

FLOCKING AND PATTERN MOTION IN A MODIFIED CUCKER-SMALE MODEL

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ABSTRACT. Self-organizing systems arise very naturally in artificial intelligence, and in physical, biological and social sciences. In this paper, we modify the classic Cucker-Smale model at both microscopic and macroscopic levels by taking the target motion pattern driving forces into consideration. Such target motion pattern driving force functions are properly defined for the line-shaped motion pattern and the ball-shaped motion pattern. For the modified Cucker-Smale model with the prescribed line-shaped motion pattern, we have analytically shown that there is a flocking pattern with an asymptotic flocking velocity. This is illustrated by numerical simulations using both symmetric and non-symmetric pairwise influence functions. For the modified Cucker-Smale model with the prescribed ball-shaped motion pattern, our simulations suggest that the solution also converges to the prescribed motion pattern.

1. Introduction

Recently the study on emerging collective behaviors in multi-agent interactions has gained increasing interest in biology, ecology, robotics and control theory, as well as sociology and economics ([2], [3], [4], [6], [7], [9], [15], [16], [17]). Several mathematical models have been proposed ([1, 8, 9, 10]) to characterize the flocking patterns. Among others, the celebrated Cucker-Smale (CS) model [8] provides a framework to explain the self-organizing behavior in various complex systems. One criterion guaranteeing flocking is that the slowly decaying influence function should have a diverging tail [14].

One essential property of flocking is that the agents all have the same velocity. However, there is no description of the motion patterns or the shape of the motion. In this work, we incorporate a target motion pattern driving force function (denoted by F) into the classic CS model to obtain a modified CS model. Such specific force functions are given for the cases where the target motion pattern is line-shaped and ball-shaped, respectively. For the former

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case, we analytically prove that flocking does exist. For the latter case, numerical simulations suggest the flocking also exists, though we do not have an analytical proof.

The rest of this paper is organized as follows. In Section 2, we introduce a modified Cucker-Smale model with two different target motion patterns driving force functions. In Section 3, we analytically prove that there is a flocking pattern for the modified Cucker-Smale model with the prescribed line-shaped motion pattern. Numerical simulations are presented in Section 4 to illustrate our results.

2. A modified CS model

Suppose a self-organizing group has N agents. Each agent i can be characterized by its position $\mathbf{x}_i \in \mathbb{R}^d$ and velocity $\mathbf{v}_i \in \mathbb{R}^d$, where $d \geq 1$ is an integer. Then the standard CS model reads as

$$(1) \quad \frac{d}{dt}\mathbf{x}_i(t) = \mathbf{v}_i(t), \quad \frac{d}{dt}\mathbf{v}_i(t) = \alpha \sum_{j \neq i} a_{ij}(\mathbf{x})(\mathbf{v}_j - \mathbf{v}_i),$$

where α measures the interaction strength and $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N)$. The function a_{ij} in [8] takes the following form

$$(2) \quad a_{ij}(\mathbf{x}) = I(|\mathbf{x}_i - \mathbf{x}_j|)/N,$$

which is used to quantify the pairwise influence of agent j on the alignment of agent i . The positive influence function I is strictly monotonically decreasing with a prototype given by $I(r) = (1 + r^2)^{-\beta}$ for $r \geq 0$, where β is a constant. More pairwise influence functions are considered in literature [14].

Now, we introduce a patterns driving force, say F , to achieve the long-term motion pattern of the self-organizing group. To this end, the self-organizing group should “learn” to converge to the final motion patterns. Thus the motion process should carry the information of the final motion patterns. So we slightly modify the classic CS model by adding the final motion pattern driving force function. Then the modified CS model gives as follows:

$$(3) \quad \frac{d}{dt}\mathbf{x}_i(t) = \mathbf{v}_i(t), \quad \frac{d}{dt}\mathbf{v}_i(t) = \alpha \sum_{j \neq i} a_{ij}(\mathbf{x})(\mathbf{v}_j - \mathbf{v}_i) + F(\mathbf{x}_i).$$

As above, if we incorporate the driving force F into the self-organized systems, then it determine the final pattern. In fact, we can show that the N -agent self-organizing system (3) converges to the line-shaped flocking pattern if driving force F is properly designed by the final line-shaped pattern.

