Bull. Korean Math. Soc. **53** (2016), No. 5, pp. 1385–1394
http://dx.doi.org/10.4134/BKMS.b150679
pISSN: 1015-8634 / eISSN: 2234-3016

THE STABILITY OF CERTAIN SETS OF ATTACHED PRIME IDEALS RELATED TO COSEQUENCE IN DIMENSION $> k$

Pham Huu Khanh

Abstract. Let $(R, \mathfrak{m})$ be a Noetherian local ring, I, J two ideals of R, and A an Artinian R-module. Let $k \geq 0$ be an integer and $r = \mathrm{Width}_{>k}(I, A)$ the supremum of lengths of A-cosequences in dimension $> k$ in I defined by Nhan-Hoang [9]. It is first shown that for each $t \leq r$ and each sequence $x_1, \ldots, x_t$ which is an A-cosequence in dimension $> k$, the set

$$\Big(\bigcup_{i=0}^{t} \mathrm{Att}_R(0 :_A (x_1^{n_1}, \ldots, x_i^{n_i}))\Big)_{\geq k}$$

is independent of the choice of $n_1, \ldots, n_t$. Let r be the eventual value of $\mathrm{Width}_{>k}(0 :_A J^n)$. Then our second result says that for each $t \leq r$ the set $(\bigcup_{i=0}^{t} \mathrm{Att}_R(\mathrm{Tor}_i^R(R/I, (0 :_A J^n))))_{\geq k}$ is stable for large n.

1. Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a Noetherian local ring, I, J two ideals of R. Let M be a finitely generated R-module, and A an Atinian R-module. For a subset T of $\mathrm{Spec}(R)$ and an integer $i \geq 0$, denote by $(T)_{\geq i}$ the set of all prime ideals $\mathfrak{p} \in T$ such that $\dim R/\mathfrak{p} \geq i$.

In 1979, M. Brodmann [1] proved that the set of associated prime ideals $\mathrm{Ass}_R(M/J^nM)$ is stable for large n. As a dual result, R. Y. Sharp [10] proved that the set of attached prime ideals $\mathrm{Att}_R(0 :_A J^n)$ is stable for large n. Hence, it is natural to ask, for a sequence $(x_1, \ldots, x_r)$ of elements in R, whether the set $\mathrm{Att}_R(0 :_A (x_1^{n_1}, \ldots, x_r^{n_r}))$ become constant for $n_1, \ldots, n_r \gg 0$?

Unfortunately, Katzman [5, Corollary 1.3] constructed an example of local ring $(R, \mathfrak{m})$ of dimension 5 and two elements $x, y \in \mathfrak{m}$ such that

$$\mathrm{Ass}_R(H^2_{(x,y)R}(R))$$

Received August 22, 2015.
2010 *Mathematics Subject Classification.* 13E05, 13E10.
Key words and phrases. asymptotic stability, attached prime ideal, A-cosequence in dimension $> k$, width in dimension $> k$.
The author is supported by the Vietnam National Foundation for Science and Technology Development (Nafosted).

is an infinite set. Therefore, the set $\bigcup_{n\in\mathbb{N}} \operatorname{Ass}_R(R/(x^n,y^n)R)$ is infinite. It follows that $\bigcup_{n\in\mathbb{N}} \operatorname{Att}_R(0:_A (x^n,y^n)R)$ is an infinite set, where $A = E(R/\mathfrak{m})$ is the injective hull of $R/\mathfrak{m}$ which is an Artinian R-module. Hence, the set $\operatorname{Att}_R(0:_A (x^n,y^n)R)$ is not stable for large n. So, ones need to find conditions on A and on $x_1,\dots,x_r$ for the set $\bigcup_{n\in\mathbb{N}} \operatorname{Att}_R(0:_A (x_1^{n_1},\dots,x_r^{n_r})R)$ to be finite.

Nhan and Hoang [9] introduced the notion of A-cosequence in dimension $> k$, which is in some sense dual to the notion of M-sequence in dimension $> k$ was introduced by Brodmann and Nhan in [2]. Following Nhan and Hoang [9], a sequence $(x_1,\dots,x_r)$ of elements in $\mathfrak{m}$ is called an A-cosequence in dimension $> k$ if $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in (\operatorname{Att}_R(0:_A (x_1,\dots,x_{i-1})R))_{>k}$ for every $i = 1,\dots,r$. Then, they showed that the set $\big(\operatorname{Att}_R(0:_A (x_1^{n_1},\dots,x_r^{n_r})R)\big)_{>k}$ is independent of the choice of $n_1,\dots,n_r$ and the set $\big(\bigcup_{n_1,\dots,n_r} \operatorname{Att}_R(0:_A (x_1^{n_1},\dots,x_r^{n_r})R)\big)_{\geq k}$ is finite, where $x_1,\dots,x_r$ is an A-cosequence in dimension $> k$, cf. [9, Theorem 1.1].

