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OPTION PRICING UNDER GENERAL GEOMETRIC RIEMANNIAN BROWNIAN MOTIONS

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Abstract. We provide a partial differential equation for European options on a stock whose price process follows a general geometric Riemannian Brownian motion. The existence and the uniqueness of solutions to the partial differential equation are investigated, and then an expression of the value for European options is obtained using the fundamental solution technique. Proper Riemannian metrics on the real number field can make the distribution of return rates of the stock induced by our model have the character of leptokurtosis and fat-tail; in addition, they can also explain option pricing bias and implied volatility smile (skew).

1. Introduction

Bachelier introduced the Brownian motion to finance, and used it to describe the stock price process [2]. Samuelson argued that the geometric Brownian motion is a good model for stock prices [17]. Under the assumption that the price process of a stock follows a geometric Brownian motion, Black and Scholes intensively studied European option pricing [3]. In their model there are two securities, the stock and the bond, whose price processes S and D satisfy the following SDEs in the sense of Itô integral,

$$\begin{aligned} \mathrm{d}S_t &= \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t, \\ \mathrm{d}D_t &= r D_t \mathrm{d}t, \end{aligned} \tag{1.1}$$

respectively. Here, σ, μ and r are some constants, and W is a standard Euclidean Brownian motion.

Under some ideal conditions, Black and Scholes obtained a pricing formula (BS formula) for European options through the method of risk hedging [3]. In their discussions, the form and the regularity of option values are *a priori* assumed, and they did not provide any mathematical reasons.

Since then many extensions of the work of Black and Scholes in [3] have been appeared. A class of the extensions is to assume that the volatility σ

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follows some stochastic processes [9, 10, 12, 18]. The option pricing PDEs obtained in [9, 10, 12, 18] are of three dimensions. Hull and White determined option prices in series form for the case that the volatility is independent of the stock price [12]. Assuming the volatility is driven by an arithmetic Ornstein-Uhlenbeck process, Elias Stein and Jeremy Stein derived an option pricing formula in terms of a double integral [18]. Under some similar assumptions to that of Elias Stein and Jeremy Stein, Heston obtained a closed-form solution for European call options by applying the method of characteristic functions [9]. Hobson and Rogers defined the volatility in terms of exponentially weighted moments of historic log-price [10]. Unlike [9, 10, 12, 18], Chan replaced W with a general Lévy process, then obtained an option pricing equation which is an integro-differential equation [4]. In this paper, we obtain an option pricing PDE of two dimensions.

Because the quadratic variation $\langle W\rangle$ of W is $\langle W\rangle_t = t$, the equations in Black and Scholes's model can be rewritten in the sense of Stratonovich integral as

$$\begin{aligned} \mathrm{d}S_t &= \mu S_t \mathrm{d}t + \sigma S_t \circ \mathrm{d}W_t, \\ \mathrm{d}D_t &= r D_t \mathrm{d}t, \end{aligned} \tag{1.2}$$

without changing the BS option pricing formula. Note that the drift μ in (1.2) is different from that in (1.1)

Since the stock price process S in (1.2) is driven by a Brownian motion on a special Riemannian manifold $(\mathbb{R}, g_0)$, where $g_0 \equiv 1$, a direct extension of the work of Black and Scholes in [3] is to replace W with a (Riemannian) Brownian motion on a general Riemannian manifold $(\mathbb{R}, g)$. In this case, we call the price process S follows a *geometric Riemannian Brownian motion.* It is shown in [16, 5] that the distribution of realistic rates of stock return has the character of leptokurtosis and fat-tail. We will demonstrate that a proper Riemannian metric g on $\mathbb{R}$ implies a distribution which has the above character.

Hull and White pointed out that the strike prices of most options are within ten percent of the security price. For this range of strike prices, options are overpriced by BS formula [12]. With a proper Riemannian metric g on the real number field $\mathbb{R}$, our model can get rid of this drawback (see Table 1).

As shown in [14, p. 316] and [8, pp. 38–39], a common occurrence is that implied volatilities derived from far in-of-the-money and out-of-the-money call options are larger than ones derived from at-the-money options. We will investigate this phenomenon (called implied volatility smile). Another phenomenon also shown in [14, p. 316] and [8, pp. 38–39] is that implied volatilities decrease monotonically as the strike price rises, which is usually regarded as implied volatility skew. We will study these two patterns concerning implied volatilities (see Figures 2 and 3).

The rest of this paper is organized as follows. First, we introduce some facts of stochastic differential geometry, and then give some concrete stochastic analyses on $(\mathbb{R}, \nabla)$, where ∇ is an affine connection. Second, we recall briefly the option pricing theory in continuous time. Third, we provide an option pricing formula (see Theorem 5.5) and the corresponding PDE (see Equation (5.8)), and show an expression of the solution to the PDE with some terminal condition by means of the fundamental solution technique (see Corollary 5.8). Fourth, we investigate leptokurtosis and fat-tail of the distribution of return rates, option pricing bias and implied volatility smile (skew) by choosing proper Riemannian metrics. Finally, we draw some conclusions.