2.1. Line-shaped target motion pattern

In this subsection, we introduce a reasonable F so that all the agents in the self-organizing system (3) converge to a line-shaped flock. The function F

takes the form

$$(4) \quad F(\mathbf{x}_i(t)) = \gamma[\langle \mathbf{x}_i(t) - \bar{\mathbf{x}}(t), \mathbf{l} \rangle \cdot \mathbf{l} - (\mathbf{x}_i(t) - \bar{\mathbf{x}}(t))], \quad \bar{\mathbf{x}}(t) = \frac{1}{N} \sum_{i=1}^N \mathbf{x}_i(t),$$

where \mathbf{l} is a unit constant vector denoting the desired final motion direction, γ is a positive constant measuring the force strength, $\langle \cdot, \cdot \rangle$ denotes the inner product in \mathbb{R}^d . $\bar{\mathbf{x}}(t)$ formulates the mass center of the self-organized system at time t and $\langle \mathbf{x}_i(t) - \bar{\mathbf{x}}(t), \mathbf{l} \rangle \cdot \mathbf{l}$ denotes the projection of the vector $\mathbf{x}_i(t) - \bar{\mathbf{x}}(t)$ on the direction \mathbf{l} . Moreover, $F(\mathbf{x}_i(t))$ denotes the inner attractive force that drives the agent i to move towards the prescribed line, which passes through the center $\bar{\mathbf{x}}$ and is parallel with \mathbf{l} . Particularly, F has the properties:

$$\begin{aligned} \langle F(\mathbf{x}_i(t)), \mathbf{l} \rangle &= 0 \quad \text{for all } t \text{ and } i, \\ \sum_{i=1}^N F(\mathbf{x}_i(t)) &= \gamma \sum_{i=1}^N [\langle \mathbf{x}_i(t) - \bar{\mathbf{x}}(t), \mathbf{l} \rangle \cdot \mathbf{l} - (\mathbf{x}_i(t) - \bar{\mathbf{x}}(t))] = \mathbf{0}. \end{aligned}$$

So if we let $\bar{\mathbf{v}}(t) = \frac{1}{N} \sum_{i=1}^N \mathbf{v}_i(t)$, then

$$(5) \quad \frac{d\bar{\mathbf{v}}(t)}{dt} = \frac{1}{N} \sum_{i=1}^N \left(\alpha \sum_{j \neq i} a_{ij}(\mathbf{x})(\mathbf{v}_j - \mathbf{v}_i) + F(\mathbf{x}_i) \right) = \mathbf{0}.$$

2.2. Ball-shaped target motion pattern

In this subsection, we suppose the prescribed motion pattern is ball-shaped. We propose the following form for the function F :

$$(6) \quad \tilde{F}(\mathbf{x}_i(t)) = \frac{\mathbf{w}_i(t)}{1 + \sum_{i=1}^N |\mathbf{w}_i(t)|} \text{ and } \mathbf{w}_i(t) = \left(1 - \frac{R}{|\mathbf{x}_i(t) - \bar{\mathbf{x}}(t)|}\right)(\bar{\mathbf{x}}(t) - \mathbf{x}_i(t),$$

where R is the radius of the target ball-shaped flock pattern and $\bar{\mathbf{x}}(t)$ denotes the mass center of all agents at t . If $|\mathbf{x}_i(t) - \bar{\mathbf{x}}(t)| < R$, then $F(\mathbf{x}_i(t))$ pushes the agent i out of the circle; and if $|\mathbf{x}_i(t) - \bar{\mathbf{x}}(t)| > R$, then $F(\mathbf{x}_i(t))$ pulls the agent i back into the circle. In general, $\sum_{i=1}^N \tilde{F}(\mathbf{x}_i(t)) \neq \mathbf{0}$.

3. Line-shaped flocking pattern

In this section, we analytically show that all solutions of the self-organizing system (3) with F given by (4) converge to a flock, and the final flocking position pattern is the prescribed line \mathbf{l} .

Theorem 3.1. *If the influence function I satisfies $\int_0^\infty I(r)dr = \infty$, then the solution of the self-organizing system (3) with F given by (4) converges to a flock. Furthermore, the asymptotic flocking velocity satisfies $\bar{\mathbf{v}}_\infty = \frac{1}{N} \sum_{i=1}^N \mathbf{v}_i(0)$. In particular, the final flocking pattern is a line parallelled to \mathbf{l} .*

Proof. Let

$$\bar{\mathbf{x}}(t) = \frac{1}{N} \sum_{i=1}^N \mathbf{x}_i(t) \quad \text{and} \quad \bar{\mathbf{v}}(t) = \frac{1}{N} \sum_{i=1}^N \mathbf{v}_i(t).$$

Noting the facts that $\sum_{i=1}^N F(\mathbf{x}_i) = \mathbf{0}$ and $a_{ij} = a_{ji}$, then

$$\frac{d}{dt} \bar{\mathbf{v}}(t) = \frac{1}{N} \sum_{i=1}^N \frac{d}{dt} \mathbf{v}_i(t) = \frac{1}{N} \sum_{i=1}^N [\alpha \sum_{j \neq i} a_{ij}(\mathbf{x})(\mathbf{v}_j - \mathbf{v}_i) + F(\mathbf{x}_i)] = \mathbf{0}.$$

Thus $\bar{\mathbf{v}}(t) \equiv \bar{\mathbf{v}}(0) = \frac{1}{N} \sum_{i=1}^N \mathbf{v}_i(0)$ for all t .

