obtain

$$F_8\left(\frac{10}{3},1;\frac{1}{3},3;x^3,y,z\right)$$

$$= \frac{-3F_1\left(\frac{4}{3},1,1;3;y,z\right)}{14(1-x^3-z)^2} + \frac{-2F_1\left(\frac{7}{3},1,1;3;y,z\right)}{7(1-x^3-z)} - \frac{18F\left(\frac{1}{3},1;2;y\right)}{14\left(-x^3-y+1\right)\left(-x^3-z+1\right)^2} + \frac{27F\left(\frac{1}{3},1;\frac{1}{3};x^3\right)}{14\left(-x^3-y+1\right)\left(-x^3-z+1\right)^2} + \frac{27\,F\left(\frac{1}{3},1;\frac{1}{3};x^3\right)}{14\left(-x^3-y+1\right)^2\left(-x^3-z+1\right)} - \frac{9\,F\left(\frac{1}{3},1;2;y\right)}{7\left(-x^3-y+1\right)^2\left(-x^3-z+1\right)}$$

$$+\frac{9\,F\left(\frac{4}{3},1;\frac{1}{3};x^3\right)}{14\left(-x^3-y+1\right)\left(-x^3-z+1\right)}-\frac{3\,F\left(\frac{4}{3},1;2;y\right)}{7\left(-x^3-y+1\right)\left(-x^3-z+1\right)}.$$

After some calculations using integral representation for F_1 function and well knows formulas $F\left(\frac{1}{3},1;2;w\right)=\frac{3\left(1-(1-w)^{2/3}\right)}{2w}$, $F\left(\frac{4}{3},1;2;w\right)=\frac{3\left(1-\sqrt[3]{1-w}\right)}{\sqrt[3]{1-w}w}$ and $F\left(\frac{4}{3},1;\frac{1}{3};w\right)=\frac{2w+1}{(1-w)^2}$ we obtain desired result. The proof for the other formulas is similar to above. □

To prove Bergman kernel formula for domain D_4 we need the following lemma.

It is possible that the following sum is represented by a hypergeometric function or is related to a combination of certain hypergeometric functions, but from the point of view of this work most interesting is its calculation.

Lemma 3.3. *For* $|x|+|y|+|z|<1$, $|4w|<1$, $w\neq 0$, $w\neq z-z^2$ *and* $2|x|+2|y|-|\sqrt{1-4w}|<1$, *we have*

$$\sum_{m,n,k,l=0}^{\infty}\frac{\Gamma\left(m+n+2\right)\Gamma\left(m+n+k+2l+6\right)}{m!n!\Gamma\left(m+n+l+3\right)\Gamma\left(k+l+3\right)}x^my^nz^kw^l$$

$$\begin{aligned}
&=\frac{z(2-2x-2y-z)}{(z-z^2-w)(z-w-1)(1-x-y)^2(1-x-y-z)^2}\\
&\quad+\frac{2z}{w(z-z^2-w)(x+y-1)^3}+\frac{z(w-z)}{(z-z^2-w)(z-w-1)(1-x-y)^2}\\
&\quad+\frac{z\left(z-w\right)\left(\sqrt{1-4w}+1\right)^2+4}{(z-z^2-w)(z-w-1)\sqrt{1-4w}\left(\sqrt{1-4w}-2x-2y+1\right)^2}\\
&\quad+\frac{6z(x+y-1)+6(x+y-1)^2+2z^2}{(z-z^2-w)(x+y-1)^3(x+y+z-1)^3}+\frac{1+x+y}{w(z-z^2-w)(1-x-y)^3}\\
&\quad+\frac{8\left(W_1(x,y,z,w)+\sqrt{1-4w}W_2(x,y,z,w)\right)}{(z-z^2-w)(1-4w)^{3/2}w\left(\sqrt{1-4w}-2x+1\right)\left(\sqrt{1-4w}-2x-2y+1\right)^3},
\end{aligned}$$

where

$$\begin{aligned}
W_1(x,y,z,w)&=w^2(8x+8y-4z+2)-(x-1)(x+y+2z+1)\\
&\quad+w\left(x^2(4z+6)+x(4yz+6y+4z-4)-2yz-7y-8z-4\right),\\
W_2(x,y,z,w)&=w(4x(x+y-1)-2yz-5y-4z-2)\\
&\quad-(x-1)(x+y+2z+1).
\end{aligned}$$