2. An elementary introduction to stochastic differential geometry

In this section, we will introduce some facts of stochastic differential geometry. For details, we refer to [11, pp. 35–49, p. 79].

Let M be a smooth differential manifold of dimension n, ∇ be an affine connection defined on the tangent bundle TM of M, and $\pi : \mathcal{F}(M) \to M$ be the frame bundle of M. A curve u_t in $\mathcal{F}(M)$ is called horizontal if for each $e \in \mathbb{R}^n$ the vector field $u_t e$ is parallel along πu_t; in this case $u'(0)$ is called a horizontal lift of the tangent vector $(\pi u)'(0)$.

Let $e_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, n$, be the coordinate unit vectors. Then we define the vector fields H_i, $i = 1, 2, \ldots, n$, by

$$H_i(u) := \text{the horizontal lift of } ue_i \in T_{\pi u}M \text{ to } u,$$

where $u \in \mathcal{F}(M)$.

Let u_t be a horizontal lift of a smooth curve x_t on M. Since $\dot{x}_t \in T_{x_t}M$, we have $u_t^{-1}\dot{x}_t \in \mathbb{R}^n$. The anti-development of the curve x_t is a curve w_t in $\mathbb{R}^n$ defined by

$$w_t = \int_0^t u_s^{-1}\dot{x}_s \mathrm{d}s.$$

Then the anti-development w_t and the horizontal lift u_t of a curve x_t on M are connected by the following ordinary differential equation on $\mathcal{F}(M)$,

$$\dot{u}_t = H_i(u_t)\dot{w}_t^i. \tag{2.1}$$

If M is a Riemannian manifold, we can define the anti-development, the horizontal lift, etc. with respect to the orthonormal frame bundle $\mathcal{O}(M)$ of M.

From now on all processes are defined on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ with $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions.

Because of (2.1), we consider the following SDE on the frame bundle $\mathcal{F}(M)$ in the sense of Stratonovish integral,

$$\mathrm{d}U_t = H_i(U_t) \circ \mathrm{d}W_t^i, \tag{2.2}$$

where $\{W_t^i\}$ an $\mathbb{R}^n$-value semimartingale.

Definition 2.1 ([11, p. 45, Definition 2.3.1])**.** (1) An $\mathcal{F}(M)$-value semimatingale U is said to be horizontal if there exists an $\mathbb{R}^n$-value semimartingle W such that the SDE (2.2) holds. The unique W (if its initial value is given) is called the anti-development of U (or of its projection $X = \pi U$).

(2) Let W be an $\mathbb{R}^n$-value semimartingale and U_0 be an $\mathcal{F}(M)$-value, $\mathscr{F}_0$-measurable random variable. The solution of the SDE (2.2) is called a development of W in $\mathcal{F}(M)$. Its projection $X = \pi U$ is called a development of W in M.

(3) Let X be an M-value semimartingale. An $\mathcal{F}(M)$-value horizontal semimartingale U such that its projection $\pi U = X$ is called a (stochastic) horizontal lift of X.

The existence of horizontal lift has been proven in [11, Section 2.3] by deriving a stochastic differential equation for it on the frame bundle $\mathcal{F}(M)$ driven by X.

Assume that M is a closed submanifold of $\mathbb{R}^N$ and regard $X = \{X^\alpha\}$ as an $\mathbb{R}^N$-value semimartinglae. For each $x \in M$, let $P(x) : \mathbb{R}^N \to T_xM$ be the orthogonal projection from $\mathbb{R}^N$ to its subspace T_xM. Then the horizontal lift U of X is the solution of the following equation on $\mathcal{F}(M)$

$$\mathrm{d}U_t = P_\alpha^*(U_t) \circ \mathrm{d}X_t^\alpha, \tag{2.3}$$

where $P_\alpha^*(u)$ is the horizontal lift of $P_\alpha(\pi u)$.

And the anti-development W of a horizontal semimartingale U is the solution of the following equation

$$\mathrm{d}W_t = U_t^{-1} P_\alpha(X_t) \circ \mathrm{d}X_t^\alpha, \tag{2.4}$$

where $X_t = \pi U_t$.

We end this section by the following definition.

Definition 2.2 ([11, p. 79, Proposition 3.2.1])**.** Let M be a Riemannian manifold. An M-value semimartingle is called a (Riemannian) Brownian motion if its anti-development with respect to the Levi-Civita connection is a standard Euclidean Brownian motion.

3. Some concrete stochastic analyses on $(\mathbb{R}, \nabla)$

In this section, we will give some concrete stochastic analyses on $(\mathbb{R}, \nabla)$.

