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WEAK HOPF ALGEBRAS CORRESPONDING TO NON-STANDARD QUANTUM GROUPS

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Abstract. We construct a weak Hopf algebra $\mathfrak{w}X_q(A_1)$ corresponding to non-standard quantum group $X_q(A_1)$. The PBW basis of $\mathfrak{w}X_q(A_1)$ is described and all the highest weight modules of $\mathfrak{w}X_q(A_1)$ are classified. Finally we give the Clebsch-Gordan decomposition of the tensor product of two highest weight modules of $\mathfrak{w}X_q(A_1)$.

Introduction

In this paper, we always assume that the base closed field is $\mathbb{F}$ with characteristic 0. All algebras, modules are over the field $\mathbb{F}$. The parameter $q \in \mathbb{F}$ is non-zero and not a root of unity.

Quantum groups play an important role in mathematics and physics. A new quantum group was constructed in [2] solving exotic solution of quantum Yang-Baxter equation. This new quantum group is called the non-standard quantum group. Jing et al. [4] derived a new quantum group $X_q(2)$ by employing the FRT method. All finite dimensional irreducible representations of $X_q(2)$ were classified. It is noted that dimensions of the irreducible representations are only one or two. In 1993, Aghamohammadi et al. (see [1]) used the method of FRT to obtain the non-standard quantum group $X_q(A_{n-1})$ corresponding to type A_{n-1}. Note that $X_q(A_1)$ is just quantum algebra $X_q(2)$. It is shown that this kind of quantum group has a Hopf algebra structure (see [3, 5]). On the other hand, Li defined a kind of weak Hopf algebra on a bialgebra with a weak antipode in [6] and many interesting results are obtained. Yang constructed weak Hopf algebras corresponding to Cartan matrices in [9] and gave their PBW bases. It is noted that finite dimensional integrable representations of $\mathfrak{w}sl_q(2)$ were described and the decomposition of the tensor product of two finite dimensional integrable modules were considered in [10].

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In this paper, we intend to study the weak Hopf algebra structure corresponding to the non-standard quantum group $X_q(A_1)$. By definition, $X_q(A_1)$ is the associative algebra over the field $\mathbb{F}$ with 1 generated by six generators $K_1^{\pm 1}, K_2^{\pm 1}, E, F$ with the following relations

$$K_iK_i^{-1} = K_i^{-1}K_i = 1, i = 1, 2,\ K_1K_2 = K_2K_1,$$

$$K_1E = q_1^{-1}EK_1,\ K_1F = q_1FK_1,$$

$$K_2E = q_2EK_2,\ K_2F = q_2^{-1}FK_2,$$

$$E^2 = F^2 = 0,$$

$$EF - FE = \frac{K_2K_1^{-1} - K_1K_2^{-1}}{q - q^{-1}},$$

where $q_1 = q$ and $q_2 = -q^{-1}$.

First we add a centeral generator J and weaken the group-likes to get an algebra $\mathfrak{w}X_q(A_1)$. It is verified that $\mathfrak{w}X_q(A_1)$ is a weak Hopf algebra but not a Hopf algebra. Then the PBW basis of $\mathfrak{w}X_q(A_1)$ is given in the similar way as [9]. We also give the sufficient and necessary conditions of the isomorphism between $\mathfrak{w}X_q(A_1)$ and $\mathfrak{w}X_p(A_1)$ as weak Hopf algebras. By applying the idea in [10] and some well-known facts, we can construct all highest weight representations of $\mathfrak{w}X_q(A_1)$ and the Clebsch-Gordan decomposition of $\mathfrak{w}X_q(A_1)$-modules. It is indicated that the indecomposable modules of $\mathfrak{w}X_q(A_1)$ are not necessarily irreducible. These results for $\mathfrak{w}X_q(A_1)$ are not the same as those in [10]. In fact they just extend the results in [4].

The paper is arranged as follows. In Section 1, we introduce some notions and define the algebra $\mathfrak{w}X_q(A_1)$, then we prove that $\mathfrak{w}X_q(A_1)$ is a weak Hopf algebra. In Section 2, We investigate the PBW basis of $\mathfrak{w}X_q(A_1)$. In Section 3, we describe the conditions of the weak Hopf isomorphisms between $\mathfrak{w}X_q(A_1)$ and $\mathfrak{w}X_p(A_1)$. In Section 4, we classify all the highest weight modules of $\mathfrak{w}X_q(A_1)$. Then in Section 5, we give the Clebsch-Gordan decomposition of tensor product of two highest weight modules of $\mathfrak{w}X_q(A_1)$.

1. Preliminaries

In this section, we construct the weak Hopf algebra $\mathfrak{w}X_q(A_1)$ by weaken K_i of $X_q(A_1)$ and the defining relation $K_iK_i^{-1} = K_i^{-1}K_i = 1$ $(i = 1, 2)$. Firstly, we replace $\{K_i,\ K_i^{-1} \mid i = 1, 2\}$ by $\{K_i,\ \overline{K}_i \mid i = 1, 2\}$ and introduce the new generator J such that

$$K_i\overline{K}_i = \overline{K}_iK_i = J \quad (i = 1, 2).$$

Secondly, we give the following the definition.

Definition 1.1 (see [9])**.** If E satisfies

$$K_1E = q_1^{-1}EK_1,\ K_2E = q_2EK_2 \ \text{ and } \ \overline{K}_1E = q_1E\overline{K}_1,\ \overline{K}_2E = q_2^{-1}E\overline{K}_2,$$

we say that E is of type I. If E satisfies

$$K_1E\overline{K}_1 = q_1^{-1}E,\ K_2E\overline{K}_2 = q_2E,$$

we say that E is of type II.

Similarly, we can define F is of type I (type II). That is, if F satisfies

$$K_1F = q_1FK_1,\ K_2F = q_2^{-1}FK_2 \ \text{ and } \ \overline{K}_1F = q_1^{-1}F\overline{K}_1,\ \overline{K}_2F = q_2F\overline{K}_2,$$

we say that F is of type I. If F satisfies

$$K_1F\overline{K}_1 = q_1F,\ K_2F\overline{K}_2 = q_2^{-1}F,$$

we say that F is of type II.

Notation. (See [9]) The notation $d = (k|\overline{k}),\ k, \overline{k} = 0$ or 1 indicated that if $k = 1$ (resp. 0), the corresponding generator E is of type I (resp. type II), and if $\overline{k} = 1$ (resp. 0), the corresponding generator F is of type II (resp. type I). The information before $|$ is related to E. The information after $|$ is related to F. E and F are said to be of type d if E and F are of type I or type II according to d.

Now, we can give the definition of the algebra $\mathfrak{w}X_q(A_1)$.

Definition 1.2. The algebra $\mathfrak{w}X_q(A_1)$ is defined as an associative algebra over the field $\mathbb{F}$ with 1 generated by $J, K_1, K_2, \overline{K}_1, \overline{K}_2, E, F$ with the relations

$$K_1K_2 = K_2K_1,\ \overline{K}_1\overline{K}_2 = \overline{K}_2\overline{K}_1,\ K_i\overline{K}_j = \overline{K}_jK_i,\ i, j = 1, 2,$$

$$K_i\overline{K}_i = J = K_i\overline{K}_i,\ K_iJ = JK_i = K_i,\ \overline{K}_iJ = J\overline{K}_i = \overline{K}_i,\ i = 1, 2,$$

$$E \text{ and } F \text{ are of type } d,$$

$$E^2 = F^2 = 0,$$

$$EF - FE = \frac{K_2\overline{K}_1 - K_1\overline{K}_2}{q - q^{-1}}.$$

In this case, we say $\mathfrak{w}X_q(A_1)$ is of type d.

Lemma 1.3. *In $\mathfrak{w}X_q(A_1)$ of type d, the following statements hold.*

(1) *$J, 1 - J$ are idempotent elements.*
(2) *J is in the center of $\mathfrak{w}X_q(A_1)$.*
(3) *If E (resp. F) is of type II, then it enjoys type I.*
(4)

$$K_1^nE^m = q_1^{-mn}E^mK_1^n,\ K_1^nF^m = q_1^{mn}F^mK_1^n,$$

$$K_2^nE^m = q_2^{mn}E^mK_2^n,\ K_2^nF^m = q_2^{-mn}F^mK_2^n,$$

$$\overline{K}_1^nE^m = q_1^{mn}E^m\overline{K}_1^n,\ \overline{K}_1^nF^m = q_1^{-mn}F^m\overline{K}_1^n,$$

$$\overline{K}_2^nE^m = q_2^{-mn}E^m\overline{K}_2^n,\ \overline{K}_2^nF^m = q_2^{mn}F^m\overline{K}_2^n.$$

Proof. (1) Easy.

(2) By definition, we have

$$K_iJ = JK_i,\ \overline{K}_iJ = J\overline{K}_i.$$

If E is type I, then

$$JE = \overline{K}_1K_1E = q_1^{-1}\overline{K}_1EK_1 = q_1q_1^{-1}E\overline{K}_1K_1 = EJ.$$

If E is type II, then

$$JE = K_1\overline{K}_1E = q_1K_1\overline{K}_1K_1E\overline{K}_1 = q_1K_1E\overline{K}_1K_1\overline{K}_1 = EK_1\overline{K}_1 = EJ.$$

It is similar to get $JF = FJ$. Therefore, J is in the center of $\mathfrak{w}X_q(A_1)$.