Furthermore,

$$\begin{aligned} \frac{d}{dt} |\langle \mathbf{x}_i - \bar{\mathbf{x}}, \mathbf{1} \rangle|^2 - \frac{d}{dt} |\mathbf{x}_i - \bar{\mathbf{x}}|^2 &= 2 \langle \langle \mathbf{x}_i - \bar{\mathbf{x}}, \mathbf{1} \rangle \cdot \mathbf{1} - (\mathbf{x}_i - \bar{\mathbf{x}}), (\mathbf{v}_i - \bar{\mathbf{v}}) \rangle \\ &= \frac{2}{\gamma} \langle F(\mathbf{x}_i), \mathbf{v}_i - \bar{\mathbf{v}} \rangle. \end{aligned}$$

As I is monotonically decreasing and from the equality (5), we can obtain that

$$\begin{aligned} & \frac{d}{dt} \sum_{i=1}^N |\mathbf{v}_i - \bar{\mathbf{v}}|^2 \\ &= 2 \sum_{i=1}^N \left\langle \frac{d\mathbf{v}_i}{dt} - \frac{d\bar{\mathbf{v}}}{dt}, \mathbf{v}_i - \bar{\mathbf{v}} \right\rangle \\ &= 2 \sum_{i=1}^N \left\langle \alpha \sum_{j=1}^N a_{ij}(\mathbf{x})(\mathbf{v}_j - \mathbf{v}_i) + F(\mathbf{x}_i), \mathbf{v}_i - \bar{\mathbf{v}} \right\rangle \\ &= \frac{\alpha}{N} \sum_{i=1}^N \sum_{j=1}^N \left(\left\langle I(|\mathbf{x}_j - \mathbf{x}_i|)(\mathbf{v}_j - \mathbf{v}_i), \mathbf{v}_i - \bar{\mathbf{v}} \right\rangle \right. \\ & \quad \left. + \left\langle I(|\mathbf{x}_j - \mathbf{x}_i|)(\mathbf{v}_i - \mathbf{v}_j), \mathbf{v}_j - \bar{\mathbf{v}} \right\rangle \right) + 2 \sum_{i=1}^N \langle F(\mathbf{x}_i), \mathbf{v}_i - \bar{\mathbf{v}} \rangle \\ &= -\frac{\alpha}{N} \sum_{i=1}^N \sum_{j=1}^N \left\langle I(|\mathbf{x}_j - \mathbf{x}_i|)(\mathbf{v}_i - \mathbf{v}_j), \mathbf{v}_i - \mathbf{v}_j \right\rangle + 2 \sum_{i=1}^N \langle F(\mathbf{x}_i), \mathbf{v}_i - \bar{\mathbf{v}} \rangle \\ &= -2\alpha \sum_{i=1}^N \left\langle I(d_X)(\mathbf{v}_i - \bar{\mathbf{v}}), \mathbf{v}_i - \bar{\mathbf{v}} \right\rangle + 2 \sum_{i=1}^N \langle F(\mathbf{x}_i), \mathbf{v}_i - \bar{\mathbf{v}} \rangle \\ &= -2\alpha \sum_{i=1}^N I(d_X) |\mathbf{v}_i - \bar{\mathbf{v}}|^2 + 2 \sum_{i=1}^N \langle F(\mathbf{x}_i), \mathbf{v}_i - \bar{\mathbf{v}} \rangle \\ &= -2\alpha \sum_{i=1}^N I(d_X) |\mathbf{v}_i - \bar{\mathbf{v}}|^2 + \gamma \sum_{i=1}^N \left(\frac{d}{dt} |\langle \mathbf{x}_i - \bar{\mathbf{x}}, \mathbf{1} \rangle|^2 - \frac{d}{dt} |\mathbf{x}_i - \bar{\mathbf{x}}|^2 \right). \end{aligned}$$

Therefore,

$$(7) \quad \frac{d}{dt} \sum_{i=1}^N |\mathbf{v}_i - \bar{\mathbf{v}}|^2 + \gamma \sum_{i=1}^N \frac{d}{dt} (|\mathbf{x}_i - \bar{\mathbf{x}}|^2 - |\langle \mathbf{x}_i - \bar{\mathbf{x}}, \mathbf{l} \rangle|^2) \leq -2\alpha \sum_{i=1}^N I(d_X) |\mathbf{v}_i - \bar{\mathbf{v}}|^2,$$

where

$$d_X(t) := \max_{1 \leq i < j \leq N} \{|\mathbf{x}_j(t) - \mathbf{x}_i(t)|\}.$$

Since $I(r)$ is non-negative and from the inequality (7), we can obtain

$$\frac{d}{dt} \sum_{i=1}^N |\mathbf{v}_i - \bar{\mathbf{v}}|^2 + \gamma \sum_{i=1}^N \frac{d}{dt} (|\mathbf{x}_i - \bar{\mathbf{x}}|^2 - |\langle \mathbf{x}_i - \bar{\mathbf{x}}, \mathbf{l} \rangle|^2) \leq 0.$$

Next, we will finish the proof by two steps:

STEP 1: Prove $\lim_{t \rightarrow \infty} \mathbf{v}_i(t) = \bar{\mathbf{v}}(0)$ for all i .