Lemma 3.1. *Let $\mathbb{R}$ be equipped with an affine connection given by $\nabla_e e = \Gamma e$, where e is the usual unit vector field on $\mathbb{R}$, i.e., for any function $f \in C^\infty(\mathbb{R})$, $e(f) = f'$, and $\Gamma \in C^\infty(\mathbb{R})$. Define*

$$G(x) = \int_0^x \Gamma(s)\mathrm{d}s, \ \phi(x) = \int_0^x \exp(G(s))\mathrm{d}s.$$

Let X be a semimartingale. Then

(1) *with the canonical isomorphism $\mathcal{F}(\mathbb{R}) \cong \mathbb{R} \times (\mathbb{R} \setminus 0)$, the horizontal lift which passes through (X_0, y) of X is given by*

$$U_t = (X_t, y \exp(G(X_0)) \exp(-G(X_t))), \tag{3.1}$$

(2) *the anti-development W_t with $W_0 = 0$ of X is given by*

$$W_t = y^{-1} \exp(-G(X_0))(\phi(X_t) - \phi(X_0)). \tag{3.2}$$

Proof. 1. Suppose that $u = (x, y) \in \mathcal{F}(\mathbb{R})$, $x_t \in C^\infty(\mathbb{R})$, and $u_t = (x_t, y_t)$ is the (determinate) horizontal lift of x_t such that $\dot{x}(0) = 1$ and $u_0 = u$. Then $\nabla_{\dot{x}_t} y_t = 0$, i.e.,

$$\dot{y}_t + \Gamma(x_t)\dot{x}_t y_t = 0.$$

2. Since the orthogonal projection $P(x) : \mathbb{R} \to T_x\mathbb{R} \cong \mathbb{R}$ is now an identity, the horizontal lift of $P(\pi u)$ at u is $P^*(u) = (\dot{x}(0), \dot{y}(0)) = (1, -y\Gamma(x))$. Therefore, according to (2.3), the (stochastic) horizontal lift $U_t = (X_t, Y_t)$ of X_t is determined by

$$\begin{cases} \mathrm{d}(X_t, Y_t) = (1, -Y_t\Gamma(X_t)) \circ \mathrm{d}X_t, \\ (X_0, Y_0) = (X_0, y). \end{cases} \tag{3.3}$$

3. Note that

$$\mathrm{d}\int_0^{X_t} \Gamma(s)\mathrm{d}s = \Gamma(X_t) \circ \mathrm{d}X_t.$$

The following identity holds,

$$\int_0^t \Gamma(X_s) \circ \mathrm{d}X_s = \int_0^{X_t} \Gamma(s)\mathrm{d}s - \int_0^{X_0} \Gamma(s)\mathrm{d}s. \tag{3.4}$$

Thus, from (3.3) and (3.4), we have

$$\begin{aligned} Y_t &= y \exp\left(-\int_0^t \Gamma(X_s) \circ \mathrm{d}X_s\right) \\ &= y \exp(G(X_0)) \exp(-G(X_t)). \end{aligned}$$

4. From Step 3 and (2.4), we find that

$$\begin{aligned} W_t &= \int_0^t y^{-1} \exp(-G(X_0)) \exp(G(X_s)) \circ \mathrm{d}X_s \\ &= y^{-1} \exp(-G(X_0))(\phi(X_t) - \phi(X_0)). \end{aligned}$$

The proof is complete. □

Corollary 3.2. *Let $\mathbb{R}$ be equipped with a Riemannian metric g. Then*

(1) *with respect to the Levi-Civita connection induced by g,*

$$\mathrm{d}W_t = y^{-1}\sqrt{\frac{g(X_t)}{g(X_0)}} \circ \mathrm{d}X_t;$$

if the horizontal lift U of X satisfies $U_0 = (X_0, \frac{1}{\sqrt{g(X_0)}})$, then

$$\mathrm{d}W_t = \sqrt{g(X_t)} \circ \mathrm{d}X_t. \tag{3.5}$$

(2) *if X is a Riemannian Brownian motion, then*

$$\mathrm{d}X_t = y\sqrt{\frac{g(X_0)}{g(X_t)}}\mathrm{d}W_t - \frac{1}{4}y^2 g(X_0)\frac{g'(X_t)}{g(X_t)^2}\mathrm{d}t;$$

if, in addition, the horizontal lift U of X satisfies $U_0 = (X_0, \frac{1}{\sqrt{g(X_0)}})$, then

$$\mathrm{d}X_t = \frac{1}{\sqrt{g(X_t)}}\mathrm{d}W_t - \frac{1}{4}\frac{g'(X_t)}{g(X_t)^2}\mathrm{d}t. \tag{3.6}$$

Proof. Noting that the Levi-Civita connection induced by g is

$$\Gamma(x) = \frac{1}{2}g(x)^{-1}\frac{\partial g}{\partial x}(x) = \frac{1}{2}\frac{\partial}{\partial x}(\log(g(x))),$$

we have

$$G(x) = \log\sqrt{\frac{g(x)}{g(0)}} \text{ and } \phi(x) = \frac{1}{\sqrt{g(0)}}\int_0^x \sqrt{g(s)}\mathrm{d}s.$$