(3) If E is type II, the relation $K_1E\overline{K}_1 = q_1^{-1}E$ implies that $K_1E\overline{K}_1K_1 = q_1^{-1}EK_1$. The left hand side is

$$K_1EJ = K_1JE = K_1E.$$

Hence, we get $K_1E = q_1^{-1}EK_1$. Similarly, $K_2E = q_2EK_2$.

For the generator F, the statement is similar to prove.

(4) Straightforward. □

The concept of weak Hopf algebra was defined by [6], and was studied by [7, 9]. By definition a weak Hopf algebra W is a bialgebra with a weak antipode T such that $T * Id * T = T$ and $Id * T * Id = Id$, where $*$ is the multiplication of convolution algebra $\mathrm{Hom}_{\mathbb{F}}(W, W)$.

In the following, we can equip a coalgebra structure with $\mathfrak{w}X_q(A_1)$ such that $\mathfrak{w}X_q(A_1)$ is a weak Hopf algebra. Indeed, we define the coalgebra structure in $\mathfrak{w}X_q(A_1)$ as follows.

The comultiplication $\Delta :\ \mathfrak{w}X_q(A_1) \longrightarrow \mathfrak{w}X_q(A_1) \otimes \mathfrak{w}X_q(A_1)$ is

$$\Delta(J) = J \otimes J\ ,\Delta(K_i) = K_i \otimes K_i,\ \Delta(\overline{K}_i) = \overline{K}_i \otimes \overline{K}_i,\ i = 1, 2;$$
$$\Delta(E) = \begin{cases} (K_1\overline{K}_2) \otimes E + E \otimes 1, & \text{if } E \text{ is of type I,} \\ (K_1\overline{K}_2) \otimes E + E \otimes J, & \text{if } E \text{ is of type II;} \end{cases}$$
$$\Delta(F) = \begin{cases} 1 \otimes F + F \otimes (K_2\overline{K}_1), & \text{if } F \text{ is of type I,} \\ J \otimes F + F \otimes (K_2\overline{K}_1), & \text{if } F \text{ is of type II.} \end{cases}$$

The counit $\varepsilon :\ \mathfrak{w}X_q(A_1) \longrightarrow \mathbb{F}$ is

$$\varepsilon(K_i) = \varepsilon(\overline{K}_i) = \varepsilon(J) = 1,\ i = 1, 2;$$
$$\varepsilon(E) = \varepsilon(F) = 0.$$

It is obvious that $\mathfrak{w}X_q(A_1)$ is a coalgebra by the definition of Δ and ε. In fact:

Theorem 1.4. *Keeping all notations as above. Then $\mathfrak{w}X_q(A_1)$ is a weak Hopf algebra with $J \neq 1$, the comultiplication Δ, counit ε and weak antipode T, but it is not a Hopf algebra.*

Proof. Indeed, it is straightforward to see that $\mathfrak{w}X_q(A_1)$ is a bialgebra (as the proof in [9, Theorem 3.1]). To see that $\mathfrak{w}X_q(A_1)$ is a weak Hopf algebra, we need to find a weak antipode T such that $T * Id * T = T$ and $Id * T * Id = Id$. For the purpose, we define $T: \mathfrak{w}X_q(A_1) \longrightarrow \mathfrak{w}X_q(A_1)$ by

$$T(J) = J,\ T(K_i) = \overline{K}_i,\ T(\overline{K}_i) = K_i,\ i = 1, 2,$$
$$T(E) = -\overline{K}_1 K_2 E,\ T(F) = -F K_1 \overline{K}_2.$$

The left is to prove T is an weak antipode of $\mathfrak{w}X_q(A_1)$. The proof is more or less the same as that in [9, Theorem 3.1].

We now prove that $\mathfrak{w}X_q(A_1)$ is not a Hopf algebra. Otherwise, we assume that $\mathfrak{w}X_q(A_1)$ is a Hopf algebra and $S : \mathfrak{w}X_q(A_1) \longrightarrow \mathfrak{w}X_q(A_1)$ is an antipode. Then $(S * id)(J) = u\varepsilon(J) = (id * S)(J)$ implies that $S(J)J = 1 = JS(J)$. It follows that J is invertible. However, $J(1 - J) = 0$ and $J \neq 1$. It is contradiction. Therefore, $\mathfrak{w}X_q(A_1)$ is a weak Hopf algebra not a Hopf algebra. □

2. The PBW basis of $\mathfrak{w}X_q(A_1)$

Let $\omega_q = \mathfrak{w}X_q(A_1)J$, $\overline{\omega}_q = \mathfrak{w}X_q(A_1)(J - 1)$, we have:

Proposition 2.1. *Assume that $\mathfrak{w}X_q(A_1)$ is of type d. Then $\mathfrak{w}X_q(A_1) = \omega_q \oplus \overline{\omega}_q$ as algebras. Furthermore, ω_q and $X_q(A_1)$ are isomorphic as Hopf algebras.*

Proof. It is easy to see that

$$\mathfrak{w}X_q(A_1) = \omega_q \oplus \overline{\omega}_q$$

as algebras for J is a center idempotent element. Consider the algebra ω_q, it can be viewed as an algebra generated by $EJ, FJ, K_1, K_2, \overline{K}_1, \overline{K}_2$, satisfying the following relations:

$$K_1K_2 = K_2K_1,\ \overline{K}_1\overline{K}_2 = \overline{K}_2\overline{K}_1,\ K_i\overline{K}_j = \overline{K}_jK_i,\ i, j = 1, 2,$$
$$K_1\overline{K}_1 = J = K_2\overline{K}_2,\ K_iJ = JK_i = K_i,\ \overline{K}_iJ = J\overline{K}_i = \overline{K}_i,\ i = 1, 2,$$
$$K_1EJ = q_1^{-1}EJK_1,\ K_1FJ = q_1FJK_1,$$
$$K_2EJ = q_2EJK_2,\ K_2FJ = q_2^{-1}FJK_2,$$
$$\overline{K}_1EJ = q_1EJ\overline{K}_1,\ \overline{K}_1FJ = q_1^{-1}FJ\overline{K}_1,$$
$$\overline{K}_2EJ = q_2^{-1}EJ\overline{K}_2,\ \overline{K}_2FJ = q_2FJ\overline{K}_2,$$
$$(EJ)^2 = (FJ)^2 = 0,$$
$$(EJ)(FJ) - (FJ)(EJ) = \frac{K_2\overline{K}_1 - K_1\overline{K}_2}{q - q^{-1}},$$

where J is the identity of ω_q. By the comultiplication of $\mathfrak{w}X_q(A_1)$, it is deduced in $\mathfrak{w}X_q(A_1)$ that

$$\Delta(K_i) = K_i \otimes K_i,\ \Delta(\overline{K}_i) = \overline{K}_i \otimes \overline{K}_i,\ i = 1, 2,$$
$$\Delta(EJ) = (K_1\overline{K}_2) \otimes EJ + EJ \otimes J,$$
$$\Delta(FJ) = J \otimes FJ + FJ \otimes (K_2\overline{K}_1),$$

$$\varepsilon(K_i) = \varepsilon(\overline{K}_i) = \varepsilon(J) = 1,\ i = 1,2,\ \varepsilon(EJ) = \varepsilon(FJ) = 0,$$
$$T(J) = J, T(K_i) = \overline{K}_i,\ T(\overline{K}_i) = K_i,\ i = 1,2,$$
$$T(EJ) = -\overline{K}_1 K_2(EJ),\ T(FJ) = -(FJ)K_1\overline{K}_2.$$

Let $\rho : X_q(A_1) \longrightarrow \omega_q$ be the map defined by

$$\rho(K_i') = K_i,\ \rho(K_i'^{-1}) = \overline{K}_i,\ i = 1,2,\ \rho(E') = EJ,\ \rho(F') = FJ,$$

where $K_i', K_i^{-1} (i = 1,2)$, E', and F' are the generators of $X_q(A_1)$. It is straightforward to see that ρ is a well-defined surjective algebra homomorphism.

Let $\phi : \mathfrak{w}X_q(A_1) \longrightarrow X_q(A_1)$ be a map given by

$$\phi(1) = 1,\ \phi(J) = 1,\ \phi(E) = E,\ \phi(F) = F,\ \phi(K_i) = K_i,\ \phi(\overline{K}_i) = K_i^{-1}.$$

We can check that ϕ is a well-defined algebra homomorphism. If we consider the restricted homomorphism $\phi|_{\omega_q}$, then we have $\phi|_{\omega_q} \circ \rho = id_{X_q(A_1)}$. Hence ρ is injective. Therefore, $\omega_q \cong X_q(A_1)$. □

It is noted that

$$\mathfrak{w}X_q(A_1)/\langle J-1 \rangle \cong X_q(A_1)$$

as Hopf algebras, where $\langle J-1 \rangle$ is the two-sided ideal generated by $J-1$ (see the proof of Proposition 2.1).

Let us describe the structure of $\overline{\omega}_q$.