The first main result of this paper is the following theorem.

Theorem 1.1. *Let $(R,\mathfrak{m})$ be a local ring and A an Artinian R-module. Let $k \geq 0$ be an integer and $x_1,\dots,x_r$ an A-cosequence in dimension $> k$. Then for each integer $t \leq r$ we have*

$$\left(\bigcup_{i=0}^{t} \operatorname{Att}_R(0:_A (x_1^{n_1},\dots,x_i^{n_i})R)\right)_{\geq k} = \left(\bigcup_{i=0}^{t} \operatorname{Att}_R(0:_A (x_1,\dots,x_i)R)\right)_{\geq k}$$

for all $n_1,\dots,n_t \in \mathbb{N}$.

It should be mentioned that Theorem 1.1 shows that for each $t \leq r$ the set

$$\left(\bigcup_{i=0}^{t} \operatorname{Att}_R(0:_A (x_1^{n_1},\dots,x_i^{n_i}))\right)_{\geq k}$$

is independent of the choice of $n_1,\dots,n_t$.

For an Artinian R-module A, denote by $\dim A$ the Krull dimension of the ring $R/\operatorname{Ann}_R A$. In [8], Nhan-Dung proved that if $\dim_R(0:_A I) > k$ then any A-cosequence in dimension $> k$ in I can be extended to a maximal one, and all maximal A-cosequences in dimension $> k$ in I have the same length. This common length is called *the width in dimension $> k$ in I with respect to A* and denoted by $\operatorname{Width}_{>k}(I,A)$. If $\dim_R(0:_A I) \leq k$, then for every positive integer r, we can choose an A-cosequence in dimension $> k$ in I of length r, in this case we set $\operatorname{Width}_{>k}(I,A) = +\infty$. Using the stability of $\operatorname{Att}_R(0:_A J^n)$ we will show that $\operatorname{Width}_{>k}(I,(0:_A J^n))$ takes a constant value for large n. The second main result of this paper is the following theorem.

Theorem 1.2. *Let $k \geq 0$ be an integer and r the eventual value of*

$$\operatorname{Width}_{>k}(I,(0:_A J^n)).$$

Then for each integer $t \leq r$ the set $\left(\bigcup_{i=0}^{t} \mathrm{Att}_R(\mathrm{Tor}_i^R(R/I, (0:_A J^n)))\right)_{\geq k}$ *is stable for large n.*

This paper is divided into four sections. In the next section, we present some preliminaries on attached prime ideals that will be used in the sequel. In Section 3, we prove Theorem 1.1. The proof of Theorem 1.2 is given in Section 4.

2. Preliminaries

Following I. G. Macdonald [7], any Artinian R-module A has a minimal secondary representation $A = A_1 + \cdots + A_r$, where A_i is $\mathfrak{p}_i$-secondary. The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ is independent of the choice of a minimal secondary representation of A, and it is denoted by $\mathrm{Att}_R(A)$.

For an ideal I of R, denote by $\mathrm{Var}(I)$ the set of all prime ideals $\mathfrak{p}$ of R containing I. The following results are well-known and will be used in the sequel.

Lemma 2.1 ([7])**.** *The set of all minimal elements of* $\mathrm{Att}_R(A)$ *is exactly the set of all minimal elements of* $\mathrm{Var}(\mathrm{Ann}_R(A))$*. Moreover,*

$$\dim_R(A) = \max\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Att}_R(A)\}.$$

Lemma 2.2 ([3, 8.24 and 8.25])**.** $\mathrm{Att}_R(A) = \{\widehat{\mathfrak{p}} \cap R \mid \widehat{\mathfrak{p}} \in \mathrm{Att}_{\widehat{R}}(A)\}$.

We denote by $E = E(R/\mathfrak{m})$ the injective envelope of $R/\mathfrak{m}$. For each R-module N, the Matlis dual $D(N)$ of N is defined by $D(N) = \mathrm{Hom}_R(N, E)$. Since A is an Artinian R-module, A has a natural structure as an Artinian $\widehat{R}$-module, and $D(A)$ is a finitely generated $\widehat{R}$-module. By Matlis Duality Theorem, $A \cong D(D(A))$ as $\widehat{R}$-modules. Moreover,

$$\mathrm{Ass}_{\widehat{R}}\, D(A) = \mathrm{Att}_{\widehat{R}}\, D(D(A)) = \mathrm{Att}_{\widehat{R}}\, A.$$

Lemma 2.3 ([11, 3.4.14])**.** *Let N be an R-module. Then*

$$D(\mathrm{Ext}_R^i(R/I, N)) \cong \mathrm{Tor}_i^R(R/I, D(N)),$$
$$\mathrm{Ext}_R^i(R/I, D(N)) \cong D(\mathrm{Tor}_i^R(R/I, N)).$$