Then, by (3.2), we find that

$$\begin{aligned}\mathrm{d}W_t &= y^{-1}\exp(-G(X_0)) \circ \mathrm{d}\phi(X_t)\\ &= y^{-1}\exp\left(-\log\sqrt{\frac{g(X_0)}{g(0)}}\right)\sqrt{\frac{g(X_t)}{g(0)}} \circ \mathrm{d}X_t\\ &= y^{-1}\sqrt{\frac{g(X_t)}{g(X_0)}} \circ \mathrm{d}X_t,\end{aligned}$$

and, if X is a Riemannian Brownian motion,

$$\begin{aligned}\mathrm{d}X_t &= y\sqrt{\frac{g(X_0)}{g(X_t)}} \circ \mathrm{d}W_t\\ &= y\sqrt{\frac{g(X_0)}{g(X_t)}}\mathrm{d}W_t + \frac{1}{2}y\sqrt{g(X_0)}\mathrm{d}\left(\frac{1}{\sqrt{g(X_t)}}\right)\mathrm{d}W_t\\ &= y\sqrt{\frac{g(X_0)}{g(X_t)}}\mathrm{d}W_t - \frac{1}{4}y^2 g(X_0)\frac{g'(X_t)}{g(X_t)^2}\mathrm{d}t.\end{aligned}$$

The proof is complete. □

Remark 3.3. (1) Although all of results are described on $\mathbb{R}$, one can obtain the same results on a regular submanifold of $\mathbb{R}$.

(2) We have pointed out in Section 2 that if a manifold M has a Riemannian metric, the anti-development, the horizontal lift, etc. can be defined with respect to the orthonormal frame bundle $\mathcal{O}(M)$ of M. Under this point of view, (3.5) and (3.6) are considered in $\mathcal{O}(\mathbb{R})$, since $g(x)^{-\frac{1}{2}}$: $(\mathbb{R}, \text{the usual Euclidean metric}) \to (T_x\mathbb{R}, g(x))$ is unitary.

4. An elementary introduction to option pricing

We will recall in this section some facts of option pricing in continuous time. For details, we refer to [13, Chapter 7].

As before all processes are defined on a fixed filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. In addition, we assume that $\mathscr{F}_0$ is the completion of the trivial σ-algebra $\{\emptyset, \Omega\}$.

The *economy* $\mathcal{E}$ consists of n assets with price process given by $A = \{A^{(j)}\}_{j=1}^n$. We assume that A is an $\mathbb{R}^n$-value semimartingale.

Now we introduce some definitions.

Definition 4.1. A self-financing trading strategy ϕ for the economy $\mathcal{E}$ is a stochastic process $\phi = \{\phi^{(j)}\}_{j=1}^n$ such that

(1) ϕ is $\{\mathscr{F}_t\}_{t\geq 0}$-predictable;

(2) ϕ has the self-financing property

$$\phi_t \cdot A_t = \phi_0 \cdot A_0 + \int_0^t \phi_u \cdot \mathrm{d}A_u.$$

Furthermore, the process

$$G_t^\phi := \int_0^t \phi_u \cdot \mathrm{d}A_u$$

is called a gain process.

Definition 4.2. A numeraire N for the economy $\mathcal{E}$ is any a.s. strictly positive $\{\mathscr{F}_t\}_{t\geq 0}$-adapted process of the form

$$N_t = N_0 + \int_0^t \alpha_u \cdot \mathrm{d}A_u = \alpha_t \cdot A_t,$$

where α is $\{\mathscr{F}_t\}_{t\geq 0}$-predictable.

Definition 4.3. The measure $\mathbb{Q}$ defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ is an equivalent martingale measure for the economy $\mathcal{E}$ if $\mathbb{Q} \sim \mathbb{P}$ and there exists some numeraire N such that $N^{-1}A$ is an $\{\mathscr{F}_t\}_{t\geq 0}$-martingale under the measure $\mathbb{Q}$. The pair $(N, \mathbb{Q})$ is then called a numeraire pair.

Definition 4.4. A self-financing trading strategy is called admissible if for any numeraire pair $(N, \mathbb{Q})$, the numeraire-rebased gain process

$$N_t^{-1} G_t^\phi = \int_0^t \phi_u \cdot \mathrm{d}(N_u^{-1} A_u)$$

is an $\{\mathscr{F}_t\}_{t\geq 0}$-martingale under $\mathbb{Q}$.

Theorem 4.5 ([13, p. 164, Corollary 7.34]). *Let $\mathcal{E}$ be an economy with a numeraire pair $(N, \mathbb{Q})$ and let V_T be some replicable claim, i.e., if there exists some admissible trading strategy ϕ such that*

$$V_T = \phi_0 \cdot A_0 + \int_0^T \phi_u \cdot \mathrm{d}A_u.$$

Then the value of this claim at time t admits the representation

$$V_t = N_t \mathbb{E}^{\mathbb{Q}}[N_T^{-1} V_T | \mathscr{F}_t].$$

5. Pricing options under geometric Riemannian Brownian motions

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a probability space filtered by the augmentation $\{\mathscr{F}_t\}_{t\geq 0}$ of the natural filtration generated by a Riemannian Brownian motion X on $(\mathbb{R}, g)$ with $X_0 = 0$.