- If E (resp. F) is of type II, then $E(1-J) = 0$ (resp. $F(1-J) = 0$). Indeed, if E is of type II, then $q_1^{-1}E = K_1E\overline{K}_1 = K_1E\overline{K}_1J = q_1^{-1}EJ$ and $E(1-J) = 0$. Similarly for F.
- If E (resp. F) is of type I, then $E(1-J) \neq 0$ (resp. $F(1-J) \neq 0$). To see this, if E and F are of type $d = (1|1)$, we apply the actions of $E(1-J)$ and $F(1-J)$ on the $\mathfrak{w}X_q(A_1)$-module $M(1,1)$ in Section 4, we have $E(1-J)X^0Y^0 = X^1Y^0 \neq 0$ and $F(1-J)X^0Y^0 = X^0Y^1 \neq 0$. Hence $E(1-J) \neq 0$ and $F(1-J) \neq 0$.

If E (resp. F) is of type I, we assume $X = E(1-J)$ (resp. $Y = F(1-J)$). There are the following four cases.

(1) If $d = (1 \mid 1)$, then $\overline{\omega}_q = \mathbb{F}X + \mathbb{F}Y + \mathbb{F}XY + \mathbb{F}(1-J)$. It is easy to see that $XY = YX$;
(2) If $d = (0 \mid 0)$, then $\overline{\omega}_q = \mathbb{F}(1-J)$;
(3) If $d = (1 \mid 0)$, then $\overline{\omega}_q = \mathbb{F}X + \mathbb{F}(1-J)$;
(4) If $d = (0 \mid 1)$, then $\overline{\omega}_q = \mathbb{F}Y + \mathbb{F}(1-J)$.

Let $X_q^+(A_1)$ (resp. $X_q^-(A_1)$, and $X_q^0(A_1)$) be the subalgebra generated by E (resp. F, and $K_1^{\pm 1}, K_2^{\pm 1}$). Considering the $X_q^+(A_1)$-module V with basis $\{v_0, v_1\}$, defined by $Ev_0 = 0, Ev_1 = v_0, 1v_i = v_i$ $(i = 0,1)$, accordingly we have $\{1, E\}$ is a basis of $X_q^+(A_1)$. Similarly, $\{1, F\}$ is a basis of $X_q^-(A_1)$. On the other hand, $X_q^0(A_1) \cong \mathbb{F}[K_1^{\pm 1}, K_2^{\pm 1}]$ as $\mathbb{F}$-algebras, where $\mathbb{F}[K_1^{\pm 1}, K_2^{\pm 1}]$ is the algebra of Laurent polynomials. Hence, $\{K_1^m K_2^n \mid m,n\in\mathbb{Z}\}$ is a basis of $X_q^0(A_1)$. Moreover, one has

$$X_q(A_1)\cong X_q^-(A_1)\otimes X_q^0(A_1)\otimes X_q^+(A_1).$$

To see these, one can refer to the statements of [3, Lemma 4.14–Theorem 4.21].

We set

$$P_i^{s_i}=\begin{cases} K_i^{s_i}, & \text{if } s_i>0,\\ J, & \text{if } s_i=0,\\ \overline{K_i}^{-s_i}, & \text{if } s_i<0.\end{cases} \tag{2.1}$$

We denote $P^{\boldsymbol{s}}=P_1^{s_1}P_2^{s_2}$ if $\boldsymbol{s}=(s_1,s_2)$. It is easy to see $P^{\boldsymbol{s}}$ is the basis of ω_q^0.

By Proposition 2.1, we have:

Proposition 2.2. *Assume that $\mathfrak{w}X_q(A_1)$ is of type d. Then the set*

$$\{F^bP^{\boldsymbol{s}}E^aJ \mid \boldsymbol{s}=(s_1,s_2)\in\mathbb{Z}\times\mathbb{Z},\ \text{and } a,b\in\mathbb{Z}_2\}\bigcup\{0\neq F^bE^a(1-J)\mid a,b\in\mathbb{Z}_2\}$$

forms a basis of $\mathfrak{w}X_q(A_1)$.

3. The isomorphisms among weak quantum algebras

We assume that $X_p(A_1)$ is generated by $E',F',K_i',K_i'^{-1},\ i=1,2$. The defining relations and comultiplications of $X_p(A_1)$ are the same as those of $X_q(A_1)$ replaced q by p.

In this section, we give the sufficient and necessary conditions as weak Hopf algebra isomorphisms between $\mathfrak{w}X_q(A_1)$ and $\mathfrak{w}X_p(A_1)$.

In first, we recall some concepts about group-like elements and primitive elements of a coalgebra.

Let C be a coalgebra, $x\in C$. If $\Delta(x)=x\otimes x$, and $\epsilon(x)=1$, then x is called a group-like element in C. Let $G(C)$ denote the set of group-like elements. Let $g,\ h\in G(C)$. If

$$\Delta(x)=g\otimes x+x\otimes h,$$

then x is called a $(g:h)$-primitive element. Let $P_{g,h}(C)$ denote the space consisting of $(g:h)$-primitive elements.

Lemma 3.1. *The space of $(K_1^{l_1}K_2^{l_2}:1)$-primitive elements of $X_q(A_1)$ is*

$$P_{K_1^{l_1}K_2^{l_2},1}(X_q(A_1))=\begin{cases}\mathbb{F}E+\mathbb{F}FK_1K_2^{-1}+\mathbb{F}(1-K_1K_2^{-1}), & \text{if } l_1=1,l_2=-1,\\ \mathbb{F}(1-K_1^{l_1}K_2^{l_2}), & \text{others.}\end{cases}$$

Proof. Assume that $x\in X_q(A_1)$ is a $(K_1^{l_1}K_2^{l_2}:1)$-primitive element, then

$$\Delta(x)=K_1^{l_1}K_2^{l_2}\otimes x+x\otimes 1.$$

We suppose that

$$x=\sum_{i,j\in\mathbb{Z}_2,\ m_1,m_2}a_{i,j,m_1,m_2}E^iF^jK_1^{m_1}K_2^{m_2},$$

we have

$$\begin{aligned}\Delta(x) &= \Delta\left(\sum_{i,j,m_1,m_2} a_{i,j,m_1,m_2} E^i F^j K_1^{m_1} K_2^{m_2}\right)\\
&= \sum_{m_1,m_2} a_{0,0,m_1,m_2} K_1^{m_1} K_2^{m_2} \otimes K_1^{m_1} K_2^{m_2}\\
&\quad + \sum_{m_1,m_2} a_{1,0,m_1,m_2} \left(K_1^{m_1+1} K_2^{m_2-1} \otimes E K_1^{m_1} K_2^{m_2} + E K_1^{m_1} K_2^{m_2} \otimes K_1^{m_1} K_2^{m_2}\right)\\
&\quad + \sum_{m_1,m_2} a_{0,1,m_1,m_2} \left(K_1^{m_1} K_2^{m_2} \otimes F K_1^{m_1} K_2^{m_2} + F K_1^{m_1} K_2^{m_2} \otimes K_1^{m_1-1} K_2^{m_2+1}\right)\\
&\quad + \sum_{m_1,m_2} a_{1,1,m_1,m_2} \big(K_1^{m_1+1} K_2^{m_2-1} \otimes EF K_1^{m_1} K_2^{m_2}\\
&\qquad + K_1 K_2^{-1} F K_1^{m_1} K_2^{m_2} \otimes E K_1^{m_1-1} K_2^{m_2+1}\\
&\qquad + E K_1^{m_1} K_2^{m_2} \otimes F K_1^{m_1} K_2^{m_2} + EF K_1^{m_1} K_2^{m_2} \otimes K_1^{m_1-1} K_2^{m_2+1}\big). \qquad (3.1)\end{aligned}$$

On the other hand

$$\begin{aligned}K_1^{l_1} K_2^{l_2} \otimes x + x \otimes 1 &= K_1^{l_1} K_2^{l_2} \otimes \sum a_{0,0,m_1,m_2} K_1^{m_1} K_2^{m_2}\\
&\quad + K_1^{l_1} K_2^{l_2} \otimes \sum a_{1,0,m_1,m_2} E K_1^{m_1} K_2^{m_2}\\
&\quad + K_1^{l_1} K_2^{l_2} \otimes \sum a_{0,1,m_1,m_2} F K_1^{m_1} K_2^{m_2}\\
&\quad + K_1^{l_1} K_2^{l_2} \otimes \sum a_{1,1,m_1,m_2} EF K_1^{m_1} K_2^{m_2}\\
&\quad + \sum a_{0,0,m_1,m_2} K_1^{m_1} K_2^{m_2} \otimes 1\\
&\quad + \sum a_{1,0,m_1,m_2} E K_1^{m_1} K_2^{m_2} \otimes 1\\
&\quad + \sum a_{0,1,m_1,m_2} F K_1^{m_1} K_2^{m_2} \otimes 1\\
&\quad + \sum a_{1,1,m_1,m_2} EF K_1^{m_1} K_2^{m_2} \otimes 1. \qquad (3.2)\end{aligned}$$

Comparing the equations (3.1) and (3.2), we have if $l_1 = 1$ and $l_2 = -1$, then x can be written as

$$aE + bFK_1K_2^{-1} + c(1 - K_1K_2^{-1}),\ a, b, c \in \mathbb{F}.$$

If $l_1 \neq 1$ or $l_2 \neq -1$, then x can be written as

$$x = d(1 - K_1^{l_1} K_2^{l_2}),\ d \in \mathbb{F}.$$

Therefore, we finish the proof. □

We now give the first main result.