Remark 2.4. (i) It follows by Lemma 2.3 and Matlis Duality Theorem that

$$\mathrm{Att}_{\widehat{R}}(\mathrm{Tor}_i^{\widehat{R}}(\widehat{R}/I\widehat{R}, A)) = \mathrm{Ass}_{\widehat{R}}(\mathrm{Ext}_{\widehat{R}}^i(\widehat{R}/I\widehat{R}, D(A))).$$

(ii) Applying Lemma 2.3 for the case $i = 0$ we have

$$\begin{aligned} D(0:_A I) &\cong D(\mathrm{Hom}_R(R/I, A)) \\ &\cong (R/I) \otimes D(A) \\ &\cong D(A)/ID(A). \end{aligned}$$

Hence, $\mathrm{Att}_{\widehat{R}}(0:_A I) = \mathrm{Ass}_{\widehat{R}}(D(A)/ID(A))$.

3. Proof of Theorem 1.1

First of all we recall the notion of poor filter sequence. An element $x \in R$ is called an *M-poor filter regular element* if $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_R(M) \setminus \{\mathfrak{m}\}$. A sequence $x_1, \ldots, x_r$ of elements in R is called an *M-poor filter sequence* if x_i is $(M/(x_1, \ldots, x_{i-1})M)$-poor filter regular for all $i = 1, \ldots, r$. It is clear that a sequence $x_1, \ldots, x_r$ of elements in R is an M-filter sequence in sense of Cuong-Schenzel-Trung [4] if and only if it is an M-poor filter sequence and $x_1, \ldots, x_r \in \mathfrak{m}$.

The following lemma plays an important role in proving our main results.

Lemma 3.1 ([6, Lemma 2.5]). *Let $k \geq 0$ be an integer. Assume that $(x_1, \ldots, x_r)$ is an A-cosequence in dimension $> k$. Then $(\bar{x}_1, \ldots, \bar{x}_r)$ is a $D(A)_{\widehat{\mathfrak{p}}}$-poor filter sequence for all $\widehat{\mathfrak{p}} \in \mathrm{Var}(\mathrm{Ann}_{\widehat{R}} A)$ satisfying $\dim(R/(\widehat{\mathfrak{p}} \cap R)) \geq k$. Here $\bar{x}_i$ is the image of x_i in $\widehat{R}_{\widehat{\mathfrak{p}}}$ for $i = 1, \ldots, r$.*

Proof of Theorem 1.1. Let $\mathfrak{p} \in \big(\bigcup_{i=0}^{t} \mathrm{Att}_R(0 :_A (x_1^{n_1}, \ldots, x_i^{n_i})R)\big)_{\geq k}$. Then $\mathfrak{p} \in \big(\mathrm{Att}_R(0 :_A (x_1^{n_1}, \ldots, x_{i_0}^{n_{i_0}})R)\big)_{\geq k}$ for some $0 \leq i_0 \leq t$. By Lemma 2.2, there exists $\widehat{\mathfrak{p}} \in \mathrm{Att}_{\widehat{R}}(0 :_A (x_1^{n_1}, \ldots, x_{i_0}^{n_{i_0}})R)$ such that $\widehat{\mathfrak{p}} \cap R = \mathfrak{p}$. It follows by Remark 2.4 that $\widehat{\mathfrak{p}} \in \mathrm{Ass}_{\widehat{R}}(D(A)/(x_1^{n_1}, \ldots, x_{i_0}^{n_{i_0}})D(A))$. Therefore

$$\widehat{\mathfrak{p}}\widehat{R}_{\widehat{\mathfrak{p}}} \in \mathrm{Ass}_{\widehat{R}_{\widehat{\mathfrak{p}}}}(D(A)_{\widehat{\mathfrak{p}}}/(x_1^{n_1}, \ldots, x_{i_0}^{n_{i_0}})D(A)_{\widehat{\mathfrak{p}}}).$$

By Lemma 3.1, $\bar{x}_1, \ldots, \bar{x}_{i_0}$ is a $D(A)_{\widehat{\mathfrak{p}}}$- filter sequence. If $\mathfrak{p} \in \mathrm{Att}_R(0 :_A (x_1, \ldots, x_i)R)$ for some $i < i_0$ then it is clear that

$$\mathfrak{p} \in \bigcup_{i=0}^{t} \mathrm{Att}_R(0 :_A (x_1, \ldots, x_i)R).$$