Let us consider a simple economy $\mathcal{E}$. In this economy, there are two assets, the stock and the bond, whose price processes S and D satisfy the following equations

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \circ \mathrm{d}X_t, \tag{5.1}$$

and

$$\mathrm{d}D_t = r D_t \mathrm{d}t, \tag{5.2}$$

respectively. Here, μ, σ and r are some constants. We assume that $\sigma > 0$.

In this section we will show a pricing formula for a European option with maturity T and payoff $h(S_T)$ for some function h (Theorem 5.5). Then we will provide a partial differential equation for European options (Theorem 5.7).

For the above purposes, we need to find a numeraire pair. This will be done in Theorem 5.3.

Let us choose the bond D with one unit payoff at the maturity T as a numeraire, i.e., $D_t = \exp(-r(T-t))$.

By (5.1), we get

$$\mathrm{d}X_t = -\frac{\mu}{\sigma}\mathrm{d}t - \frac{\sigma}{2}\mathrm{d}\langle X\rangle_t + \frac{1}{\sigma S_t}\mathrm{d}S_t.$$

From (3.6), we have

$$\mathrm{d}\langle X\rangle_t = \frac{1}{g(X_t)}\mathrm{d}t.$$

Inserting these two equalities into (3.6), we find that

$$\frac{\mathrm{d}S_t}{S_t} = \left(\mu + \frac{\sigma^2}{2g(X_t)} - \frac{\sigma}{4}\frac{g'(X_t)}{g(X_t)^2}\right)\mathrm{d}t + \frac{\sigma}{\sqrt{g(X_t)}}\mathrm{d}W_t, \tag{5.3}$$

where W is a standard Euclidean Brownian motion, which is the anti-development of X.

For any process Y, let us denote the numeraire-rebased process $D^{-1}Y$ by Y^D. Then

$$\frac{\mathrm{d}S_t^D}{S_t^D} = \left(\mu - r + \frac{\sigma^2}{2g(X_t)} - \frac{\sigma}{4}\frac{g'(X_t)}{g(X_t)^2}\right)\mathrm{d}t + \frac{\sigma}{\sqrt{g(X_t)}}\mathrm{d}W_t. \tag{5.4}$$

Define a process $\widetilde{W}$ by

$$\widetilde{W}_t := W_t - \int_0^t C_u \mathrm{d}u, \tag{5.5}$$

where the process C will be determined later (see Lemma 5.1).

We insert (5.5) into (5.4), then

$$\frac{\mathrm{d}S_t^D}{S_t^D} = \left(\mu - r + \frac{\sigma^2}{2g(X_t)} - \frac{\sigma}{4}\frac{g'(X_t)}{g(X_t)^2} + \frac{\sigma}{\sqrt{g(X_t)}}C_t\right)\mathrm{d}t + \frac{\sigma}{\sqrt{g(X_t)}}\mathrm{d}\widetilde{W}_t. \tag{5.6}$$

Now we need the following lemma.

Lemma 5.1. *Assume that* $\mathbb{E}^{\mathbb{P}}\left[\left(\exp(\frac{1}{2}\int_0^T C_u^2\mathrm{d}u)\right)\right] < \infty$, *where*

$$C_t := \sqrt{g(X_t)}\left(\frac{r-\mu}{\sigma} - \frac{\sigma}{2g(X_t)} + \frac{1}{4}\frac{g'(X_t)}{g(X_t)^2}\right).$$

Then the measure $\mathbb{Q}$ *determined by*

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(\int_0^T C_u \mathrm{d}W_u - \frac{1}{2}\int_0^T C_u^2\mathrm{d}u\right)$$

is a probability measure on $(\Omega, \mathscr{F}_T)$ *satisfying* $\mathbb{Q} \sim \mathbb{P}$, *and* $(\widetilde{W}_t,\ 0 \le t \le T)$ *is a standard Euclidean Brownian motion under* $\mathbb{Q}$.

Proof. The proof is straight by Novikov Theorem (see [15, p. 198, Proposition 5.12]) and Girsanov Theorem (see [15, p. 191, Theorem 5.1]). □

Corollary 5.2. *The following identity holds,*

$$\mathrm{d}X_t = \left(\frac{r-\mu}{\sigma} - \frac{\sigma}{2g(X_t)}\right)\mathrm{d}t + \frac{1}{\sqrt{g(X_t)}}\mathrm{d}\widetilde{W}_t.$$

Proof. By plugging (5.5) into (3.6) and noting that

$$C_t = \sqrt{g(X_t)}\left(\frac{r-\mu}{\sigma} - \frac{\sigma}{2g(X_t)} + \frac{1}{4}\frac{g'(X_t)}{g(X_t)^2}\right),$$

we complete the proof. □

Theorem 5.3. *Assume that*

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T C_u^2\mathrm{d}u\right)\right] < \infty \text{ and } \mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \sigma^2 g(X_u)^{-1}\mathrm{d}u\right)\right] < \infty,$$

then $(D, \mathbb{Q})$ *is a numeraire pair of the economy* $\mathcal{E}$.