Firstly, we simplify the above formulations by denoting

$$s_i^2 := |\mathbf{x}_i - \bar{\mathbf{x}}|^2 - |\langle \mathbf{x}_i - \bar{\mathbf{x}}, \mathbf{l} \rangle|^2, \quad (s_i \geq 0),$$

$$s^2 := \gamma \sum_{i=1}^N s_i^2, \quad (s \geq 0),$$

$$\mathbf{v}^* := (\mathbf{v}_1 - \bar{\mathbf{v}}, \mathbf{v}_2 - \bar{\mathbf{v}}, \dots, \mathbf{v}_N - \bar{\mathbf{v}}).$$

The norm of \mathbf{v}^* is defined as usual

$$|\mathbf{v}^*| := \left(\sum_{i=1}^N |\mathbf{v}_i - \bar{\mathbf{v}}|^2 \right)^{\frac{1}{2}}.$$

Then the inequality (7) is

$$\frac{d|\mathbf{v}^*|^2}{dt} + \frac{d(s^2)}{dt} \leq 0.$$

Thus $|\mathbf{v}^*|^2 + s^2$ is non-increasing so that

$$|\mathbf{v}^*(t)|^2 + s^2(t) \leq |\mathbf{v}^*(0)|^2 + s^2(0).$$

Since $|\mathbf{v}^*(t)|^2 \geq 0$, we achieve that

$$s^2(t) \leq |\mathbf{v}^*(0)|^2 + s^2(0) := M^* \quad (t \geq 0).$$

This implies

$$\gamma \sum_{i=1}^N s_i^2 \leq M^*.$$

Applying the arithmetic-geometric inequality, we have

$$\frac{\gamma}{N} \left(\sum_{i=1}^N s_i \right)^2 \leq \gamma \sum_{i=1}^N s_i^2.$$

Based on the above two inequalities, it's easy to see that

$$(8) \quad s_i \leq \sum_{i=1}^N s_i \leq \sqrt{\frac{N}{\gamma}} M^* := M.$$

In fact, the inequality (8) indicates that if we project the whole system into the plane which is perpendicular to the prescribed direction, then the projective system is bounded. So we only need to consider the one-dimension space which is parallel with \mathbf{l} . If in this one-dimension space, the projective system is bounded, then the original system we consider is bounded. To this end, we deduce a new system by a projection process. (inner-product both sides of the original equation (3) by \mathbf{l})

$$\frac{d\tilde{\mathbf{x}}_i(t)}{dt} = \tilde{\mathbf{v}}_i(t), \quad \frac{d\tilde{\mathbf{v}}_i(t)}{dt} = \alpha \sum_{j \neq i} a_{ij}(\mathbf{x})(\tilde{\mathbf{v}}_j - \tilde{\mathbf{v}}_i),$$

where $\tilde{\mathbf{x}}_i = (\mathbf{x}_i, l)$, $\tilde{\mathbf{v}}_i = (\mathbf{v}_i, \mathbf{l})$, $i = 1, 2, \dots, N$. For the new system, similarly, we introduce

$$d_X := \max_{1 \leq i < j \leq N} |\mathbf{x}_j - \mathbf{x}_i|, \quad d_{\tilde{X}} := \max_{1 \leq i < j \leq N} |\tilde{\mathbf{x}}_j - \tilde{\mathbf{x}}_i|.$$