Proof. Since $\Gamma(a+1)=a\Gamma(a)$

$$S:=\sum_{m,n,k,l=0}^{\infty}\frac{\Gamma\left(m+n+2\right)\Gamma\left(m+n+k+2l+6\right)}{m!n!\Gamma\left(m+n+l+3\right)\Gamma\left(k+l+3\right)}x^my^nz^kw^l$$

$$= \sum_{m,n,k,l=0}^{\infty} \frac{(m+n+k+2l+5)\,\Gamma(m+n+2)\,\Gamma(m+n+k+2l+5)}{m!n!\Gamma(m+n+l+3)\,\Gamma(k+l+3)} x^m y^n z^k w^l.$$

Hence

$$\begin{aligned} S = S_1 &:= \sum_{m,n,k,l=0}^{\infty} \frac{\Gamma(m+n+2)\,\Gamma(m+n+k+2l+5)}{m!n!\Gamma(m+n+l+3)\,\Gamma(k+l+3)} x^m y^n z^k w^l \\ &+ S_2 := \sum_{m,n,k,l=0}^{\infty} \frac{\Gamma(m+n+2)\,\Gamma(m+n+k+2l+5)}{m!n!\Gamma(m+n+l+2)\,\Gamma(k+l+3)} x^m y^n z^k w^l \\ &+ S_3 := \sum_{m,n,k,l=0}^{\infty} \frac{\Gamma(m+n+2)\,\Gamma(m+n+k+2l+5)}{m!n!\Gamma(m+n+l+3)\,\Gamma(k+l+2)} x^m y^n z^k w^l. \end{aligned}$$

By proceeding in a similar manner as in the case of the sum of S, we have

$$\begin{aligned} S_1 &= \sum_{m,n,k,l=0}^{\infty} \frac{\Gamma(m+n+2)\,\Gamma(m+n+k+2l+5)}{m!n!\Gamma(m+n+l+3)\,\Gamma(k+l+3)} x^m y^n z^k w^l \\ &= \sum_{m,n,k,l=0}^{\infty} \frac{\Gamma(m+n+2)\,\Gamma(m+n+k+2l+4)}{m!n!\Gamma(m+n+l+2)\,\Gamma(k+l+3)} x^m y^n z^k w^l \\ &\quad + \sum_{m,n,k,l=0}^{\infty} \frac{\Gamma(m+n+2)\,\Gamma(m+n+k+2l+4)}{m!n!\Gamma(m+n+l+3)\,\Gamma(k+l+2)} x^m y^n z^k w^l. \end{aligned}$$

Using the identity $\Gamma(a+1) = a\Gamma(a)$, after a little simplification, we obtain

$$\begin{aligned} \left(1 - \frac{w}{z} - z\right) S_1 &= \frac{1}{z} \sum_{m,n,k=0}^{\infty} \frac{\Gamma(m+n+2)\,\Gamma(m+n+k+3)}{m!n!\Gamma(m+n+2)\,\Gamma(k+2)} x^m y^n z^k \\ &\quad - \frac{1}{z} \sum_{m,n,l=0}^{\infty} \frac{\Gamma(m+n+2)\,\Gamma(m+n+2l+3)}{m!n!\Gamma(m+n+l+2)\,\Gamma(l+2)} x^m y^n w^l \\ &\quad + \sum_{m,n,l=0}^{\infty} \frac{\Gamma(m+n+2)\,\Gamma(m+n+2l+4)}{m!n!\Gamma(m+n+l+3)\,\Gamma(l+2)} x^m y^n w^l. \end{aligned}$$

After some calculations, we obtain

$$\begin{aligned} \left(1 - \frac{w}{z} - z\right) S_1 &= \frac{2 - 2x - 2y - z}{z(1-x-y)^2(1-x-y-z)^2} \\ &\quad + (1-1/z) \sum_{m,n,l=0}^{\infty} \frac{\Gamma(m+n+2)\,\Gamma(m+n+2l+3)}{m!n!\Gamma(m+n+l+2)\,\Gamma(l+2)} x^m y^n w^l \\ &\quad + \sum_{m,n,l=0}^{\infty} \frac{\Gamma(m+n+2)\,\Gamma(m+n+2l+3)}{m!n!\Gamma(m+n+l+3)\,\Gamma(l+1)} x^m y^n w^l. \end{aligned}$$

Shifting (changing) the summation index l, we have

$$\begin{aligned}&\left(1-\frac{w}{z}-z\right)S_1\\ =&\frac{2-2x-2y-z}{z(1-x-y)^2(1-x-y-z)^2}\\ &+\left(\frac{1}{w}-\frac{1}{zw}\right)\sum_{m,n,l=0}^{\infty}\frac{\Gamma(m+n+2)\,\Gamma(m+n+2l+1)}{m!n!\Gamma(m+n+l+1)\,\Gamma(l+1)}x^m y^n w^l\\ &-\left(\frac{1}{w}-\frac{1}{zw}\right)\sum_{m,n=0}^{\infty}\frac{\Gamma(m+n+2)}{m!n!}x^m y^n\\ &+\sum_{m,n,l=0}^{\infty}\frac{\Gamma(m+n+2)\,\Gamma(m+n+2l+3)}{m!n!\Gamma(m+n+l+3)\,\Gamma(l+1)}x^m y^n w^l.\end{aligned}$$