Proof. Under the numeraire D, the price of the bound D is identically equal to one, and $(S_t^D,\ 0 \le t \le T)$ is a martingale under $\mathbb{Q}$ by Novikov Theorem (see [15, p. 198, Proposition 5.12]). □

Corollary 5.4. *Assume that*

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T C_u^2\mathrm{d}u\right)\right] < \infty \text{ and the metric } g \text{ has a positive lower bound.}$$

Let h *be an at most linear growth function. Then the contingent claim* $h(S_T)$ *is replicable.*

Proof. 1. Note that $S_T \in L^2(\Omega, \mathscr{F}_T, \mathbb{Q})$ according to [1, p. 373, Corollary 6.2.4]. We have $h(S_T) \in L^2(\Omega, \mathscr{F}_T, \mathbb{Q})$, since h is an at most linear growth function. Then we can appeal to the martingale representation theorem (see [1, p. 303, Theorem 5.3.5]) in the probability space $(\Omega, \mathscr{F}_T, \mathbb{Q})$ to deduce that there is a square integrable process $\rho := (\rho_t, 0 \leq t \leq T)$ such that

$$\mathrm{d}H_t^D = \rho_t \mathrm{d}\widetilde{W}_t,$$

where $H_t := \exp(-r(T-t))\mathbb{E}^{\mathbb{Q}}[h(S_T)|\mathscr{F}_t]$.

2. Define $\phi_t^1 := \rho_t \sqrt{g(X_t)}/(\sigma S_t^D)$ and $\phi_t^2 := H_t^D - \phi_t^1 S_t^D$. Then we find that

$$h(S_T) = H_T^D = \phi_T^1 S_T + \phi_T^2 D_T$$

and

$$\phi^1(t) S_t^D + \phi_t^2 = H_t^D,$$

which implies that the numeraire-rebased gain process $(\phi_t^1 S_t^D + \phi_t^2, 0 \leq t \leq T)$ is a martingale under the probability measure $\mathbb{Q}$.

Therefore, in order to prove the claim $h(S_T)$ is replicable, we only need to verify that the process $\phi := (\phi^1, \phi^2)$ is self-financing.

Note that $H_t = D_t H_t^D$. Then we have

$$\begin{aligned}
\mathrm{d}H_t &= H_t^D \mathrm{d}D_t + D_t \mathrm{d}H_t^D \\
&= (\phi_t^1 S_t^D + \phi_t^2)\mathrm{d}D_t + D_t \phi_t^1 \frac{\sigma S_t^D}{\sqrt{g(X_t)}} \mathrm{d}\widetilde{W}_t \\
&= \phi_t^1 \big(S_t^D \mathrm{d}D_t + D_t \frac{\sigma S_t^D}{\sqrt{g(X_t)}} \mathrm{d}\widetilde{W}_t\big) + \phi_t^2 \mathrm{d}D_t \\
&= \phi_t^1 \mathrm{d}(D_t S_t^D) + \phi_t^2 \mathrm{d}D_t \\
&= \phi_t^1 \mathrm{d}S_t + \phi_t^2 \mathrm{d}D_t,
\end{aligned}$$

where we have used (5.6) for the third equality.

Thus we have verified that the process ϕ is self-financing. □

Theorem 5.5. *Let V be a replicable European option with maturity T and payoff $h(S_T)$ for some function h. Then the value of V at time zero is given by*

$$V_0 = \exp(-rT)\mathbb{E}^{\mathbb{Q}}[h(S_T)],$$

where

$$S_T = \exp(rT) S_0 \exp\left(\int_0^T \frac{\sigma}{\sqrt{g(X_u)}} \mathrm{d}\widetilde{W}_u - \frac{1}{2}\int_0^T \frac{\sigma^2}{g(X_u)} \mathrm{d}u\right). \tag{5.7}$$

Proof. By (5.6), we have

$$S_T^D = S_0^D \exp\left(\int_0^T \frac{\sigma}{\sqrt{g(X_u)}} \mathrm{d}\widetilde{W}_u - \frac{1}{2}\int_0^T \frac{\sigma^2}{g(X_u)} \mathrm{d}u\right).$$

Note that

$$S_T^D = S_T, \ S_0^D = \exp(rT) S_0.$$

These imply (5.7). The proof is completed by Theorem 4.5. □

In the rest of this section, we will provide a partial differential equation for European options. The following lemma tells us the form of European option values.