Otherwise, we have $\mathfrak{p} \notin \mathrm{Att}_R(0 :_A (x_1, \ldots, x_i)R)$ for all $i < i_0$. In this case we have

$$\widehat{\mathfrak{p}}\widehat{R}_{\widehat{\mathfrak{p}}} \notin \mathrm{Ass}_{\widehat{R}_{\widehat{\mathfrak{p}}}}(D(A)_{\widehat{\mathfrak{p}}}/(x_1, \ldots, x_i)D(A)_{\widehat{\mathfrak{p}}})$$

for all $i < i_0$. Therefore in this case, $\bar{x}_1, \ldots, \bar{x}_{i_0}$ is a $D(A)_{\widehat{\mathfrak{p}}}$- regular sequence. Hence

$$\mathrm{Ass}_{\widehat{R}_{\widehat{\mathfrak{p}}}}(D(A)_{\widehat{\mathfrak{p}}}/(x_1^{n_1}, \ldots, x_{i_0}^{n_{i_0}})D(A)_{\widehat{\mathfrak{p}}}) = \mathrm{Ass}_{\widehat{R}_{\widehat{\mathfrak{p}}}}(D(A)_{\widehat{\mathfrak{p}}}/(x_1, \ldots, x_{i_0})D(A)_{\widehat{\mathfrak{p}}})$$

and so

$$\widehat{\mathfrak{p}}\widehat{R}_{\widehat{\mathfrak{p}}} \in \mathrm{Ass}_{\widehat{R}_{\widehat{\mathfrak{p}}}}(D(A)_{\widehat{\mathfrak{p}}}/(x_1, \ldots, x_{i_0})D(A)_{\widehat{\mathfrak{p}}}).$$

From this we have

$$\widehat{\mathfrak{p}} \in \mathrm{Ass}_{\widehat{R}}(D(A)/(x_1, \ldots, x_{i_0})D(A)) = \mathrm{Att}_{\widehat{R}}(0 :_A (x_1, \ldots, x_{i_0})R).$$

By Lemma 2.2, $\mathfrak{p} = \widehat{\mathfrak{p}} \cap R \in \big(\mathrm{Att}_R(0 :_A (x_1, \ldots, x_{i_0})R)\big)_{\geq k}$. In any case we obtain the inclusion

$$\left(\bigcup_{i=0}^{t} \mathrm{Att}_R(0 :_A (x_1^{n_1}, \ldots, x_i^{n_i})R)\right)_{\geq k} \subseteq \left(\bigcup_{i=0}^{t} \mathrm{Att}_R(0 :_A (x_1, \ldots, x_i)R)\right)_{\geq k}.$$

Similarly we can show that

$$\left(\bigcup_{i=0}^{t} \mathrm{Att}_R(0 :_A (x_1, \ldots, x_i)R)\right)_{\geq k} \subseteq \left(\bigcup_{i=0}^{t} \mathrm{Att}_R(0 :_A (x_1^{n_1}, \ldots, x_i^{n_i})R)\right)_{\geq k}.$$

Hence,

$$\left(\bigcup_{i=0}^{t} \mathrm{Att}_R(0 :_A (x_1^{n_1}, \ldots, x_i^{n_i})R)\right)_{\geq k} = \left(\bigcup_{i=0}^{t} \mathrm{Att}_R(0 :_A (x_1, \ldots, x_i)R)\right)_{\geq k}.$$

The proof is completed. □

An immediate consequence of Theorem 1.1 is the following result.

Corollary 3.2 ([9, Theorem 1.1(ii)]). *Let $k \geq 0$ be an integer and $x_1, \ldots, x_r$ an A-cosequence in dimension $> k$. Then* $\left(\bigcup_{n_1,\ldots,n_r} \mathrm{Att}_R(0 :_A (x_1^{n_1}, \ldots, x_r^{n_r})R)\right)_{\geq k}$ *is a finite set.*

Remark 3.3. Theorem 1.1 may be stated as follows.

Let $k \geq 0$ be an integer and $x_1, \ldots, x_r$ an A-cosequence in dimension $> k$. Then for each integer $t \leq r$ the set

$$\left(\bigcup_{i=0}^{t} \mathrm{Att}_R(0 :_A (x_1^{n_1}, \ldots, x_i^{n_i})R)\right)_{\geq k}$$

is independent of the choice of $n_1, \ldots, n_t$.

In the case $k = 0$, by Remark 3.3 we obtain the following result.

Corollary 3.4. *Let $x_1, \ldots, x_r$ be an A-cosequence in dimension > 0. Then for each integer $t \leq r$ the set*

$$\bigcup_{i=0}^{t} \mathrm{Att}_R(0 :_A (x_1^{n_1}, \ldots, x_i^{n_i})R)$$

is independent of the choice of $n_1, \ldots, n_t$.

4. Proof of Theorem 1.2

In order to prove Theorem 1.2 we need some auxiliary results.

Firstly, the following result on the asymptotic stability of set of attached prime ideals was proved by Sharp.

Lemma 4.1 ([10, Theorem 3.1]). *The set* $\operatorname{Att}_R(0 :_A J^n)$ *is stable for large* n.

Now we apply Lemma 4.1 to prove the stability of width in dimension $> k$ in I of $(0 :_A J^n)$.