Sum out of l

$$\begin{aligned}&\left(1-\frac{w}{z}-z\right)S_1\\ =&\frac{2-2x-2y-z}{z(1-x-y)^2(1-x-y-z)^2}\\ &+\sum_{m,n=0}^{\infty}\frac{(z-1)\Gamma(m+n+2)\,F(\frac{m+n+1}{2},\frac{m+n+2}{2};m+n+1;4w)}{zwm!n!}x^m y^n\\ &+\sum_{m,n=0}^{\infty}\frac{(1-z)\Gamma(m+n+2)}{zwm!n!}x^m y^n\\ &+\sum_{m,n=0}^{\infty}\frac{\Gamma(m+n+2)\,F(\frac{m+n+3}{2},\frac{m+n+4}{2};m+n+3;4w)}{m!n!}x^m y^n.\end{aligned}$$

Using the following well know formula

$$F\left(a,a+\frac{1}{2};2a;z\right)=\frac{2^{2a-1}}{\sqrt{1-z}(\sqrt{1-z}+1)^{2a-1}},$$

we obtain

$$\begin{aligned}&\left(1-\frac{w}{z}-z\right)S_1\\ =&\frac{2-2x-2y-z}{z(1-x-y)^2(1-x-y-z)^2}\\ &+\left(\frac{1}{w}-\frac{1}{zw}\right)\sum_{m,n=0}^{\infty}\frac{\Gamma(m+n+2)\,2^{m+n}}{m!n!\sqrt{1-4w}(\sqrt{1-4w}+1)^{m+n}}x^m y^n\\ &-\left(\frac{1}{w}-\frac{1}{zw}\right)\sum_{m,n=0}^{\infty}\frac{\Gamma(m+n+2)}{m!n!}x^m y^n\end{aligned}$$

$$+\sum_{m,n=0}^{\infty}\frac{\Gamma(m+n+2)\,2^{m+n+2}}{m!n!\sqrt{1-4w}(\sqrt{1-4w}+1)^{m+n+2}}x^m y^n.$$

Sum out of m and n variables, we get

$$\left(1-\frac{w}{z}-z\right)S_1=\frac{2-2x-2y-z}{z(1-x-y)^2(1-x-y-z)^2}-\left(\frac{1}{w}-\frac{1}{zw}\right)\frac{1}{(1-x-y)^2}$$
$$+\frac{\left(\frac{1}{w}-\frac{1}{zw}\right)\left(\sqrt{1-4w}+1\right)^2+4}{\sqrt{1-4w}\left(\sqrt{1-4w}-2x-2y+1\right)^2}.$$

Hence

$$S_1=\frac{2-2x-2y-z}{(z-w-1)(1-x-y)^2(1-x-y-z)^2}-\frac{z-w}{(z-w-1)(1-x-y)^2}$$
$$+\frac{(z-w)\left(\sqrt{1-4w}+1\right)^2+4}{(z-w-1)\sqrt{1-4w}\left(\sqrt{1-4w}-2x-2y+1\right)^2}.$$

Now analogous maneuvers for sum S_2 lead us to

$$S_2=\frac{w}{z}S+\frac{1}{z}\sum_{m,n,k=0}^{\infty}\frac{\Gamma(m+n+k+4)}{m!n!\Gamma(k+2)}x^m y^n z^k$$
$$-\frac{1}{z}\sum_{m,n,l=0}^{\infty}\frac{\Gamma(m+n+2)\Gamma(m+n+2l+4)}{m!n!\Gamma(m+n+l+2)\Gamma(l+2)}x^m y^n w^l.$$

Similar as in S_1 case, we have

$$S_2=\frac{w}{z}S+\frac{6z(x+y-1)+6(x+y-1)^2+2z^2}{z(x+y-1)^3(x+y+z-1)^3}$$
$$-\frac{1}{zw}\sum_{m,n,l=0}^{\infty}\frac{\Gamma(m+n+2)\Gamma(m+n+2l+2)}{m!n!\Gamma(m+n+l+1)\Gamma(l+1)}x^m y^n w^l$$
$$+\frac{1}{zw}\sum_{m,n=0}^{\infty}\frac{\Gamma(m+n+2)\Gamma(m+n+2)}{m!n!\Gamma(m+n+1)}x^m y^n.$$