Proposition 3.2. *$X_p(A_1) \cong X_q(A_1)$ as Hopf algebras if and only if $p = \pm q^{\pm 1}$.*

Proof. ($\Rightarrow$) Let $\phi : X_p(A_1) \longrightarrow X_q(A_1)$ be a Hopf algebra isomorphism. Then ϕ must map group-like elements to group-like elements. Therefore we can assume that

$$\phi(K_1') = K_1^{m_1}K_2^{m_2},\ \phi(K_2') = K_1^{n_1}K_2^{n_2}.$$

Then we have

$$\begin{aligned}\Delta(\phi(E')) &= (\phi\otimes\phi)(\Delta(E')) = \phi({K'}_1{K'}_2^{-1})\otimes\phi(E') + \phi(E')\otimes 1\\ &= K_1^{m_1-n_1}K_2^{m_2-n_2}\otimes\phi(E') + \phi(E')\otimes 1.\end{aligned}$$

So $\phi(E')$ is a $(K_1^{m_1-n_1}K_2^{m_2-n_2} : 1)$-primitive element. By Lemma 3.1, if $m_1 - n_1 \neq 1$, or $m_2 - n_2 \neq -1$, we can assume $\phi(E') = d(1 - K_1^{m_1-n_1}K_2^{m_2-n_2}) \neq 0$. This contradicts to the fact that $\phi(K_1')\phi(E') = p^{-1}\phi(E')\phi(K_1')$.

Now, we focus on $m_1 - n_1 = 1, m_2 - n_2 = -1$. By Lemma 3.1, we can assume that

$$\phi(E') = aE + bFK_1K_2^{-1} + c(1 - K_1K_2^{-1}).$$

Applying the algebra isomorphism ϕ to the relation $K_1'E' = p^{-1}E'K_1'$, we get

$$\begin{aligned}\phi(K_1')\phi(E') &= K_1^{m_1}K_2^{m_2}(aE + bFK_1K_2^{-1} + c(1 - K_1K_2^{-1}))\\ &= aK_1^{m_1}K_2^{m_2}E + bK_1^{m_1}K_2^{m_2}FK_1K_2^{-1}\\ &\quad + cK_1^{m_1}K_2^{m_2}(1 - K_1K_2^{-1})\\ &= (-1)^{-m_2}aq^{-m_1-m_2}EK_1^{m_1}K_2^{m_2}\\ &\quad + (-1)^{m_2}bq^{m_1+m_2}FK_1^{m_1+1}K_2^{m_2-1}\\ &\quad + cK_1^{m_1}K_2^{m_2}(1 - K_1K_2^{-1}),\\ p^{-1}\phi(E')\phi(K_1') &= K_1^{m_1}K_2^{m_2}(aE + bFK_1K_2^{-1} + c(1 - K_1K_2^{-1}))\\ &= p^{-1}(aE + bFK_1K_2^{-1} + c(1 - K_1K_2^{-1}))K_1^{m_1}K_2^{m_2}\\ &= p^{-1}aEK_1^{m_1}K_2^{m_2} + p^{-1}bFK_1^{m_1+1}K_2^{m_2-1}\\ &\quad + p^{-1}c(1 - K_1K_2^{-1})K_1^{m_1}K_2^{m_2}\\ \Longrightarrow \qquad (-1)^{-m_2}aq^{-m_1-m_2} &= p^{-1}a,\ (-1)^{m_2}bq^{m_1+m_2} = p^{-1}b,\ c = p^{-1}c.\end{aligned}$$

Hence $c = 0$ since p and q are not a root of unity.

(1) If $a \neq 0$, then

$$(-1)^{m_2}q^{m_1+m_2} = p,\ b = 0,\ \phi(E') = aE.$$

Let us determine $\phi(F')$ as follows. Since $F'K_1'K_2'^{-1}$ is a $(K_1'K_2'^{-1} : 1)$-primitive element, we can assume that

$$\phi(F'K_1'K_2'^{-1}) = a'E + b'FK_1K_2^{-1} + c'(1 - K_1K_2^{-1}) = \phi(F')K_1K_2^{-1}.$$

This implies that

$$\phi(F') = b'FK_1^{1-(m_1-n_1)}K_2^{-1-(m_2-n_2)} = b'F$$

by the defining relations. Moreover, applying ϕ to the relation

$$E'F' - F'E' = \frac{K_2'K_1'^{-1} - K_1'K_2'^{-1}}{p - p^{-1}},$$

we get that

$$b' = \frac{q - q^{-1}}{a(p - p^{-1})}, \text{ and that } \phi(F') = \frac{q - q^{-1}}{a(p - p^{-1})}F.$$

Therefore, we may assume that

$$m_1 + m_2 = n_1 + n_2 = l, m_2 = m.$$

Then $(-1)^m q^l = p$, the corresponding isomorphism has the form

$$\phi(K_1') = K_1^{l-m}K_2^m,\ \phi(K_2') = K_1^{l-m-1}K_2^{m+1},$$
$$\phi(E') = aE,\ \phi(F') = \frac{q - q^{-1}}{a(p - p^{-1})}F,\ (a \neq 0).$$

This isomorphism forces that there are $a, b \in \mathbb{Z}$ such that

$$\phi({K_1'}^a)\phi({K_2'}^b) = K_1 \text{ or } \phi({K_1'}^a)\phi({K_2'}^b) = K_2.$$

It concludes that $a(l - m) + b(l - m - 1) = 1$, $am + b(m + 1) = 0$ or $a(l - m) + b(l - m - 1) = 0$, $am + b(m + 1) = 1$. For the first case, we have $l = 1$, $a = 1 + m$, $b = -m$, or $l = -1$, $a = -1 - m$, $b = m$. For the last case, we have $l = 1$, $a = m$, $b = 1 - m$, or $l = -1$, $a = -2 - m$, $b = m + 1$. Therefore $p = (-1)^m q^{\pm 1}$.

If $p = (-1)^m q$, then we get the weak Hopf algebra isomorphism

$$\phi(K_1') = K_1^{1-m}K_2^m,\ \phi(K_2') = K_1^{-m}K_2^{m+1},$$
$$\phi(E') = aE,\ \phi(F') = (-1)^m a^{-1}F,\ (a \neq 0).$$

The inverse ϕ' of ϕ is

$$\phi'(K_1) = (K_1')^{1+m}(K_2')^{-m},\ \phi'(K_2) = (K_1')^m(K_2')^{1-m},$$
$$\phi'(E) = a^{-1}E',\ \phi'(F) = (-1)^m aF'.$$

If $p = (-1)^m q^{-1}$, then we get the weak Hopf algebra isomorphism

$$\phi(K_1') = K_1^{-1-m}K_2^m,\ \phi(K_2') = K_1^{-2-m}K_2^{m+1},$$
$$\phi(E') = aE,\ \phi(F') = (-1)^{m+1}a^{-1}F,\ (a \neq 0).$$

The inverse ϕ' of ϕ is

$$\phi'(K_1) = (K_1')^{-1-m}(K_2')^m,\ \phi'(K_2) = (K_1')^{-2-m}(K_2')^{m+1},$$
$$\phi'(E) = a^{-1}E',\ \phi'(F) = (-1)^{m+1}aF'.$$

(2) If $b \neq 0$, then

$$(-1)^{m_2}q^{m_1+m_2} = p^{-1},\ a = 0,\ \phi(E') = bFK_1K_2^{-1}.$$

We assume that

$$\phi(F'K_1'K_2'^{-1}) = a'E + b'FK_1K_2^{-1} + c'(1 - K_1K_2^{-1}).$$

By the defining relations and more or less than the above discussion, we have

$$\phi(F') = a'EK_1^{-1}K_2.$$

In fact,

$$a' = \frac{q - q^{-1}}{b(p - p^{-1})}$$

by applying the isomorphism ϕ to the relation

$$E'F' - F'E' = \frac{K_2'K_1'^{-1} - K_1'K_2'^{-1}}{p - p^{-1}}.$$

Therefore, we have that in this case

$$\phi(F') = \frac{q - q^{-1}}{b(p - p^{-1})}K_1^{-1}K_2E.$$

Let $m_1 + m_2 = l, m_2 = m$, then $p = (-1)^m q^{-l}$, the corresponding isomorphism

$$\phi(K_1') = K_1^{l-m}K_2^m,\ \phi(K_2') = K_1^{l-m-1}K_2^{m+1},$$
$$\phi(E') = bFK_1K_2^{-1},\ \phi(F') = \frac{q - q^{-1}}{b(p - p^{-1})}EK_1^{-1}K_2, (b \neq 0).$$

The similar arguments as the case (1) show that $p = (-1)^m q^{\pm 1}$.