Lemma 4.2. *Let* $k \geq 0$ *be an integer. Then* $\operatorname{Width}_{>k}(I, (0 :_A J^n))$ *takes a constant value for large* n.

Proof. Note that $(0 :_{(0:_A J^n)} I) = (0 :_{(0:_A I)} J^n)$. Therefore, by Lemma 4.1, there exists an integer n_0 such that $\operatorname{Att}_R(0 :_{(0:_A J^n)} I)$ are stable for all $n \geq n_0$. Hence, $\dim(0 :_{(0:_A J^n)} I)$ takes a constant value d for all $n \geq n_0$. We consider two cases as follows.

Case 1. If $d \leq k$ then $\operatorname{Width}_{>k}(I, (0 :_A J^n)) = +\infty$ for all $n \geq n_0$. The conclusion is true.

Case 2. Assume that $d > k$. In this case, we set $r = \liminf_{n\to\infty} \operatorname{Width}_{>k}(I, (0 :_A J^n))$ and note that r is an integer. We prove that $r = \operatorname{Width}_{>k}(I, (0 :_A J^n))$ for large n by induction on r. By Lemma 4.1, there exists an integer $a \geq n_0$ such that $\operatorname{Att}_R(0 :_A J^n)$ is constant for all $n \geq a$.

If $r = 0$ then we have $I \subseteq (\bigcup_{\mathfrak{p}\in \operatorname{Att}_R(0:_A J^n)} \mathfrak{p})_{>k}$ for infinitely many n. If $I \not\subseteq (\bigcup_{\mathfrak{p}\in \operatorname{Att}_R(0:_A J^n)} \mathfrak{p})_{>k}$ for some $n \geq a$ then $I \not\subseteq (\bigcup_{\mathfrak{p}\in \operatorname{Att}_R(0:_A J^n)} \mathfrak{p})_{>k}$ for all $n \geq a$, so in this case there are at most finitely many numbers n in $\{1, 2, \ldots, a-1\}$ satisfying $I \subseteq (\bigcup_{\mathfrak{p}\in \operatorname{Att}_R(0:_A J^n)} \mathfrak{p})_{>k}$. This is impossible. Therefore $I \subseteq (\bigcup_{\mathfrak{p}\in \operatorname{Att}_R(0:_A J^n)} \mathfrak{p})_{>k}$ for all $n \geq a$ and hence $\operatorname{Width}_{>k}(I, (0 :_A J^n)) = 0$ for all $n \geq a$. So, the result is true for $r = 0$.

If $r > 0$, since $\operatorname{Width}_{>k}(I, (0 :_A J^n)) = r \geq 1$ for infinitely many n and $\operatorname{Att}_R(0 :_A J^n)$ is constant for all $n \geq a$, there exists $x_1 \in I$ and $x_1 \notin (\bigcup_{\mathfrak{p}\in \operatorname{Att}_R(0:_A J^n)} \mathfrak{p})_{>k}$ for all $n \geq a$. Now, set $\bar{r} = \liminf_{n\to\infty} \operatorname{Width}_{>k}(I, (0 :_{(0:_A J^n)} x_1R))$. Note that

$$\operatorname{Width}_{>k}(I, (0 :_{(0:_A J^n)} x_1R)) = \operatorname{Width}_{>k}(I, (0 :_A J^n)) - 1 \text{ and } \bar{r} = r - 1.$$

By induction assumptions $\operatorname{Width}_{>k}(I, (0 :_{(0:_A J^n)} x_1R)) = \bar{r}$ for large n. It follows that $\operatorname{Width}_{>k}(I, (0 :_A J^n)) = r$ for large n, as required. □

Lemma 4.3. *Let* $k \geq 0$ *be an integer and* r *is the stable value of*

$$\operatorname{Width}_{>k}(I, (0 :_A J^n)).$$

Assume that $1 \leq r < \infty$. *Then there exists a sequence* $x_1, \ldots, x_r$ *in* I *which is an* $(0 :_A J^n)$*-cosequence in dimension* $> k$ *for all large* n.

Proof. We will show by induction on r that there exists a sequence of r elements in I satisfying the conclusion of the lemma. By Lemma 4.1 and Lemma 4.2 we may assume that $\mathrm{Att}_R(0:_A J^n)$ is stable and $r = \mathrm{Width}_{>k}(I, (0:_A J^n))$ for all $n \geq a$ for some integer a.

If $r = 1$, there exists $x_1 \in I$ and $x_1 \notin (\bigcup_{\mathfrak{p} \in \mathrm{Att}_R(0:_A J^n)} \mathfrak{p})_{>k}$ for all $n \geq a$. Therefore, x_1 is an element in I satisfying our lemma.