Sum out of l

$$S_2=\frac{w}{z}S+\frac{6z(x+y-1)+6(x+y-1)^2+2z^2}{z(x+y-1)^3(x+y+z-1)^3}$$
$$-\frac{1}{zw}\sum_{m,n=0}^{\infty}\frac{(\Gamma(m+n+2))^2F\left(\frac{m+n+2}{2},\frac{m+n+3}{2};m+n+1;4w\right)}{m!n!\Gamma(m+n+1)}x^m y^n$$
$$+\frac{1}{zw}\sum_{m,n,l=0}^{\infty}\frac{(m+n+1)\Gamma(m+n+2)}{m!n!}x^m y^n.$$

Using the following formula

$$F\left(\frac{a+2}{2},\frac{a+3}{2};a+1;z\right)=\frac{2^a((a-1)(\sqrt{1-z}-1)+az)}{(a+1)z(1-z)^{3/2}(\sqrt{1-z}+1)^{a-1}},$$

we get

$$\begin{aligned}S_2 = {} & \frac{w}{z}S+\frac{6z(x+y-1)+6(x+y-1)^2+2z^2}{z(x+y-1)^3(x+y+z-1)^3}\\ & -\sum_{m,n=0}^{\infty}\frac{\Gamma(m+n+2)((m+n-1)(\sqrt{1-4w}-1)+(m+n)4w)}{m!n!4zw^2(1-4w)^{3/2}(\sqrt{1-4w}+1)^{m+n-1}}(2x)^m(2y)^n\\ & +\frac{1}{zw}\sum_{m,n,l=0}^{\infty}\frac{(m+n+1)\,\Gamma\,(m+n+2)}{m!n!}x^my^n.\end{aligned}$$

Sum out of m and n variables, we have

$$\begin{aligned}S_2 = {} & \frac{w}{z}S+\frac{6z(x+y-1)+6(x+y-1)^2+2z^2}{z(x+y-1)^3(x+y+z-1)^3}+\frac{1+x+y}{wz(1-x-y)^3}\\ & -\frac{3\left(\sqrt{1-4w}-1\right)\left(\sqrt{1-4w}+1\right)^3x(x+y)}{(1-4w)^{3/2}w^2z\left(\sqrt{1-4w}-2x+1\right)\left(\sqrt{1-4w}-2x-2y+1\right)^3}\\ & +\frac{\left(\sqrt{1-4w}+1\right)^3\left(\sqrt{1-4w}-8x^2+1\right)}{(1-4w)^{3/2}wz\left(\sqrt{1-4w}-2x+1\right)\left(\sqrt{1-4w}-2x-2y+1\right)^3}\\ & +\frac{\left(\sqrt{1-4w}+1\right)^3\left(4x\left(\sqrt{1-4w}-2y-1\right)+4\sqrt{1-4w}y-2y\right)}{(1-4w)^{3/2}wz\left(\sqrt{1-4w}-2x+1\right)\left(\sqrt{1-4w}-2x-2y+1\right)^3}.\end{aligned}$$

In S_3 case, we have

$$\begin{aligned}S_3 = {} & zS-\frac{1}{w}\sum_{m,n=0}^{\infty}\frac{\Gamma\,(m+n+3)}{m!n!}x^my^n\\ & +\frac{1}{w}\sum_{m,n,l=0}^{\infty}\frac{\Gamma\,(m+n+2)\,\Gamma\,(m+n+2l+3)}{m!n!\Gamma\,(m+n+l+2)\,\Gamma\,(l+1)}x^my^nw^l.\end{aligned}$$

Hence

$$\begin{aligned}S_3 = {} & zS-\frac{1}{w}\sum_{m,n=0}^{\infty}\frac{\Gamma\,(m+n+3)}{m!n!}x^my^n\\ & +\sum_{m,n=0}^{\infty}\frac{\Gamma\,(m+n+3)\,((m+n)(\sqrt{1-4w}-1)+(m+n+1)4w)}{m!n!(m+n+2)2w^2(1-4w)^{3/2}(\sqrt{1-4w}+1)^{m+n}}(2x)^m(2y)^n.\end{aligned}$$

After a little calculations, we have

$$S_3 = zS+\frac{2}{w(x+y-1)^3}$$

$$+\frac{16\left(\left(\sqrt{1-4w}+1\right)(1-x)-2w^2\right)}{(1-4w)^{3/2}w\left(\sqrt{1-4w}-2x+1\right)\left(\sqrt{1-4w}-2x-2y+1\right)^3}$$
$$-\frac{16\left(\sqrt{1-4w}y+2\sqrt{1-4w}-2x(x+y+1)+y+4\right)}{(1-4w)^{3/2}\left(\sqrt{1-4w}-2x+1\right)\left(\sqrt{1-4w}-2x-2y+1\right)^3}.$$