From the equation (8), we have

$$d_X \leq \sqrt{d_{\tilde{X}}^2 + M^2}.$$

Thus,

$$\begin{aligned} \frac{d}{dt} \sum_{i=1}^N |\tilde{\mathbf{v}}_i - \bar{\mathbf{v}}|^2 &= 2 \sum_{i=1}^N \left\langle \frac{d}{dt} (\tilde{\mathbf{v}} - \bar{\mathbf{v}}), \tilde{\mathbf{v}}_i - \bar{\mathbf{v}} \right\rangle \\ &= -\frac{\alpha}{N} \sum_{i=1}^N \sum_{j=1}^N \left\langle I(|\mathbf{x}_j - \mathbf{x}_i|) (\tilde{\mathbf{v}}_i - \tilde{\mathbf{v}}_j), \tilde{\mathbf{v}}_i - \tilde{\mathbf{v}}_j \right\rangle \\ &= -2\alpha \sum_{i=1}^N \left\langle I(|\mathbf{x}_j - \mathbf{x}_i|) (\tilde{\mathbf{v}}_i - \bar{\mathbf{v}}), \tilde{\mathbf{v}}_i - \bar{\mathbf{v}} \right\rangle \\ &\leq -2\alpha \sum_{i=1}^N I(d_X) |\tilde{\mathbf{v}}_i - \bar{\mathbf{v}}|^2 \\ &\leq -2\alpha \sum_{i=1}^N I(\sqrt{d_{\tilde{X}}^2 + M^2}) |\tilde{\mathbf{v}}_i - \bar{\mathbf{v}}|^2. \end{aligned}$$

Considering the function $G(r) := I(\sqrt{r^2 + M^2})$ for $r > 0$, we can obtain

$$\frac{d}{dt} \sum_{i=1}^N |\tilde{\mathbf{v}}_i - \bar{\mathbf{v}}|^2 \leq -2\alpha \sum_{i=1}^N G(d_{\tilde{X}}) |\tilde{\mathbf{v}}_i - \bar{\mathbf{v}}|^2.$$

In [11], by using Lyapunov functional method, we see that if

$$(9) \quad \int_0^\infty G(r)dr = \infty,$$

then there exists $d_* > 0$ such that for all $t > 0$,

$$(10) \quad d_{\bar{X}}(t) \leq d_*.$$

Following the fact that $I(r + M) < G(r) < I(r)$, we see $\int_0^\infty I(r)dr = \infty$ is equivalent to (9). At this stage, let us look back to equations (7) and (10),

$$(11) \quad \frac{d|\mathbf{v}^*|^2}{dt} + \frac{d(s^2)}{dt} \leq -2\alpha I(d_*)|\mathbf{v}^*|^2.$$

Integrating both sides of the above inequality from 0 to ∞ , we find the left-hand side is bounded, so the right-hand side $\int_0^\infty |\mathbf{v}^*|^2 dt$ must be bounded as well. This means

$$(12) \quad \lim_{t \rightarrow \infty} |\mathbf{v}^*|^2 = 0,$$

since $\frac{d|\mathbf{v}^*|^2}{dt}$ is bounded.

Thus, for all i ,

$$(13) \quad \lim_{t \rightarrow \infty} \mathbf{v}_i(t) = \bar{\mathbf{v}}(0).$$

STEP 2: Prove $\lim_{t \rightarrow \infty} [(\mathbf{x}_i - \bar{\mathbf{x}}, \mathbf{l}) \cdot \mathbf{l} - (\mathbf{x}_i - \bar{\mathbf{x}})] = 0$.

Noting equation (11) and $|\mathbf{v}^*|^2 + s^2$ is monotonically decreasing, we see that $\lim_{t \rightarrow \infty} (|\mathbf{v}^*|^2 + s^2)$ exists. Thus

$$\lim_{t \rightarrow \infty} (|\mathbf{v}^*|^2 + s^2) = \lim_{t \rightarrow \infty} s^2.$$

We claim that $\lim_{t \rightarrow \infty} s(t) = 0$. In fact, if there is a constant δ such that $\lim_{t \rightarrow \infty} s(t) = \delta > 0$, then it follows from the definition of $s(t)$ and $s_i(t)$ that there is at least one $s_i(t)$ satisfying $\overline{\lim}_{t \rightarrow \infty} s_i = \delta_i^* > 0$ for some δ_i^* . This implies two cases: one is $\underline{\lim}_{t \rightarrow \infty} s_i = \delta_i > 0$ for some δ_i , the other is $\underline{\lim}_{t \rightarrow \infty} s_i = 0$.

For the first case, it follows from (13) that there is a $t_* > 0$ such that whenever $t \geq t_*$,

$$|\alpha \sum_{j \neq i} a_{ij}(\mathbf{x})(\mathbf{v}_j - \mathbf{v}_i)| < \frac{1}{3} \gamma \delta_i^*$$

and

$$\delta_i - s_i \leq \frac{1}{3} \delta_i^*.$$

Thus

$$\left| \frac{d\mathbf{v}_i}{dt} \right| > \frac{1}{3} \gamma \delta_i^* \quad \text{for all } t \geq t_*.$$

This is impossible and contrary to equation (13).

For the second case, similarly, by using the classical mathematical analysis, we see that $\underline{\lim}_{t \rightarrow \infty} s_i = 0 \neq \overline{\lim}_{t \rightarrow \infty} s_i$ is also impossible.