Assume that $r > 1$, by the same arguments as in the above, we can choose an element $x_1 \in I$ such that x_1 is an $(0:_A J^n)$-coregular element in dimension $> k$ for all $n \geq a$. Note that

$$\mathrm{Width}_{>k}(I, (0:_{(0:_A J^n)} x_1 R)) = \mathrm{Width}_{>k}(I, (0:_A J^n)) - 1 = r - 1$$

for all $n \geq a$. It follows by induction assumptions that there exists $x_2, \ldots, x_r$ in I which is an $(0:_{(0:_A J^n)} x_1 R)$-cosequence in dimension $> k$ for all large n. Hence, $x_1, \ldots, x_r$ in I which is an $(0:_A J^n)$-cosequence in dimension $> k$ for all large n. The proof is completed. □

The following result on the asymptotic stability of set of associated prime ideals was proved by Brodmann.

Lemma 4.4 ([1]). *Let M be a finitely generated R-module. Then the set $\mathrm{Ass}_R(M/J^n M)$ is stable for large n.*

Lemma 4.5. *Let M be a finitely generated R-module and I, J two ideals of R. Set $M_n = M/J^n M$. If $\dim(M_n/IM_n) \leq k$, then for each integer $t \geq 0$ the set*

$$\Big(\bigcup_{i=0}^{t} \mathrm{Ass}_R(\mathrm{Ext}_R^i(R/I, M_n))\Big)_{\geq k}$$

is stable for large n.

Proof. Note that $M_n/IM_n \cong (M/IM)/J^n(M/IM)$. Therefore, by Lemma 4.4, there exists an integer a such that $\mathrm{Ass}_R(M_n/IM_n) = \mathrm{Ass}_R(M_a/IM_a)$ for all $n \geq a$. Hence $\dim(M_n/IM_n)$ is stable for all $n \geq a$. Set $\dim(M_n/IM_n) = d$ for $n \geq a$. Note that $(\mathrm{Supp}_R(M_n/IM_n))_d = (\mathrm{Ass}_R(M_n/IM_n))_d$ for all $n \geq a$.

If $d < k$, then $\big(\bigcup_{i=0}^{t} \mathrm{Ass}_R(\mathrm{Ext}_R^i(R/I, M_n))\big)_{\geq k} = \emptyset$, for all $n \geq a$.

Let $d = k$. Since $\mathrm{Var}(\mathrm{Ann}_R(M_n/IM_n)) = \mathrm{Var}(I + \mathrm{Ann}_R M_n)$, it follows that $(\mathrm{Supp}_R(M_n/IM_n))_d$ contains all prime ideals $\mathfrak{p}$ such that $\dim(R/\mathfrak{p}) = d$ and $\mathfrak{p} \supseteq I + \mathrm{Ann}_R M_n$. Hence

$$\begin{aligned}\Big(\bigcup_{i=0}^{t} \mathrm{Ass}_R(\mathrm{Ext}_R^i(R/I, M_n))\Big)_{\geq d} &\subseteq \big(\mathrm{Supp}_R(M_n/IM_n)\big)_{\geq d} \\ &= \big(\mathrm{Ass}_R(M_n/IM_n)\big)_{\geq d}\end{aligned}$$

for all $n \geq a$. It implies that $\Omega = \bigcup_{n \geq a} \big(\bigcup_{i=0}^{t} \mathrm{Ass}_R(\mathrm{Ext}_R^i(R/I, M_n)) \big)_{\geq d}$ is finite set. Therefore we can choose a large enough such that for each $\mathfrak{p} \in \Omega$ we have $\mathfrak{p} \in \big(\bigcup_{i=0}^{t} \mathrm{Ass}_R(\mathrm{Ext}_R^i(R/I, M_n)) \big)_{\geq d}$ for infinitely many $n \geq a$.

Now for each $\mathfrak{p} \in \Omega$, let $s(\mathfrak{p})$ be the eventual value of $\mathrm{depth}(I_\mathfrak{p}, (M_n)_\mathfrak{p})$. Then there exists an integer $n(\mathfrak{p}) \geq a$ such that $s(\mathfrak{p}) = \mathrm{depth}(I_\mathfrak{p}, (M_n)_\mathfrak{p})$ for all $n \geq n(\mathfrak{p})$. Hence, $(\mathrm{Ext}_R^{s(\mathfrak{p})}(R/I, M_n))_\mathfrak{p} \neq 0$ for all $n \geq n(\mathfrak{p})$ and so $\mathfrak{p}$ is a minimal element of $\mathrm{Supp}_R(\mathrm{Ext}_R^{s(\mathfrak{p})}(R/I, M_n))$. Therefore $\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{Ext}_R^{s(\mathfrak{p})}(R/I, M_n))$. If $s(\mathfrak{p}) > t$ then $(\mathrm{Ext}_R^i(R/I, M_n))_\mathfrak{p} = 0$ for all $i \leq t$, for all $n \geq n(\mathfrak{p})$. It follows that $\mathfrak{p} \notin \bigcup_{i=0}^{t} \mathrm{Ass}_R(\mathrm{Ext}_R^i(R/I, M_n))$ for all $n \geq n(\mathfrak{p})$. This is impossible. So $s(\mathfrak{p}) \leq t$ and we have $\mathfrak{p} \in \big(\bigcup_{i=0}^{t} \mathrm{Ass}_R(\mathrm{Ext}_R^i(R/I, M_n)) \big)_{\geq d}$ for all $n \geq n(\mathfrak{p})$.