This completes the proof of Lemma 3.3. □

4. Computation of the kernel

For Reinhardt domains it is a standard method for computing the Bergman kernel to use series representation, since we can choose $\phi_\alpha(z) = \frac{z^\alpha}{\|z^\alpha\|}$. Put $\Phi_\alpha(\zeta) = z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}$. It is well known, that function f holomorphic in a Reinhardt domain $D \subset \mathbb{C}^n$ has a global expansion into a Laurent series $f(z) = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha z^\alpha$, $z \in D$ (see Proposition 1.7.15(c) in [10]). Moreover if $D \cap (\mathbb{C}^{j-1} \times \{0\} \times \mathbb{C}^{n-j}) \neq \emptyset$, $j = 1, \ldots, n$, then $a_\alpha = 0$ for $\alpha \in \mathbb{Z}^n \setminus \mathbb{Z}_+^n$ (see Proposition 1.6.5(c) in [10]). Therefore $\{\Phi_\alpha\}$ such that each $\alpha_i \geq 0$ is a complete orthogonal set for $L^2(D_1)$ and $L^2(D_2)$.

If D is a Reinhardt domain, $f \in L_a^2(D) := \mathcal{O}(D) \cap L^2(D)$, $f(z) = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha z^\alpha$, then $\{z^\alpha \colon \alpha \in \sum(f)\} \subset L_a^2(D)$, where $\sum(f) := \{\alpha \in \mathbb{Z}^n \colon a_\alpha \neq 0\}$ (for proof see [10] p. 67). Thus it is easy to check, that the set $\{z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} \colon \alpha_2 \geq 0, \alpha_3 \geq 0, \alpha_1 \geq -1-\alpha_3\}$ is a complete orthogonal set for $L_a^2(D_3)$.

Proposition 4.1. *Let $\alpha_i \in \mathbb{Z}_+$ for $i = 1, 2, 3$. Then, we have*

$$\|z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}\|^2_{L^2(D_1)} = \frac{\pi^3\Gamma(\alpha_1+1)\Gamma(\alpha_2+\alpha_3+3)}{(\alpha_2+1)(\alpha_3+1)\Gamma(\alpha_1+\alpha_2+\alpha_3+4)}.$$

Proposition 4.2. *Let $\alpha_i \in \mathbb{Z}_+$ for $i = 1, 2, 3$. Then, we have*

$$\|z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}\|^2_{L^2(D_2)} = \frac{2\pi^3\Gamma(\frac{\alpha_1+1}{3})\Gamma(\alpha_2+\alpha_3+3)}{3(\alpha_2+1)(\alpha_3+1)\Gamma(\frac{\alpha_1+1}{3}+\alpha_2+\alpha_3+3)}.$$

Proposition 4.3. *For $\alpha_2, \alpha_3 \in \mathbb{Z}_+$ and $\alpha_1 \geq -1-\alpha_3$, we have*

$$\|z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}\|^2_{L^2(D_3)} = \frac{\pi^3\Gamma(\alpha_1+\alpha_3+2)\Gamma(\alpha_2+\alpha_3+3)}{(\alpha_2+1)(\alpha_3+1)\Gamma(\alpha_1+\alpha_2+2\alpha_3+5)}.$$

Proposition 4.4. *Let $\alpha_i \in \mathbb{Z}_+$ for $i = 1, 2, 3, 4$. Then, we have*

$$\|z^\alpha\|^2_{L^2(D_4)} = \frac{\pi^4\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(\alpha_1+\alpha_2+\alpha_4+3)\Gamma(\alpha_3+\alpha_4+3)}{(\alpha_3+1)(\alpha_4+1)\Gamma(\alpha_1+\alpha_2+2)\Gamma(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+6)}.$$

The proof of all above propositions is similar, and so we only prove Proposition 4.2.