Lemma 5.6. *The value of the option V in Theorem 5.5 at time t takes the following form*

$$V_t = H(t, S_t) \text{ for some function } H.$$

Proof. 1. Note that $\sigma > 0$. Then from Equation (5.1), we have

$$\mathrm{d}X_t = -\frac{\mu}{\sigma}\mathrm{d}t + \frac{1}{\sigma S_t} \circ \mathrm{d}S_t.$$

It follows that $\mathscr{F}_t \subset \mathscr{F}_t^S$, where $\{\mathscr{F}_t^S\}_{t\geq 0}$ is the augmentation of the natural filtration generated by the process S. On the other hand, Equation (5.1) implies $\mathscr{F}_t^S \subset \mathscr{F}_t$. Thus $\mathscr{F}_t^S = \mathscr{F}_t$.

2. Now we have

$$\begin{aligned} V_t &= \exp(-r(T-t))\mathbb{E}^{\mathbb{Q}}[h(S_T)|\mathscr{F}_t] \\ &= \exp(-r(T-t))\mathbb{E}^{\mathbb{Q}}[h(S_T)|\mathscr{F}_t^S] \\ &= \exp(-r(T-t))\mathbb{E}^{\mathbb{Q}}[h(S_T)|S_t] \\ &= H(t, S_t) \quad \text{for some function } H, \end{aligned}$$

where we have used the Markov property of the process S for the third equality and Doob-Dynkin lemma for the last equality. □

Theorem 5.7. *Assume that g, the derivative g' and the inverse $1/g$ of g are bounded smooth functions and h is an at most linear growth function. Then the function H in Lemma 5.6 is the unique solution of the following PDE*

$$\frac{\partial H}{\partial t} + \frac{1}{2}\frac{\sigma^2 x^2}{g(\frac{1}{\sigma}(\log x - \log S_0 - \mu t))}\frac{\partial^2 H}{\partial x^2} + rx\frac{\partial H}{\partial x} - rH = 0 \tag{5.8}$$

with the terminal condition $H(T, x) = h(x)$.

Proof. 1. Note that H is a solution of Equation (5.8) with the terminal condition $H(T, x) = h(x)$ if and only if $F(t, y) := H(T-t, \exp(y))$ is a solution of the following initial value problem

$$\begin{cases} \dfrac{\partial F}{\partial t} = \dfrac{1}{2}\dfrac{\sigma^2}{g(\frac{1}{\sigma}(y - \log S_0 - \mu(T-t)))}\dfrac{\partial^2 F}{\partial y^2} \\ \qquad\qquad + \left(r - \dfrac{1}{2}\dfrac{\sigma^2}{g(\frac{1}{\sigma}(y - \log S_0 - \mu(T-t)))}\right)\dfrac{\partial F}{\partial y} - rF \\ F(0, y) = h(\exp(y)). \end{cases}$$

According to [7, pp. 141–142, Theorem 4.5], the above initial value problem has a solution, say F. In addition, F is at most exponential growth with respect to y. Thus there exists a solution, say H, to Equation (5.8) with the terminal condition $H(T, x) = h(x)$. Furthermore, H is of at most linear growth with respect to x.

2. Equation (5.3) can be written as

$$\frac{\mathrm{d}S_t}{S_t} = r\mathrm{d}t + \frac{\sigma}{\sqrt{g(\frac{1}{\sigma}(\log S_t - \log S_0 - \mu t))}}\mathrm{d}\widetilde{W}_t.$$

Therefore, by Feynman-Kac formula (see, for example, [15, p. 366, Theorem 7.6]), the function H in Lemma 5.6 is the unique solution of Equation (5.8) with the terminal condition $H(T, x) = h(x)$. □

It is not easy to obtain explicit solutions of Equation 5.8 for general Riemannian metric g. We here provide a semi-explicit expression for the solution using the fundamental solution technique.

Define $U(t, x) := \exp(rt)H(T - t, S_0 \exp(\mu(T - t) + \sigma x))$. Then U satisfies

$$\begin{cases} \dfrac{\partial U}{\partial t} = \dfrac{1}{2g(x)}\dfrac{\partial^2 U}{\partial x^2} - \dfrac{\sigma}{2g(x)}\dfrac{\partial U}{\partial x} \\ U(0, y) = h(S_0 \exp(rT + \sigma x)). \end{cases}$$

We follow [6, Chapter 9] to construct the fundamental solution of the equation

$$\frac{\partial U}{\partial t} = \frac{1}{2g(x)}\frac{\partial^2 U}{\partial x^2} - \frac{\sigma}{2g(x)}\frac{\partial U}{\partial x}. \tag{5.9}$$

First, let $Z(t, x - \xi; \tau, y)$ be the fundamental solution of the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2g(y)}\frac{\partial^2 u}{\partial x^2},$$

i.e.,

$$Z(t, x - \xi; \tau, y) = \sqrt{\frac{g(y)}{2\pi(t - \tau)}} \exp\left[-\frac{g(y)(x - \xi)^2}{2(t - \tau)}\right].$$

Second, define K as follows:

$$K(t, x; \tau, \xi) := \left(\frac{1}{2g(x)}\frac{\partial^2}{\partial x^2} - \frac{\sigma}{2g(x)}\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right) Z(t, x - \xi; \tau, \xi).$$