If $p = (-1)^m q$, we get the weak Hopf algebra isomorphism

$$\phi(K_1') = K_1^{-1-m}K_2^m,\ \phi(K_2') = K_1^{-2-m}K_2^{m+1},$$
$$\phi(E') = bFK_1K_2^{-1},\ \phi(F') = (-1)^m b^{-1}EK_1^{-1}K_2,\ (b \neq 0).$$

The inverse ϕ' of ϕ is

$$\phi'(K_1) = (K_1')^{-1-m}(K_2')^m,\ \phi'(K_2) = (K_1')^{-2-m}(K_2')^{m+1},$$
$$\phi'(E) = (-1)^m bF'K_1'(K_2')^{-1},\ \phi'(F) = b^{-1}E'(K_1')^{-1}K_2'.$$

If $p = (-1)^m q^{-1}$, then we get the weak Hopf algebra isomorphism

$$\phi(K_1') = K_1^{1-m}K_2^m,\ \phi(K_2') = K_1^{-m}K_2^{m+1},$$
$$\phi(E') = bFK_1K_2^{-1},\ \phi(F') = (-1)^{m+1}b^{-1}EK_1^{-1}K_2,\ (b \neq 0).$$

The inverse ϕ' of ϕ is

$$\phi'(K_1) = (K_1')^{1+m}(K_2')^{-m},\ \phi'(K_2) = (K_1')^m(K_2')^{1-m},$$
$$\phi'(E) = (-1)^{m+1}bF'K_1'(K_2')^{-1},\ \phi'(F) = b^{-1}E'(K_1')^{-1}K_2'.$$

($\Leftarrow$) If $p = \pm q^{\pm 1}$, we can assume that $p = (-1)^m q^n (n = \pm 1)$ and define the map $\psi : X_p(A_1) \longrightarrow X_q(A_1)$ as

$$\psi(K_1') = K_1^{n-m}K_2^m,\ \psi(K_2') = K_1^{n-m-1}K_2^{m+1},$$
$$\psi(E') = aE,\ \psi(F') = (-1)^{m+\delta_{-1,n}}a^{-1}F,$$

where

$$\delta_{-1,n} = \begin{cases} 1, & \text{if } n = -1, \\ 0, & \text{if } n \neq -1. \end{cases}$$

It is easy to see that ψ is a Hopf algebra isomorphism. □

Recall that

$$\mathfrak{w}X_q(A_1) \cong \omega_q \oplus \overline{w}_q.$$

Let us consider the weak Hopf algebra isomorphism between $\mathfrak{w}X_q(A_1)$ and $\mathfrak{w}X_p(A_1)$.

Theorem 3.3. *For the weak Hopf algebra $\mathfrak{w}X_q(A_1)$ of type $(1|1)$, we have $\mathfrak{w}X_p(A_1) \cong \mathfrak{w}X_q(A_1)$ as weak Hopf algebras if and only if $p = \pm q^{\pm 1}$.*

Proof. Let $\gamma : \mathfrak{w}X_p(A_1) \longrightarrow \mathfrak{w}X_q(A_1)$ be an isomorphism of weak Hopf algebra. It is easy to see that $\gamma(J') = J$ since γ sends group-likes to group-likes.

By Proposition 2.1 it is well-known that

$$\mathfrak{w}X_p(A_1) = w_p \oplus \overline{w}_p,\ \mathfrak{w}X_q(A_1) = w_q \oplus \overline{w}_q,$$

and $w_p \cong X_p(A_1),\ w_q \cong X_q(A_1)$. Note that $\overline{w}_p$ is spanned by $\{E'^i F'^j (1 - J) \,|\, i, j = 0, 1\}$, and $\overline{w}_q$ is spanned by $\{E^i F^j (1 - J) \,|\, i, j = 0, 1\}$.

Assume that $\mathrm{inj}_p : w_p \longrightarrow \mathfrak{w}X_p(A_1)$ is defined by

$$J' \longmapsto J',\ E'J' \longmapsto E'J',\ F'J' \longmapsto F'J',\ K'_i \longmapsto K'_i,\ \overline{K'}_i \longmapsto \overline{K'}_i,\ i = 1, 2.$$

It is easy to see that inj_p is a bialgebra homomorphism (see [8]). Moreover, we have $w_q = \gamma \circ \mathrm{inj}_p(w_p)$. Since $\mathfrak{w}X_p(A_1) \cong \mathfrak{w}X_q(A_1)$, it follows that $X_p(A_1) \cong X_q(A_1)$. By Proposition 3.2, $p = \pm q^{\pm 1}$.

($\Leftarrow$) Assume that $p = \pm q^{\pm 1}$. Without loss of generality, we assume that $p = (-1)^m q^n (n = \pm 1)$ and define the map $\gamma : \mathfrak{w}X_p(A_1) \longrightarrow \mathfrak{w}X_q(A_1)$ as follows

$$\begin{aligned} &\gamma(1) = 1,\ \gamma(J') = J \\ &\gamma(P'_1) = P_1^{n-m} P_2^m,\ \gamma(P'_2) = P_1^{n-m-1} P_2^{m+1}, \\ &\gamma(E') = E,\ \gamma(F') = (-1)^{m+\delta_{-1,n}} F, \end{aligned}$$

where P_i and P'_i are defined by (2.1) respectively. It is straightforward to see that γ indeed can be extended to a weak Hopf algebra isomorphism.

The proof is finished. □

Remark 3.4. In general, if E, F are of type $(1|0)$, $(0|1)$, or $(0|0)$, more or less the same arguments show that Theorem 3.3 also hold.

4. The representations of $\mathfrak{w}X_q(A_1)$

In this section, we consider the representation theory of $\mathfrak{w}X_q(A_1)$ of type d.

Let V be a $\mathfrak{w}X_q(A_1)$-module and $0 \neq v \in V$. If $K_1v = \lambda_1 v, K_2v = \lambda_2 v$, then $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ is called a weight of V and v is called a weight vector. The subspace

$$\{0\} \neq V_\lambda = \{v \in V \mid K_1v = \lambda_1 v, K_2v = \lambda_2 v\}$$

is called a weight space of $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$. If

$$Ev = 0,\ K_1v = \lambda_1 v,\ K_2v = \lambda_2 v,$$

then v is called a highest weight vector of $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$. If $V = \mathfrak{w}X_q(A_1)v$ and v is a highest weight vector, then V is called a highest weight module of $\mathfrak{w}X_q(A_1)$ generated by the highest weight vector v.

Lemma 4.1. *Let $\mathfrak{w}X_q(A_1)$ be the weak Hopf algebra of type d, V be a $\mathfrak{w}X_q(A_1)$-module and $0 \neq v \in V$. If $K_iv = \lambda_i v,\ i = 1, 2,\ \lambda_i \in \mathbb{F}$, then there are elements $\overline{\lambda}_i \in \mathbb{F}$ such that $\overline{K}_iv = \overline{\lambda}_i v$. Moreover, if $\lambda_i \neq 0$, then $\overline{\lambda}_i = \lambda_i^{-1}$; if $\lambda_i = 0$, then $\overline{\lambda}_i = 0$.*

Proof. Since $K_iv = \lambda_i v$, we have $K_iv = K_i\overline{K}_iK_iv = \overline{K}_i\lambda_i^2 v = \lambda_i v$. Therefore, if $\lambda_i \neq 0$, $\overline{K}_iv = \lambda_i^{-1}v$. If $\lambda_i = 0$, then $\overline{K}_iv = \overline{K}_iK_i\overline{K}_iv = 0$. Hence $\overline{\lambda}_i = 0$. □

Assume that $(\lambda_1, \lambda_2, \delta) \in \mathbb{F}^* \times \mathbb{F}^* \times \{0, 1\}$, $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$, where

$$\delta = \begin{cases} 1, & \text{if } \lambda_1^2 = \lambda_2^2, \\ 0, & \text{if } \lambda_1^2 \neq \lambda_2^2. \end{cases}$$

Suppose $\lambda_1\lambda_2 \neq 0$ let $V_{\lambda_1,\lambda_2,\delta}(n)(n = 0, 1)$ be the $(n+1)$-dimensional vector space with the basis $\{v_i \mid 0 \leq i \leq n\}$. The module structure of $V_{\lambda_1,\lambda_2,\delta}(0)$ is a one-dimensional highest weight $\mathfrak{w}X_q(A_1)$-module with $\delta = 1$ and relations

$$Ev_0 = Fv_0 = 0,\ K_iv_0 = \lambda_i v_0,\ \overline{K}_iv_0 = \overline{\lambda}_i v_0,\ i = 1, 2.$$

The module structure of $V_{\lambda_1,\lambda_2,\delta}(1)$ is defined by

$$K_1v_0 = \lambda_1 v_0,\ \overline{K}_1v_0 = \overline{\lambda}_1 v_0,\ K_2v_0 = \lambda_2 v_0,\ \overline{K}_2v_0 = \overline{\lambda}_2 v_0,$$
$$K_1v_1 = q\lambda_1 v_1,\ \overline{K}_1v_1 = q^{-1}\overline{\lambda}_1 v_1,\ K_2v_1 = -q\lambda_2 v_1,\ \overline{K}_2v_1 = -q^{-1}\overline{\lambda}_2 v_1,$$
$$Ev_0 = 0,\ Ev_1 = \frac{\overline{\lambda}_1\lambda_2 - \lambda_1\overline{\lambda}_2}{q - q^{-1}}v_0,$$
$$Fv_0 = v_1,\ Fv_1 = 0.$$

In fact, when $\lambda_1\lambda_2 \neq 0$, we have $\overline{\lambda}_1\lambda_2 = \lambda_1\overline{\lambda}_2 \Leftrightarrow \lambda_1^2 = \lambda_2^2$.