Proof.

$$\|z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}\|^2_{L^2(D_2)} = \int_{D_2} |z_1|^{2\alpha_1}|z_2|^{2\alpha_2}|z_3|^{2\alpha_3}\,dV(z)$$

we introduce polar coordinate in each variable by putting $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, $z_3 = r_3 e^{i\theta_3}$. After doing so, and integrating out the angular variables we have

$$(2\pi)^3 \int_0^1 \int_0^{\sqrt{1-r_1^6}} \int_0^{\sqrt{1-r_1^6}} r_1^{2\alpha_1+1} r_2^{2\alpha_2+1} r_3^{2\alpha_3+1}\, dr_1 dr_2 dr_3.$$

Integrating out of r_2 and r_3 variables, we obtain

$$\frac{(2\pi)^3}{(2\alpha_2+2)(2\alpha_3+2)} \int_0^1 r_1^{2\alpha_1+1}(1-r_1^6)^{\alpha_2+\alpha_3+2}\, dr_1.$$

After little calculation using well known fact

$$\int_0^1 x^a (1-x^6)^b\, dx = \frac{\Gamma((a+1)/6)\Gamma(b+1)}{3\Gamma((a+1)/6+b+1)},$$

we obtain desired result. □

Now by series representation of the Bergman kernel function, we have

$$K_{D_1}(z,w) = \frac{1}{\pi^3} \sum_{\alpha_1,\alpha_2,\alpha_3=0}^{\infty} \frac{(\alpha_2+1)(\alpha_3+1)\Gamma(\alpha_1+\alpha_2+\alpha_3+4)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+\alpha_3+3)} \nu_1^{\alpha_1}\nu_2^{\alpha_2}\nu_3^{\alpha_3},$$

where $\nu_1 = z_1\overline{w}_1$, $\nu_2 = z_2\overline{w}_2$, $\nu_3 = z_3\overline{w}_3$. Sum out of ν_1 variable, we have

$$\frac{1}{\pi^3(1-\nu_1)^4} \sum_{\alpha_2,\alpha_3=0}^{\infty} \frac{(\alpha_2+1)(\alpha_3+1)\Gamma(\alpha_2+\alpha_3+4)}{\Gamma(\alpha_2+\alpha_3+3)} \left(\frac{\nu_2}{1-\nu_1}\right)^{\alpha_2} \left(\frac{\nu_3}{1-\nu_1}\right)^{\alpha_3}.$$

Since $\Gamma(\alpha_2+\alpha_3+4) = (\alpha_2+\alpha_3+3)\Gamma(\alpha_2+\alpha_3+3)$, that

$$\frac{1}{\pi^3(1-\nu_1)^4} \sum_{\alpha_2,\alpha_3=0}^{\infty} (\alpha_2+1)(\alpha_3+1)(\alpha_2+\alpha_3+3) \left(\frac{\nu_2}{1-\nu_1}\right)^{\alpha_2} \left(\frac{\nu_3}{1-\nu_1}\right)^{\alpha_3}.$$

Finally using

$$\sum_{n,k=0}^{\infty} (n+1)(k+1)(n+k+3)x^n y^k = \frac{3-x-y-xy}{(1-x)^3(1-y)^3},$$

we obtain explicit formula for domain D_1. Similarly we can obtain Bergman kernel for domains

$$D_{p,q} = \{(z_1,z_2,z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^p < 1, \quad |z_1|^2 + |z_3|^q < 1\},$$

where p, q are positive reals numbers.

Now we consider domain D_2. As before by Proposition 4.2, we have

$$K_{D_2}(z,w) = \sum_{\alpha_1,\alpha_2,\alpha_3=0}^{\infty} \frac{3(\alpha_2+1)(\alpha_3+1)\Gamma(\frac{\alpha_1+1}{3}+\alpha_2+\alpha_3+3)}{2\pi^3\Gamma(\frac{\alpha_1+1}{3})\Gamma(\alpha_2+\alpha_3+3)} \nu_1^{\alpha_1}\nu_2^{\alpha_2}\nu_3^{\alpha_3},$$

where $\nu_1 = z_1\overline{w}_1$, $\nu_2 = z_2\overline{w}_2$, $\nu_3 = z_3\overline{w}_3$.

Using partial derivative notation we can write

$$\frac{3}{2\pi^3}\frac{\partial^2}{\partial\nu_2\partial\nu_3}\sum_{\alpha_1,\alpha_2,\alpha_3=0}^{\infty}\frac{\Gamma(\frac{\alpha_1+1}{3}+\alpha_2+\alpha_3+3)}{\Gamma(\frac{\alpha_1+1}{3})\Gamma(\alpha_2+\alpha_3+3)}\nu_1^{\alpha_1}\nu_2^{\alpha_2+1}\nu_3^{\alpha_3+1}.$$

Now we can express above sum in F_8 hypergeometric function terms separating α_1 modulo 3.

$$\frac{3}{2\pi^3}\frac{\partial^2}{\partial\nu_2\partial\nu_3}\left\{\nu_2\nu_3\sum_{j=1}^{3}\nu_1^{j-1}C_jF_8\left(3+\frac{j}{3},1,1,1;\frac{j}{3},3;\nu_1^3,\nu_2,\nu_3\right)\right\},$$

where $C_i = \frac{\Gamma(3+\frac{i}{3})}{2\Gamma(\frac{i}{3})}$ for $i = 1,2,3$. After some calculations using explicit formulas from lemma 3.2 we obtain desired result.