Summarizing from above two cases, we see that $\lim_{t \rightarrow \infty} s(t) = 0$. This leads to

$$\lim_{t \rightarrow \infty} [(\mathbf{x}_i - \bar{\mathbf{x}}, \mathbf{l}) \cdot \mathbf{l} - (\mathbf{x}_i - \bar{\mathbf{x}})] = 0.$$

Thus there is a constant c_i such that

$$\lim_{t \rightarrow \infty} (\mathbf{x}_i - \bar{\mathbf{x}}) = c_i \mathbf{l} \text{ for all } i.$$

Based on above two steps, we conclude that the solution of the self-organizing system (3) and (4) converges to a flock if $\int_0^\infty I(r)dr = \infty$ holds. Moreover, the asymptotic flocking velocity satisfies $\bar{\mathbf{v}}_\infty = \frac{1}{N} \sum_{i=1}^N \mathbf{v}_i(0)$. In particular, the final flocking pattern is a line paralleled to \mathbf{l} . This completes the proof of Theorem 3.1. \square

Remark 3.1. From Theorem 3.1, we see that the driving force F given by (4) does not change the final velocity of the flocking, which is also the average of the initial velocity, but changes the final flocking pattern.

Remark 3.2. We can also formulate the kinetic model with a target motion pattern driving force function. To this end, as in [5, 12], we proceed the model through a mean-field limit. The key idea of the mean-field limit is to derive a single evolutionary equation for f^N , and the empirical distribution function is defined as

$$f^N(\mathbf{x}, \mathbf{v}, t) = \frac{1}{N} \sum_{i=1}^N \delta(\mathbf{x} - \mathbf{x}_i(t)) \delta(\mathbf{v} - \mathbf{v}_i(t)).$$

The density distribution function is then

$$\rho^N(\mathbf{x}, t) = \int_{\mathbb{R}^d} f^N(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}.$$

Thus the model of line-shaped motion pattern at the mesoscopic level takes the form of

$$\partial_t f^N + \mathbf{v} \cdot \nabla_x f^N = -\nabla_v \cdot (E(\mathbf{x}, \mathbf{v}) f^N),$$

where

$$\begin{aligned} E(\mathbf{x}, \mathbf{v}) &= \alpha \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} I(\mathbf{x} - \mathbf{y})(\mathbf{v} - \mathbf{w}) f^N(\mathbf{y}, \mathbf{w}) d\mathbf{y} d\mathbf{w} \\ &\quad + \gamma \int_{\mathbb{R}^d} [(\mathbf{x} - \mathbf{y}, \mathbf{l}) \mathbf{l} - (\mathbf{x} - \mathbf{y})] \rho^N(\mathbf{y}, t) d\mathbf{y}. \end{aligned}$$

Similarly, the model of ball-shaped formation motion at the mesoscopic level reads as

$$\partial_t f^N + \mathbf{v} \cdot \nabla_x f^N = -\nabla_v \cdot (\tilde{E}(\mathbf{x}, \mathbf{v}) f^N),$$

where

$$\tilde{E}(\mathbf{x}, \mathbf{v}) = \alpha \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} I(\mathbf{x} - \mathbf{y})(\mathbf{v} - \mathbf{w}) f^N(\mathbf{y}, \mathbf{w}) d\mathbf{y} d\mathbf{w}$$

$$+ \frac{(1 - \frac{R}{\int_{\mathbb{R}^d} |\mathbf{x}-\mathbf{y}| \rho^N(\mathbf{y},t) d\mathbf{y}}) \int_{\mathbb{R}^d} (\mathbf{y} - \mathbf{x}) \rho^N(\mathbf{y},t) d\mathbf{y}}{1 + \int_{\mathbb{R}^d} |1 - \frac{R}{\int_{\mathbb{R}^d} |\mathbf{x}-\mathbf{y}| \rho^N(\mathbf{y},t) d\mathbf{y}}| \int_{\mathbb{R}^d} |\mathbf{y} - \mathbf{x}| \rho^N(\mathbf{y},t) d\mathbf{y} d\mathbf{x}}.$$