Third, define Φ through

$$\Phi(t, x; \tau, \xi) = \sum_{n=1}^{\infty} K_n(t, x; \tau, \xi),$$

where $K_1 = K$ and for $n \geq 2$,

$$K_n(t, x; \tau, \xi) = \int_\tau^t \int_{\mathbb{R}} K_1(t, x; \theta, y)K_{n-1}(\theta, y; \tau, \xi)\mathrm{d}y\mathrm{d}\theta.$$

Finally, the fundamental solution Γ of Equation (5.9) is

$$\Gamma(t, x; \tau, \xi) = Z(t, x - \xi; \tau, \xi) + \int_\tau^t \int_{\mathbb{R}} Z(t, x - y; \theta, y)\Phi(\theta, y; \tau, \xi)\mathrm{d}y\mathrm{d}\theta.$$

Now we get a consequence which shows a form of the function H in Theorem 5.7.

Corollary 5.8. *Assume that the conditions in Theorem 5.7 are fulfilled. Then the following equality holds,*

$$H(t,x) = \exp(-r(T-t)) \int_{\mathbb{R}} \Gamma(T-t, \sigma^{-1}(\log(x/S_0) - \mu t); 0, \xi) \times h(S_0 \exp(rT + \sigma\xi))\mathrm{d}\xi.$$

6. Choices of Riemannian metric on $\mathbb{R}$

The distribution of realistic rates of stock return has a character of leptokurtosis and fat-tail. A proper Riemannian metric g on $\mathbb{R}$ implies a distribution which has the above character. Here is a numerical experiment.

In this numerical experiment, we choose the parameters as follows, $\mu = 0.09$, $\sigma = 0.2$, and $g(x) = 0.1 + 3\sin^2(8x)$. See Figure 1.

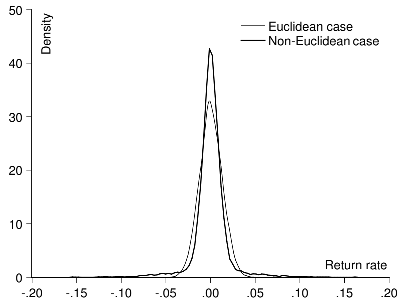

FIGURE 1. Leptokurtosis and fat-tail

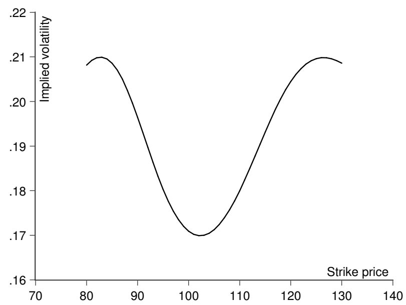

FIGURE 2. Implied volatility smile

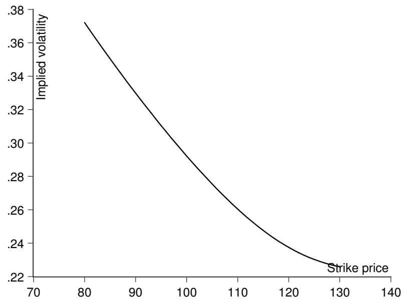

FIGURE 3. Implied volatility skew

We investigate option pricing bias and implied volatility smile. In this numerical experiment, we consider a European call option and choose the parameters as follows, $\mu = 0.09$, $\sigma = 0.2$, $r = 0.08, S_0 = 100$, $T = 0.25$ and $g(x) = 1.5 - \sin^2(2x)$. See Table 1 and Figure 2.

TABLE 1. Pricing options by BS formula and Equation (5.8)

Strike price	90	92	94	96	98	100
BS formula	12.2533	10.5627	8.9767	7.5143	6.1909	5.0170
Equation (5.8)	12.2233	10.4518	8.7539	7.1636	5.7193	4.4562
Strike price	102	104	106	108	110	
BS formula	3.9972	3.1302	2.4088	1.8216	1.3537	
Equation (5.8)	3.3976	2.5484	1.8943	1.4069	1.0514	

We examine implied volatility skew by considering a European call option and choosing the parameters as follows, $\mu = 0.09$, $\sigma = 0.2$, $r = 0.08, S_0 = 100$, $T = 0.25$ and $g(x) = 1/(1 + (x - 1)^2)$. See Figure 3.

7. Conclusions

In this paper, we have introduced a new class of models for stock prices (see Equation (5.1)). The dynamic equation (5.1) of price processes is chosen by the motivation that the model satisfies the three criteria: (1) explaining the character of leptokurtosis and fat-tail of the distribution of realistic rates of stock return, (2) explaining option pricing bias and implied volatility smile (skew), and (3) obtaining a simple pricing equation for European options (see Equation (5.8)).

With proper choices of parameters, our model achieves (1) and (2) mentioned above (see Section 6). We also have investigated the existence and the uniqueness of solutions to the partial differential equation (5.8), and provided an expression of the solution to the equation with some terminal condition by means of the fundamental solution technique.

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