Since Ω is a finite set, we can put $n_0 = \max\{n(\mathfrak{p}) |\ \mathfrak{p} \in \Omega\}$ then we have $\Omega = \big(\bigcup_{i=0}^{t} \mathrm{Ass}_R(\mathrm{Ext}_R^i(R/I, M_n)) \big)_{\geq d}$ for all $n \geq n_0$. Hence, the set

$$\Big(\bigcup_{i=0}^{t} \mathrm{Ass}_R(\mathrm{Ext}_R^i(R/I, M_n)) \Big)_{\geq d}$$

is stable for large n. The proof is completed. $\square$

By the proof of [6, Theorem 3.4], we have the following lemma.

Lemma 4.6. *Let $(x_1, \ldots, x_r)$ be an A-cosequence in dimension $> k$ in I. Then*

$$\left(\bigcup_{i=0}^{t} \mathrm{Att}_R(\mathrm{Tor}_i^R(R/I, A)) \right)_{\geq k} = \Big(\bigcup_{i=0}^{t} \mathrm{Ass}_R(D(A)/(x_1, \ldots, x_i)D(A)) \Big)_{\geq k} \cap \mathrm{Var}(I),$$

for all $t \leq r$.

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We consider three cases as follows.
Case 1. If $r = \infty$, then $\dim_R(0 :_{(0:_A J^n)} I) \leq k$ for large n. Since

$$\begin{aligned} \dim_R(0 :_{(0:_A J^n)} I) &= \dim_R D(0 :_{(0:_A J^n)} I) \\ &= \dim_R(D(0 :_A J^n)/ID(0 :_A J^n)) \\ &= \dim_R((D(A)/J^n D(A))/I(D(A)/J^n D(A))), \end{aligned}$$

we obtain $\dim_R((D(A)/J^n D(A))/I(D(A)/J^n D(A))) \leq k$ for large n.

By Remark 2.4, we have

$$\left(\bigcup_{i=0}^{t} \mathrm{Att}_{\widehat{R}}(\mathrm{Tor}_i^{\widehat{R}}(\widehat{R}/I\widehat{R}, (0 :_A J^n))) \right)_{\geq k}$$

$$= \left(\bigcup_{i=0}^{t} \mathrm{Ass}_{\widehat{R}}(\mathrm{Ext}^i_{\widehat{R}}(\widehat{R}/I\widehat{R}, D(0:_A J^n)))\right)_{\geq k}$$
$$= \left(\bigcup_{i=0}^{t} \mathrm{Ass}_{\widehat{R}}(\mathrm{Ext}^i_{\widehat{R}}(\widehat{R}/I\widehat{R}, D(A)/J^n D(A)))\right)_{\geq k}.$$

Therefore, $\big(\bigcup_{i=0}^{t} \mathrm{Att}_{\widehat{R}}(\mathrm{Tor}_i^{\widehat{R}}(\widehat{R}/I\widehat{R}, (0 :_A J^n)))\big)_{\geq k}$ is stable for large n, by Lemma 4.5. It follows by Lemma 2.2 that

$$\big(\bigcup_{i=0}^{t} \mathrm{Att}_R(\mathrm{Tor}_i^R(R/I, (0:_A J^n)))\big)_{\geq k}$$
$$= \{\widehat{\mathfrak{p}} \cap R \mid \widehat{\mathfrak{p}} \in \big(\bigcup_{i=0}^{t} \mathrm{Att}_{\widehat{R}}(\mathrm{Tor}_i^{\widehat{R}}(\widehat{R}/I\widehat{R}, (0:_A J^n)))\big)_{\geq k}\},$$

which is stable for large n.

Case 2. If $r = 0$, then

$$\begin{aligned}\mathrm{Att}_{\widehat{R}}(\mathrm{Tor}_0^{\widehat{R}}(\widehat{R}/I\widehat{R}, (0:_A J^n))) &= \mathrm{Ass}_{\widehat{R}}(\mathrm{Ext}^0_{\widehat{R}}(\widehat{R}/I\widehat{R}, D(0:_A J^n)))\\ &= \mathrm{Ass}_{\widehat{R}}(\mathrm{Ext}^0_{\widehat{R}}(\widehat{R}/I\widehat{R}, D(A)/J^n D(A)))\\ &= \mathrm{Ass}_{\widehat{R}}(D(A)/J^n D(A)) \cap \mathrm{V}(I\widehat{R}).\end{aligned}$$

It follows by Lemma 4.4 that $\mathrm{Att}_{\widehat{R}}(\mathrm{Tor}_0^{\widehat{R}}(\widehat{R}/I\widehat{R}, (0:_A J^n)))$ is stable for large n. Using Lemma 2.2 again, we obtain that the set $\mathrm{Tor}_0^R(R/I, (0:_A J^n))$ is stable for large n.