4. Simulations

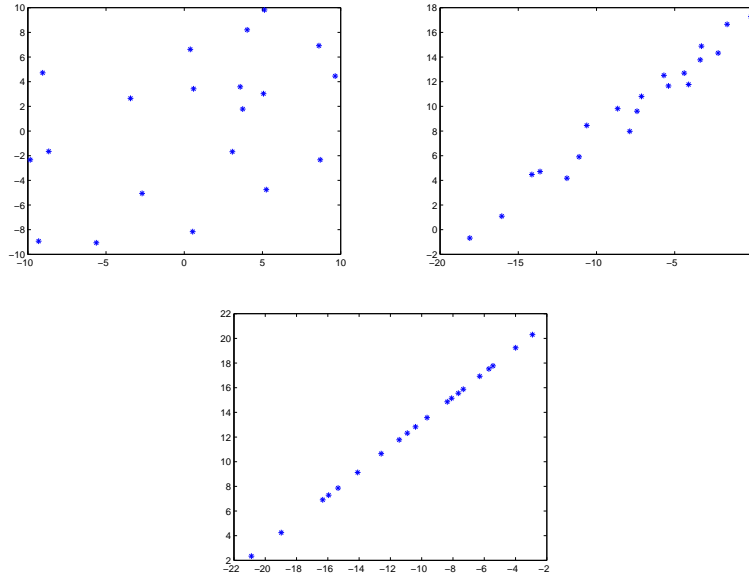


FIGURE 1. Line-shaped flocking pattern with symmetric pairwise influence functions. The left panel denotes the initial positions of the agents, the middle is the position pattern at half runtime and the right one is the final positions with the designed flocking pattern. The values of parameters are: $d = 2$, $\beta = 1/10$, $\mathbf{1} = \frac{1}{\sqrt{2}}(1, 1)$, $\alpha = 1$, $N = 20$.

In this section, by using Matlab, we numerically explore the relationships among the flocking patterns, the pairwise influence function a_{ij} and the target motion pattern driving force function F . To this end, we simulate our models with both symmetric and non-symmetric pairwise influence functions a_{ij} . Simulations are also performed for both the line-shaped and the ball-shaped target motion patterns. We let $I(r) = (1 + r^2)^{-\beta}$ and consider models (3) with symmetric pairwise influence functions

$$a_{ij}(\mathbf{x}) = I(|\mathbf{x}_i - \mathbf{x}_j|)/N,$$

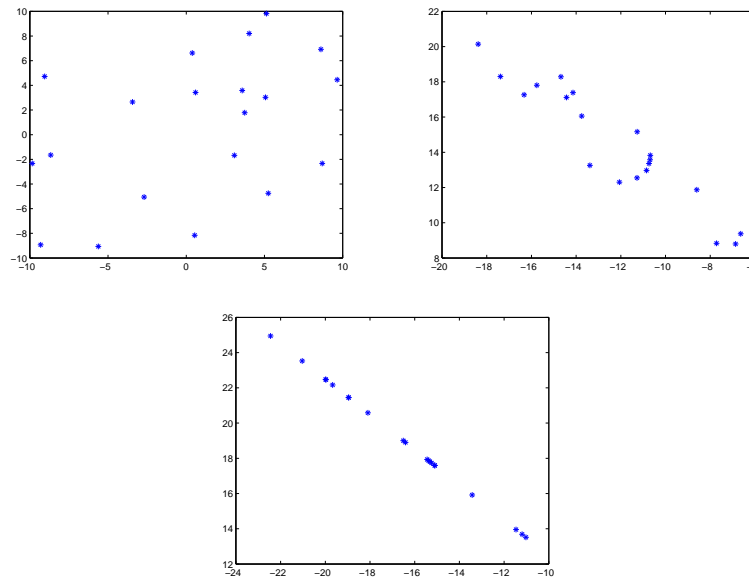


FIGURE 2. Line-shaped flocking pattern with nonsymmetric pairwise influence. The left sub-figure denotes the initial positions of the agents, the middle is the position pattern at half runtime and the right is the final position pattern. The values of parameters is given by: $d = 2$, $\beta = 1/10$, $\mathbf{l} = \frac{1}{\sqrt{2}}(-1, 1)$, $\alpha = 1$, $N = 20$.

and the line-shaped target motion pattern driving force function

$$(14) \quad F(x_i) = \frac{\mathbf{w}_i}{1 + \sum_{i=1}^N |\mathbf{w}_i|}, \quad \mathbf{w}_i = \langle \mathbf{x}_i - \bar{\mathbf{x}}, \mathbf{l} \rangle \mathbf{l} - (\mathbf{x}_i - \bar{\mathbf{x}}).$$

We take $d = 2$, $\beta = \frac{1}{10}$, $\mathbf{l} = \frac{1}{\sqrt{2}}(1, 1)$, $\alpha = 1$, $N = 20$. Simulation results presented in Fig. 1 show that the solution converges to a flock with the prescribed line shape.

For the non-symmetric pairwise influence functions ([14])

$$a_{ij}(\mathbf{x}) = I(|\mathbf{x}_i - \mathbf{x}_j|) / \sum_k I(|\mathbf{x}_i - \mathbf{x}_k|)$$

and F given by (14), we take the same initial value as symmetric case and set $d = 2$, $\beta = \frac{1}{10}$, $\mathbf{l} = \frac{1}{\sqrt{2}}(-1, 1)$, $\alpha = 1$, $N = 20$. Then our simulations (Fig. 2) also confirm that the solution of self-organizing system (3) converges to a flock with the prescribed motion pattern.