Lemma 4.2. *Assume that $\mathfrak{w}X_q(A_1)$ is the weak Hopf algebra of any type d and $\lambda_1\lambda_2 \neq 0$. Let V be a highest weight $\mathfrak{w}X_q(A_1)$-module generated by a highest weight vector v_0 with weight $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$. Then*

(1) $V \cong V_{\lambda_1,\lambda_2,\delta}(n)(n = 0, 1)$;

(2) $V_{\lambda_1,\lambda_2,\delta}(n) \cong V_{\lambda_1',\lambda_2',\delta'}(n)(n = 0, 1)$ *as $\mathfrak{w}X_q(A_1)$-modules if and only if $(\lambda_1, \lambda_2, \delta) = (\lambda_1', \lambda_2', \delta')$.*

Proof. Straightforward. □

Assume that $\lambda_1\lambda_2 = 0$ and $\mathfrak{w}X_q(A_1)$ is a weak Hopf algebra of type $d = (0|1)$ or $(1|1)$. Let $W(n)(n = 0, 1)$ be the $(n + 1)$-dimensional vector space with the basis $\{w_i | 0 \leq i \leq n\}$. It is noted that if $\lambda_1\lambda_2 = 0$ and $W(n)$ is a $\mathfrak{w}X_q(A_1)$-module, both λ_1 and λ_2 must be zero since $K_1\overline{K}_1 = \overline{K}_2K_2 = J$. In this case, the $\mathfrak{w}X_q(A_1)$-module structure on $W(n)$ is given as follows

$$K_1w_i = K_2w_i = 0,\ \overline{K}_1w_i = \overline{K}_2w_i = 0,\ 0 \leq i \leq n,$$
$$Ew_i = 0,\ 0 \leq i \leq n,$$
$$Fw_j = w_{j+1},\ 0 \leq j \leq n-1,$$
$$Fw_n = 0.$$

Remark 4.3. If $\mathfrak{w}X_q(A_1)$ is a weak Hopf algebra with $d = (1|0)$ or $(0|0)$, we only can define the $\mathfrak{w}X_q(A_1)$-module $W(0)$. For, if F is of type II, then $K_1F\overline{K}_1w_0 = q_1Fw_0 = 0$ and $Fw_0 = 0$. On the other hand, if $\mathfrak{w}X_q(A_1)$ is of type $d = (0|1)$ or $(1|1)$, then $W(1)$ is an indecomposable $\mathfrak{w}X_q(A_1)$-module of dimension 2, but is not simple since $W(0)$ is a proper submodule of $W(1)$.

Theorem 4.4. *Assume that $\mathfrak{w}X_q(A_1)$ is the weak Hopf algebra of type $d = (k|\bar{k})$. Let M be a highest weight $\mathfrak{w}X_q(A_1)$-module. Then $M \cong W(t)(0 \leq t \leq \bar{k})$ or $M \cong V_{\lambda_1,\lambda_2,\delta}(n)$, where $n = 0, 1$.*

Proof. Since M is a highest weight $\mathfrak{w}X_q(A_1)$-module, M has a highest weight vector v_0 such that $M = \mathfrak{w}X_q(A_1)v_0$, and

$$Ev_0 = 0,\ K_iv_0 = \lambda_iv_0,\ i = 1, 2.$$

Let $\lambda_1\lambda_2 \neq 0$. By Lemma 4.2, we have $M \cong V_{\lambda_1,\lambda_2,\delta}(n)(n = 0, 1)$.

Let $\lambda_1\lambda_2 = 0$. If F is of type II, then we have $Fv_0 = 0$ because of the relations $K_1F\overline{K}_1 = q_1F$ and $K_2F\overline{K}_2 = q_2^{-1}F$. Hence we obtain $M \cong W(0)$. If F is of type I, it is easy to check that $M \cong W(0)$ when $\dim M = 1$. If $\dim M \neq 1$, we have $Fv_0 \neq 0$ by Proposition 2.2. If $Fv_0 = av_0$ for some non-zero $a \in \mathbb{F}$, then $FFv_0 = a^2v_0 = 0$ and it is a contradiction. So $\{v_0, Fv_0\}$ is linearly independent. If we take $v_1 = Fv_0$, then we have

$$Ev_0 = 0,\ Ev_1 = EFv_0 = FEv_0 = 0,$$
$$Fv_0 = v_1,\ Fv_1 = 0.$$

Since M is generated by v_0, we have $M \cong W(1)$.

In conclusion, $M \cong W(t)(0 \leq t \leq \bar{k})$ or $M \cong V_{\lambda_1,\lambda_2,\delta}(n),\ n = 0, 1$. □

Assume $\eta_1^2 = \eta_2^2$, $\mathfrak{w}X_q(A_1)$ is of type $d = (k|\overline{k})$. Let $M_{\eta_1,\eta_2}(m, n)$ be a vector space spanned by $\{X^iY^j \,|\, 0 \leq i \leq m, 0 \leq j \leq n\}$, where $0 \leq m \leq k,\ 0 \leq n \leq \overline{k}$. Then it is straightforward to see that $M_{\eta_1,\eta_2}(m, n)$ is a $\mathfrak{w}X_q(A_1)$-module defined by

$$K_1(X^iY^j) = q^{j-i}\eta_1X^iY^j,\ K_2(X^iY^j) = (-q)^{j-i}\eta_2X^iY^j,$$
$$\overline{K}_1(X^iY^j) = q^{i-j}\overline{\eta}_1X^iY^j,\ \overline{K}_2(X^iY^j) = (-q)^{i-j}\overline{\eta}_2X^iY^j,$$

$$E(X^iY^j) = X^{i+1}Y^j,\ 0 \le i < m,\ E(X^mY^j) = 0,$$
$$F(X^iY^j) = X^iY^{j+1},\ 0 \le j < n,\ F(X^iY^n) = 0.$$

Remark 4.5. If $\eta_1 = \eta_2 = 0$, we denote $M_{0,0}(m,n)$ by $M(m,n)$ for simplicity. Specially, $M(0,n) \cong W(n)$. Under the condition of $\eta_1 = \eta_2 = 0$, if $\mathfrak{w}X_q(A_1)$ is of type $d = (1|1)$, we can define the $\mathfrak{w}X_q(A_1)$-modules $M(0,0)$, $M(1,0)$, $M(0,1)$, $M(1,1)$; if $\mathfrak{w}X_q(A_1)$ is of type $d = (1|0)$, we can define $M(0,0)$, $M(1,0)$; if $\mathfrak{w}X_q(A_1)$ is of type $d = (0|1)$, we can define $M(0,0)$, $M(0,1)$; if $\mathfrak{w}X_q(A_1)$ is of type $d = (0|0)$, we can only define $M(0,0)$.

If we can define $\mathfrak{w}X_q(A_1)$-modules $M_{\eta_1,\eta_2}(1,0), M_{\eta_1,\eta_2}(0,1), M_{\eta_1,\eta_2}(1,1)$ for some type d, then they are indecomposable and $M_{\eta_1,\eta_2}(0,0)$ is simple. For example, assume that $\mathfrak{w}X_q(A_1)$ is of type $d = (1|1)$. Let $0 \ne M_1$ be any submodule of $M_{\eta_1,\eta_2}(1,1)$. For any $0 \ne x \in M_1$, x can be written as

$$x = a_{00}X^0Y^0 + a_{10}X^1Y^0 + a_{01}X^0Y^1 + a_{11}X^1Y^1.$$

There is at least a nonzero coefficient. It yields that $X^1Y^1 \in M_1$ for all cases. This means that $\mathbb{F}X^1Y^1$ is the submodule of any non-zero submodule of $M_{\eta_1,\eta_2}(1,1)$. Hence $M_{\eta_1,\eta_2}(1,1)$ is indecomposable. The other cases are similar to see.

5. The Clebsch-Gordan decomposition for $\mathfrak{w}X_q(A_1)$

In this section, we assume that the weak Hopf algebra $\mathfrak{w}X_q(A_1)$ is of type $(1|1)$ and consider tensor products of their two the highest weight $\mathfrak{w}X_q(A_1)$-modules.