Case 3. $1 \leq r < \infty$. By Lemma 4.2 and Lemma 4.3, there exists a positive integer a such that for all $n \geq a$ the followings are true:

(i) $r = \mathrm{Width}_{>k}(I, (0:_A J^n))$, and

(ii) there exists a sequence $x_1, \ldots, x_r \in I$ which is a cosequence in dimension $> k$ of $(0:_A J^n)$. Then

$$\begin{aligned}&D(0:_A J^n)/(x_1,\ldots,x_i)D(0:_A J^n)\\ &= \big(D(A)/J^n D(A)\big)/(x_1,\ldots,x_i)\big(D(A)/J^n D(A)\big)\\ &= \big(D(A)/(x_1,\ldots,x_i)D(A)\big)/J^n\big(D(A)/(x_1,\ldots,x_i)D(A)\big)\end{aligned}$$

for all $i = 1, \ldots, r$, for all $n \geq a$. From this and by Lemma 4.6 we get the equality

$$\big(\bigcup_{i=0}^{t} \mathrm{Att}_R(\mathrm{Tor}_i^R(R/I, (0:_A J^n)))\big)_{\geq k}$$
$$= \big(\bigcup_{i=0}^{t} \mathrm{Ass}_R(D(0:_A J^n)/(x_1,\ldots,x_i)D(0:_A J^n))\big)_{\geq k} \cap \mathrm{Var}(I)$$

$$= \Big(\bigcup_{i=0}^{t} \mathrm{Ass}_R((D(A)/(x_1,\ldots,x_i)D(A))/J^n(D(A)/(x_1,\ldots,x_i)D(A)))\Big)_{\geq k} \cap \mathrm{Var}(I)$$

for all $t \leq r$ and for all $n \geq a$, and so $\Big(\bigcup_{i=0}^{t} \mathrm{Att}_R(\mathrm{Tor}_i^R(R/I,(0:_A J^n)))\Big)_{\geq k}$ is stable for large n by Lemma 4.4. □

In the case $k = 0$, the following result is an immediate consequence of Theorem 1.1.

Corollary 4.7. *Let r be the eventual value of* $\mathrm{Width}_{>0}(I,(0:_A J^n))$. *Then for each $t \leq r$ the set* $\bigcup_{i=0}^{t} \mathrm{Att}_R(\mathrm{Tor}_i^R(R/I,(0:_A J^n)))$ *is stable for large n.*

References

[1] M. Brodmann, *Asymptotic stability of* $\mathrm{Ass}_R(M/I^nM)$, Proc. Amer. Math. Soc. **74** (1979), no. 1, 16–18.

[2] M. Brodmann and L. T. Nhan, *A finiteness result for associated primes of certain Ext-modules*, Comm. Algebra **36** (2008), no. 4, 1527–1536.

[3] M. Brodmann and R. Y. Sharp, *Local Cohomology: an algebraic introduction with geometric applications*, Cambridge University Press, 1998.

[4] N. T. Cuong, P. Schenzel, and N. V. Trung, *Verallgemeinerte Cohen-Macaulay moduln*, Math. Nachr. **85** (1978), 57–73.

[5] M. Katzman, *An example of an infinite set of associated primes of a local cohomology module*, J. Algebra **252** (2002), no. 1, 161–166.

[6] P. H. Khanh, *An independent result for attached primes of certain Tor-modules*, Bull. Korean Math. Soc. **52** (2015), no. 2, 531–540.

[7] I. G. Macdonald, *Secondary representation of modules over a commutative ring*, Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971), pp. 23-3. Academic Press, London, 1973.

[8] L. T. Nhan and N. T. Dung, *A finiteness result for attached primes of certain tor-modules*, Algebra Colloq. **19** (2012), no. 1, 787–796.

[9] L. T. Nhan and N. V. Hoang, *A finiteness result for attached primes of Artinian local cohomology*, J. Algebra Appl. **13** (2014), no. 1, 14 pages.

[10] R. Y. Sharp, *Asymptotic behaviour of certain sets of attached prime ideals*, J. London Math. Soc. (2) **34** (1986), no. 2, 212–218.

[11] J. Strooker, *Homological Question in Local Algebra*, Cambridge University Press, 1990.

Pham Huu Khanh
Department of mathematics
Tay Nguyen University
Dak Lak, Viet Nam
E-mail address: phkhanhlt@gmail.com