It is also possible in analogous way, compute Bergman kernel function for

$$\{(z_1,z_2,z_3)\in\mathbb{C}^3 : |z_1|^r+|z_2|^2<1, \quad |z_1|^2+|z_3|^2<1\},$$

for any rational number r and for

$$\{(z_1,z_2,z_3)\in\mathbb{C}^3 : |z_1|^4+|z_2|^4<1, \quad |z_1|^4+|z_3|^4<1\}.$$

Now we consider domain D_3. From Proposition 4.3, we have

$$K_{D_3}(z,w)=\sum_{\alpha_2,\alpha_3,\alpha_1=0}^{\infty}\frac{(\alpha_3+1)(\alpha_2+1)\Gamma(\alpha_1+\alpha_2+\alpha_3+4)}{\pi^3\nu_1\Gamma(\alpha_1+1)\Gamma(\alpha_2+\alpha_3+3)}\nu_1^{\alpha_1}\nu_2^{\alpha_2}\left(\frac{\nu_3}{\nu_1}\right)^{\alpha_3}.$$

Sum out of ν_1 variable, we have

$$\sum_{\alpha_2,\alpha_3=0}^{\infty}\frac{(\alpha_3+1)(\alpha_2+1)\Gamma(\alpha_2+\alpha_3+4)}{(1-\nu_1)^4\pi^3\nu_1\Gamma(\alpha_2+\alpha_3+3)}\left(\frac{\nu_2}{1-\nu_1}\right)^{\alpha_2}\left(\frac{\nu_3}{(1-\nu_1)\nu_1}\right)^{\alpha_3}.$$

Sum out of ν_2 variable, we get

$$\sum_{\alpha_3=0}^{\infty}\frac{(\alpha_3+1)(\alpha_3\nu_1+\alpha_3\nu_2-\alpha_3+3\nu_1+\nu_2-3)}{\pi^3\nu_1(1-\nu_1)^2(\nu_1+\nu_2-1)^3}\left(\frac{\nu_3}{(1-\nu_1)\nu_1}\right)^{\alpha_3}.$$

Finally sum out of α_3 variable, we obtain the desired result.

In order to prove formula for domain D_4 is sufficient to use Lemma 3.3.

5. Lu Qi-Kengs problem

A domain $\Omega\subset\mathbb{C}^n$ is called a Lu Qi-Keng domain if $K_\Omega(z,\overline{w})\neq 0$ for all $z,w\in\Omega$. Obviously, a biholomorphic image of a Lu Qi-Keng domain is a Lu Qi-Keng domain due to the rule of the Bergman kernel transformation between two biholomorphic equivalent domains. A Cartesian product of two Lu Qi-Keng domains is a Lu Qi-Keng domain. If $K_\Omega\neq \mathit{const}$ and Ω is the sum of an increasing sequence of Lu Qi-Keng domains Ω_m, then Ω is a Lu Qi-Keng domain due to the Ramadanov theorem and Hurwitz theorem. However, it is not always easy to determine whether or not a given domain is Lu Qi-Keng domain. In 1969, M. Skwarczynski [20] gave the first example that the

Bergman kernel on an annulus in the complex plane $\Omega = \{r < |z| < 1\}$ has zeros if $0 < r < e^{-2}$. Since the Bergman kernel for a bounded symmetric domain is a negative power of a certain polynomial, it has no zeros anywhere. In 1996, Boas [3] proved that bounded Lu Qi-Keng domains of holomorphy in $\mathbb{C}^n$ form a nowhere dense subset of all bounded domains of holomorphy. Since then the concrete forms of non-Lu Qi-Keng domains have been found in the various classes of domains in $\mathbb{C}^n$. The minimal ball [16] with $n \geq 4$ and the symmetrized polydisks [15] with $n \geq 3$ are not Lu Qi-Keng domains. For the complex ellipsoids, see [4, 23].