Let V and W be two $\mathfrak{w}X_q(A_1)$-modules, recall that $V\otimes W$ is also a $\mathfrak{w}X_q(A_1)$-module defined by

$$E(v\otimes w) = K_1\overline{K}_2 v \otimes Ew + Ev \otimes w,$$
$$F(v\otimes w) = v\otimes Fw + Fv \otimes \overline{K}_1K_2 w,$$
$$K_i(v\otimes w) = K_iv \otimes K_iw,$$
$$\overline{K}_i(v\otimes w) = \overline{K}_iv \otimes \overline{K}_iw.$$

We denote

$$mW(n) = \underbrace{W(n)\oplus W(n) \oplus \cdots \oplus W(n)}_{m\ copies}.$$

Theorem 5.1. *Assume that the weak Hopf algebra $\mathfrak{w}X_q(A_1)$ is of type $d = (1|1)$. Then*

(1) $V_{\lambda_1,\lambda_2,\delta}(m) \otimes V_{\lambda_1',\lambda_2',\delta'}(n) \cong V_{\lambda_1\lambda_1',\lambda_2\lambda_2',\delta\delta'}(m+n),\ m+n \le 1;$

(2) *If $\lambda_1^2{\lambda_1'}^2 \ne \lambda_2^2{\lambda_2'}^2$, then*

$$V_{\lambda_1,\lambda_2,0}(1)\otimes V_{\lambda_1',\lambda_2',\delta'}(1) \cong V_{\lambda_1\lambda_1',\lambda_2\lambda_2',0}(1) \oplus V_{q\lambda_1\lambda_1',(-q)\lambda_2\lambda_2',0}(1);$$

if $\lambda_1^2{\lambda_1'}^2 = \lambda_2^2{\lambda_2'}^2$, then

$$V_{\lambda_1,\lambda_2,0}(1)\otimes V_{\lambda_1',\lambda_2',\delta'}(1) \cong M_{q\lambda_1\lambda_1',(-q)\lambda_2\lambda_2'}(1,1);$$

(3) $V_{\lambda_1,\lambda_2,1}(1) \otimes V_{\lambda'_1,\lambda'_2,\delta'}(1) \cong V_{\lambda_1\lambda'_1,\lambda_2\lambda'_2,\delta'}(1) \oplus V_{q\lambda_1\lambda'_1,(-q)\lambda_2\lambda'_2,\delta'}(1)$;
(4) $V_{\lambda_1,\lambda_2,1}(m) \otimes W(n) \cong (m+1)W(n)$, $V_{\lambda_1,\lambda_2,0}(1) \otimes W(n) \cong M(1,n)$;
(5) $W(0) \otimes V_{\lambda_1,\lambda_2,\delta}(n) \cong W(n)$, $W(1) \otimes V_{\lambda_1,\lambda_2,\delta}(n) \cong (n+1)W(1)$;
(6) $W(m) \otimes W(n) \cong (m+1)W(n)$,

where $m, n = 0$ *or* 1.

Proof. Keeping all notations as Section 4.

(1) We consider the following cases, the others can be obtained in a similar way.

Case 1. For $V_{\lambda_1,\lambda_2,1}(0) \otimes V_{\lambda'_1,\lambda'_2,1}(1)$, we have

$$K_i(v_0 \otimes v'_0) = \lambda_i\lambda'_i v_0 \otimes v'_0, \overline{K}_i(v_0 \otimes v'_0) = \overline{\lambda}_i\overline{\lambda}'_i v_0 \otimes v'_0,$$
$$E(v_0 \otimes v'_0) = 0, E(v_0 \otimes v'_1) = 0, F(v_0 \otimes v'_0) = v_0 \otimes v'_1, F(v_0 \otimes v'_1) = 0.$$

So

$$V_{\lambda_1,\lambda_2,1}(0) \otimes V_{\lambda'_1,\lambda'_2,1}(1) \cong V_{\lambda_1\lambda'_1,\lambda_2\lambda'_2,1}(1).$$

Case 2. For $V_{\lambda_1,\lambda_2,1}(0) \otimes V_{\lambda'_1,\lambda'_2,0}(1)$, note that

$$K_i(v_0 \otimes v'_0) = \lambda_i\lambda'_i v_0 \otimes v'_0,\ \overline{K}_i(v_0 \otimes v'_0) = \overline{\lambda}_i\overline{\lambda}'_i v_0 \otimes v'_0,$$
$$E(v_0 \otimes v'_0) = 0,\ F(v_0 \otimes v'_0) = v_0 \otimes v'_1,\ F(v_0 \otimes v'_1) = 0,$$
$$E(v_0 \otimes v'_1) = K_1\overline{K}_2 v_0 \otimes Ev'_1 = \frac{\overline{\lambda}_1\overline{\lambda}'_1\lambda_2\lambda'_2 - \lambda_1\lambda'_1\overline{\lambda}_2\overline{\lambda}'_2}{q-q^{-1}} v_0 \otimes v'_0 \neq 0.$$

Then

$$V_{\lambda_1,\lambda_2,1}(0) \otimes V_{\lambda'_1,\lambda'_2,0}(1) \cong V_{\lambda_1\lambda'_1,\lambda_2\lambda'_2,0}(1).$$

Case 3. Considering $V_{\lambda_1,\lambda_2,0}(1) \otimes V_{\lambda'_1,\lambda'_2,1}(0)$, note that

$$K_i(v_0 \otimes v'_0) = \lambda_i\lambda'_i v_0 \otimes v'_0,\ \overline{K}_i(v_0 \otimes v'_0) = \overline{\lambda}_i\overline{\lambda}'_i v_0 \otimes v'_0,$$
$$E(v_0 \otimes v'_0) = 0,\ F(v_0 \otimes v'_0) = \overline{\lambda}'_1\overline{\lambda}'_2 v_1 \otimes v'_0,\ F(\overline{\lambda}'_1\overline{\lambda}'_2 v_1 \otimes v'_0) = 0,$$
$$E(F(v_0 \otimes v'_0)) = \overline{\lambda}'_1\overline{\lambda}'_2(Ev_1 \otimes v'_0) = \frac{\overline{\lambda}_1\overline{\lambda}'_1\lambda_2\lambda'_2 - \lambda_1\lambda'_1\overline{\lambda}_2\overline{\lambda}'_2}{q-q^{-1}} v_0 \otimes v'_0 \neq 0.$$

So

$$V_{\lambda_1,\lambda_2,0}(1) \otimes V_{\lambda'_1,\lambda'_2,1}(0) \cong V_{\lambda_1\lambda'_1,\lambda_2\lambda'_2,0}(1).$$

For $V_{\lambda_1,\lambda_2,1}(0) \otimes V_{\lambda'_1,\lambda'_2,1}(0)$ and $V_{\lambda_1,\lambda_2,1}(1) \otimes V_{\lambda'_1,\lambda'_2,1}(0)$, we also can get the similar result.

It follows that

$$V_{\lambda_1,\lambda_2,\delta}(m) \otimes V_{\lambda'_1,\lambda'_2,\delta'}(n) \cong V_{\lambda_1\lambda'_1,\lambda_2\lambda'_2,\delta\delta'}(m+n),\ m+n \leq 1.$$

(2) Considering $V_{\lambda_1,\lambda_2,\delta}(1) \otimes V_{\lambda'_1,\lambda'_2,\delta'}(1)$, we have

$$K_i(v_0 \otimes v'_0) = \lambda_i\lambda'_i v_0 \otimes v'_0,\ \overline{K}_i(v_0 \otimes v'_0) = \overline{\lambda}_i\overline{\lambda}'_i v_0 \otimes v'_0,$$
$$E(v_0 \otimes v'_0) = K_1\overline{K}_2 v_0 \otimes Ev'_0 + Ev_0 \otimes v'_0 = 0,$$
$$F(v_0 \otimes v'_0) = v_0 \otimes Fv'_0 + Fv_0 \otimes \overline{K}_1K_2 v'_0 = v_0 \otimes v'_1 + \overline{\lambda}'_1\lambda'_2 v_1 \otimes v'_0,$$

$$E(F(v_0 \otimes v_0')) = E(v_0 \otimes v_1' + \overline{\lambda}_1'\lambda_2' v_1 \otimes v_0') = \frac{\overline{\lambda}_1\overline{\lambda}_1'\lambda_2\lambda_2' - \lambda_1\lambda_1'\overline{\lambda}_2\overline{\lambda}_2'}{q-q^{-1}} v_0 \otimes v_0',$$

$$F(F(v_0 \otimes v_0')) = F(v_0 \otimes v_1' + \overline{\lambda}_1'\lambda_2' v_1 \otimes v_0') = 0.$$

So $v_0 \otimes v_0'$ is a $\mathfrak{w}X_q(A_1)$-module highest weight vector and

$$\mathfrak{w}X_q(A_1)(v_0 \otimes v_0') \cong V_{\lambda_1\lambda_1', \lambda_2\lambda_2', \delta''}(1),$$

where

$$\delta'' = \begin{cases} 1, & \text{if } \lambda_1^2{\lambda_1'}^2 = \lambda_2^2{\lambda_2'}^2, \\ 0, & \text{if } \lambda_1^2{\lambda_1'}^2 \neq \lambda_2^2{\lambda_2'}^2. \end{cases}$$

Now we consider other submodules of

$$V_{\lambda_1,\lambda_2,\delta}(1) \otimes V_{\lambda_1',\lambda_2',\delta'}(1).$$

If $\delta = 0$, this means that $\overline{\lambda}_1\lambda_2 - \lambda_1\overline{\lambda}_2 \neq 0$, we take

$$\nu_0 = (\overline{\lambda}_1\lambda_2 - \lambda_1\overline{\lambda}_2)v_0 \otimes v_1' - (\overline{\lambda}_1'\lambda_2' - \lambda_1'\overline{\lambda}_2')\lambda_1\overline{\lambda}_2 v_1 \otimes v_0' \neq 0.$$