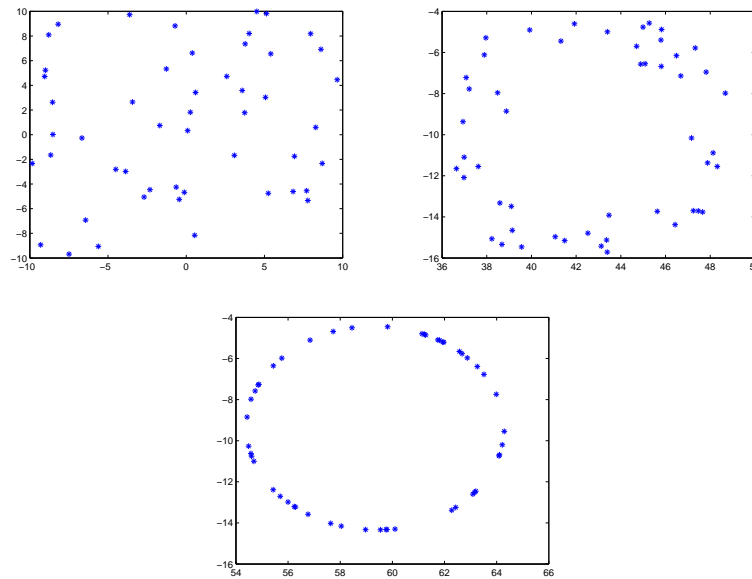


FIGURE 3. Circle-shaped flocking pattern with symmetric pairwise influence. The left sub-figure denotes the initial positions of the agents, the middle is the position pattern at half runtime and the right one is the final flocking pattern. The values of parameters is given by: $d = 2$, $\beta=1/6$, $R = 5$, $\alpha = 1$, $N = 45$.

Next we consider the symmetric pairwise influence functions

$$a_{ij}(\mathbf{x}) = I(|\mathbf{x}_i - \mathbf{x}_j|)/N,$$

and

$$(15) \quad \tilde{F}(x_i) = \frac{\mathbf{w}_i}{1 + \sum_{i=1}^N |\mathbf{w}_i|}, \quad \mathbf{w}_i = \left(1 - \frac{R}{|\mathbf{x}_i - \bar{\mathbf{x}}|}\right)(\bar{\mathbf{x}} - \mathbf{x}_i).$$

We take $d = 2$, $\beta = \frac{1}{6}$, $R = 5$, $\alpha = 1$, $N = 45$. Then our simulations (Fig. 3) suggest that the solution of self-organizing system (3) converges to a flock with the prescribed circle-shaped pattern.

With the same initial values as above, we take $d = 2$, $\beta = \frac{1}{10}$, $R = 4$, $\alpha = 1$, $N = 45$ and consider the non-symmetric pairwise influence

$$a_{ij}(\mathbf{x}) = I(|\mathbf{x}_i - \mathbf{x}_j|) / \sum_k I(|\mathbf{x}_i - \mathbf{x}_k|)$$

and F . Again, the solution of the self-organizing system (3) is shown to converge to a flock with the prescribed circle-shaped pattern as shown in Fig. 4.

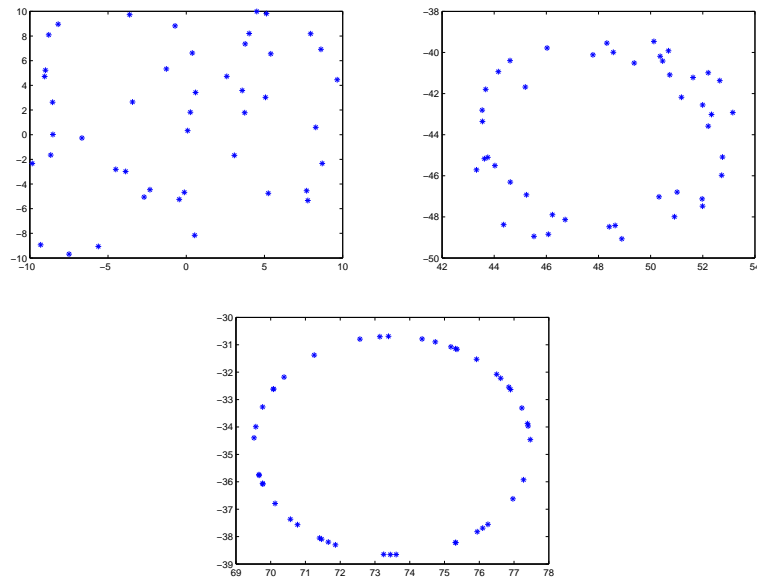


FIGURE 4. Circle-shaped flocking pattern with nonsymmetric pairwise influence. The left sub-figure denotes the initial positions of the agents, the middle is the position pattern at half runtime and the right one is the final position with the designed flocking pattern. The values of parameters is given by: $d = 2$, $\beta=1/10$, $R = 4$, $\alpha = 1$, $N = 45$.

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