The explicit formula of the Bergman kernel function for the domain D enables us to investigate whether the Bergman kernel has zeros in $D \times D$ or not. We will call this kind of problem a Lu Qi-Keng problem. The motivation of this problem comes from the Riemann mapping theorem. If $n \geq 2$, then there is no analogue of Riemann mapping theorem in $\mathbb{C}^n$. Thus the following natural question arises: Are there canonical representatives of biholomorphic equivalence classes of domains? In higher dimensions, Bergman himself [1] introduced a representative domain to which a given domain may be mapped by representative coordinates. Let $K(z, w)$ be the Bergman kernel for a bounded domain $D \in \mathbb{C}^n$, and define

$$T_{i\bar{j}}(z) = (g_{ij}) := \frac{\partial^2}{\partial \zeta_i \partial \overline{\zeta}_j} \log K(\zeta, \zeta)_{|\zeta = z}.$$

Then its converse is $T^{\bar{j}i}(z) = (g_{ji}^{-1})$. Hence the local representative coordinate $f(z) = (f_1, \ldots, f_n)$ based at the point z_0 is given by

$$f_i(z) = \sum_{i=1}^{n} T^{\bar{j}i}(z_0) \frac{\partial}{\partial \overline{\zeta}_j} \log \frac{K(z, \zeta)}{K(\zeta, \zeta)} \Big|_{\zeta = z_0}$$

for $i = 1, \ldots, n$. In 1966, Lu Qi-Keng [12] observed the following phenomenon: It is necessary that the Bergman kernel $K(z, w)$ has no zeros in order to define the Bergman representative coordinates.

Corollary 5.1. *The domains D_1 and D_3 are Lu Qi-Kengs domains.*

Proof. Suppose that the Bergman function for D_1 is zero at (z, w). It means that there exist $z = (z_1, z_2, z_3) \in D_1$ and $w = (w_1, w_2, w_3) \in D_1$, such that $K_{D_1}(z, w) = 0$. Then by proof of Theorem 2.1, we have

$$K_{D_1}(z, w) = \frac{1}{\pi^3 (1 - \nu_1)^4} \frac{3 - x - y - xy}{(1 - x)^3 (1 - y)^3},$$

where $x = \frac{\nu_2}{1 - \nu_1}$, $y = \frac{\nu_3}{1 - \nu_1}$. Since

$$|z_1|^2 + |z_2|^2 < 1, \quad |z_1|^2 + |z_3|^2 < 1, \quad |w_1|^2 + |w_2|^2 < 1, \quad |w_1|^2 + |w_3|^2 < 1,$$

then

$$(|z_1|^2 + |z_2|^2)(|w_1|^2 + |w_2|^2) < 1 \quad \text{and} \quad (|z_1|^2 + |z_3|^2)(|w_1|^2 + |w_3|^2) < 1.$$

Hence from the Cauchy-Schwarz inequality, readily follows

$$|\nu_1| + |\nu_2| < 1, \quad |\nu_1| + |\nu_3| < 1.$$

Therefore $|\nu_2| < |1 - \nu_1|$ and $|\nu_3| < |1 - \nu_1|$. By $K_{D_1}(z, w) = 0$, we have $3 = x + y + xy$. On the other hand side

$$3 = |x + y + xy| \leq |x| + |y| + |xy| < 3$$

which is a contradiction.

Similarly as for D_1, we have

$$K_{D_3}(z, w) = \frac{1}{\pi^3 \nu_1 (1 - \nu_1)^4} \frac{3 - x - y - xy}{(1 - x)^3 (1 - y)^3},$$

where $x = \frac{\nu_2}{1-\nu_1}$, $y = \frac{\nu_3}{(1-\nu_1)\nu_1}$. Since

$$|z_1|^2 + |z_2|^2 < 1, \; |z_1|^4 + |z_3|^2 < |z_1|^2, \quad |w_1|^2 + |w_2|^2 < 1, \; |w_1|^4 + |w_3|^2 < |w_1|^2,$$

then

$$(|z_1|^2 + |z_2|^2)(|w_1|^2 + |w_2|^2) < 1 \quad \text{and} \quad (|z_1|^4 + |z_3|^2)(|w_1|^4 + |w_3|^2) < |z_1 w_1|^2.$$

As a consequence of the Cauchy-Schwarz inequality, we have

$$|\nu_1| + |\nu_2| < 1, \quad |\nu_1|^2 + |\nu_3| < |\nu_1|.$$

Therefore $|\nu_2| < |1 - \nu_1|$ and $|\nu_3| < |(1 - \nu_1)\nu_1|$. This provide the same contradiction as in above case. □

It is well known fact, that Bergman kernel function for unit ball in $\mathbb{C}^2$ is zero free. In view of the above lemma, we can ask the following question: Is there a relationship between the existence of zeros of the Bergman kernel function for domains

$$\{z \in \mathbb{C}^2 \colon |z_1|^p + |z_2|^q < 1\} \quad \text{and} \quad \{z \in \mathbb{C}^3 \colon |z_1|^p + |z_2|^q < 1, \; |z_1|^p + |z_3|^q < 1\}\,?$$

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