Then

$$K_1\nu_0 = q\lambda_1\lambda_1'\nu_0, \ K_2(\nu) = -q\lambda_2\lambda_2'\nu_0,$$
$$\overline{K}_1\nu_0 = q^{-1}\overline{\lambda}_1\overline{\lambda}_1'\nu_0, \ \overline{K}_2\nu_0 = -q^{-1}\overline{\lambda}_2\overline{\lambda}_2'\nu_0,$$

and

$$E\nu_0 = 0,$$
$$F\nu_0 = (\lambda_1\lambda_1'\overline{\lambda}_2\overline{\lambda}_2' - \overline{\lambda}_1\overline{\lambda}_1'\lambda_2\lambda_2')v_1 \otimes v_1' := \nu_1,$$
$$E(\nu_1) = E(F(\nu_0)) = \frac{\lambda_1\lambda_1'\overline{\lambda}_2\overline{\lambda}_2' - \overline{\lambda}_1\overline{\lambda}_1'\lambda_2\lambda_2'}{q-q^{-1}}\nu_0,$$
$$F(F(\nu_0)) = 0.$$

If $\lambda_1\lambda_1'\overline{\lambda}_2\overline{\lambda}_2' - \overline{\lambda}_1\overline{\lambda}_1'\lambda_2\lambda_2' \neq 0$, hence $\delta'' = 0$, then ν_0 is another $\mathfrak{w}X_q(A_1)$-module highest weight vector and

$$\mathfrak{w}X_q(A_1)\nu_0 \cong V_{q\lambda_1\lambda_1', (-q)\lambda_2\lambda_2', 0}(1).$$

It follows that

$$V_{\lambda_1,\lambda_2,0}(1) \otimes V_{\lambda_1',\lambda_2',\delta'}(1) \cong V_{\lambda_1\lambda_1',\lambda_2\lambda_2',0}(1) \oplus V_{q\lambda_1\lambda_1',(-q)\lambda_2\lambda_2',0}(1).$$

If $\lambda_1\lambda_1'\overline{\lambda}_2\overline{\lambda}_2' - \overline{\lambda}_1\overline{\lambda}_1'\lambda_2\lambda_2' = 0$, hence $\delta'' = 1$, then ν_0 is a constant multiple of $F(v_0 \otimes v_0')$. We have

$$K_1(v_1 \otimes v_0') = q\lambda_1\lambda_1' v_1 \otimes v_0', \ K_2(v_1 \otimes v_0') = -q\lambda_2\lambda_2' v_1 \otimes v_0',$$

$$E(v_1 \otimes v_0') = Ev_1 \otimes v_0' = \frac{\overline{\lambda}_1\lambda_2 - \lambda_1\overline{\lambda}_2}{q-q^{-1}} v_0 \otimes v_0', E(Ev_1 \otimes v_0') = 0,$$

$$F(v_1 \otimes v_0') = v_1 \otimes Fv_0' = v_1 \otimes v_1', \ F(v_1 \otimes v_1') = 0,$$

$$F(E(v_1 \otimes v_0')) = \frac{\overline{\lambda}_1\lambda_2 - \lambda_1\overline{\lambda}_2}{q-q^{-1}} F(v_0 \otimes v_0'),$$

$$E(v_1 \otimes v_1') = E(F(v_1 \otimes v_0')) = \frac{\overline{\lambda}_1\lambda_2 - \lambda_1\overline{\lambda}_2}{q-q^{-1}} v_0 \otimes v_1' - \frac{\overline{\lambda}_1'\lambda_2' - \lambda_1'\overline{\lambda}_2'}{q-q^{-1}} \lambda_1\overline{\lambda}_2 v_1 \otimes v_0'$$
$$= \frac{\overline{\lambda}_1\lambda_2 - \lambda_1\overline{\lambda}_2}{q-q^{-1}} (v_0 \otimes v_1' + \overline{\lambda}_1'\lambda_2' v_1 \otimes v_0') = F(E(v_1 \otimes v_0')).$$

Let $X^iY^j = E^iF^j(v_1 \otimes v_0')$, where $i, j = 0$ or 1.

$$K_1(X^0Y^0) = q\lambda_1\lambda_1' X^0Y^0,\ K_2(X^0Y^0) = -q\lambda_2\lambda_2' X^0Y^0,$$
$$E(X^0Y^0) = X^1Y^0 = E(v_1 \otimes v_0'),\ E(X^1Y^0) = 0,$$
$$F(X^0Y^0) = X^0Y^1 = F(v_1 \otimes v_0'),\ F(X^0Y^1) = 0,$$
$$E(X^0Y^1) = E(F(v_1 \otimes v_0')) = X^1Y^1 = E(v_1 \otimes v_1'),\ E(X^1Y^1) = 0,$$
$$F(X^1Y^0) = F(E(v_1 \otimes v_0')) = X^1Y^1,\ F(X^1Y^1) = 0.$$

Thus

$$V_{\lambda_1,\lambda_2,0}(1) \otimes V_{\lambda_1',\lambda_2',\delta'}(1) \cong M_{q\lambda_1\lambda_1', -q\lambda_2\lambda_2'}(1,1).$$

(3) Assume that $\lambda_1^2 = \lambda_2^2$, this means that $\delta = 1$. We have

$$K_1(v_1 \otimes v_0') = q\lambda_1\lambda_1' v_1 \otimes v_0', K_2(v_1 \otimes v_0') = -q\lambda_2\lambda_2' v_1 \otimes v_0',$$
$$\overline{K}_1(v_1 \otimes v_0') = q^{-1}\overline{\lambda}_1\overline{\lambda}_1' v_1 \otimes v_0', \overline{K}_2(v_1 \otimes v_0') = -q^{-1}\overline{\lambda}_2\overline{\lambda}_2' v_1 \otimes v_0',$$
$$E(v_1 \otimes v_0') = 0,$$
$$F(v_1 \otimes v_0') = v_1 \otimes v_1',$$
$$E(F(v_1 \otimes v_0')) = E(v_1 \otimes v_1') = \frac{\lambda_1\lambda_1'\overline{\lambda}_2\overline{\lambda}_2' - \overline{\lambda}_1\overline{\lambda}_1'\lambda_2\lambda_2'}{q-q^{-1}} v_1 \otimes v_0',$$
$$F(F(v_1 \otimes v_0')) = 0.$$

So $v_1 \otimes v_0'$ is a $\mathfrak{w}X_q(A_1)$-module highest weight vector and

$$\mathfrak{w}X_q(A_1)(v_1 \otimes v_0') \cong V_{q\lambda_1\lambda_1', (-q)\lambda_2\lambda_2', \delta'}(1).$$

On the other hand, from the proof of the statement (2) we see that

$$\mathfrak{w}X_q(A_1)(v_0 \otimes v_0') \cong V_{\lambda_1\lambda_1', \lambda_2\lambda_2', \delta'}(1).$$

It follows that

$$V_{\lambda_1,\lambda_2,1}(1) \otimes V_{\lambda_1',\lambda_2',\delta'}(1) \cong V_{\lambda_1\lambda_1', \lambda_2\lambda_2', \delta'}(1) \oplus V_{q\lambda_1\lambda_1', (-q)\lambda_2\lambda_2', \delta'}(1).$$

(4) We consider the following cases.

Case 1. For $V_{\lambda_1,\lambda_2,1}(0) \otimes W(0)$, we have

$$K_i(v_0 \otimes w_0) = 0,$$
$$E(v_0 \otimes w_0) = K_1\overline{K}_2 v_0 \otimes Ew_0 + Ev_0 \otimes w_0 = 0,$$
$$F(v_0 \otimes w_0) = v_0 \otimes Fw_0 + Fv_0 \otimes \overline{K}_1K_2 w_0 = 0,$$

hence

$$V_{\lambda_1,\lambda_2,1}(0) \otimes W(0) \cong W(0).$$

Case 2. For $V_{\lambda_1,\lambda_2,1}(1) \otimes W(1)$, we get

$$K_i(v_0 \otimes w_0) = 0,\ K_i(v_1 \otimes w_0) = 0,$$

$$E(v_0 \otimes w_0) = 0,\ F(v_0 \otimes w_0) = v_0 \otimes w_1,$$
$$E(v_0 \otimes w_1) = 0,\ F(v_0 \otimes w_1) = 0,$$
$$E(v_1 \otimes w_0) = 0,\ F(v_1 \otimes w_0) = v_1 \otimes w_1,$$
$$E(v_1 \otimes w_1) = 0,\ F(v_1 \otimes w_1) = 0.$$

Thus

$$V_{\lambda_1,\lambda_2,1}(1) \otimes W(1) \cong 2W(1).$$

Case 3. Considering the case $V_{\lambda_1,\lambda_2,0}(1) \otimes W(0)$. Note that $\lambda_1\overline{\lambda}_2 \neq \overline{\lambda}_1\lambda_2$, we have

$$K_i(v_0 \otimes w_0) = 0,\ K_i(v_1 \otimes w_0) = 0,$$
$$E(v_0 \otimes w_0) = 0,\ F(v_0 \otimes w_0) = 0,$$
$$E(v_1 \otimes w_0) = Ev_1 \otimes w_0 = \frac{\overline{\lambda}_1\lambda_2 - \lambda_1\overline{\lambda}_2}{q - q^{-1}} v_0 \otimes w_0 \neq 0,$$
$$E(E(v_1 \otimes w_0)) = 0,\ F(v_1 \otimes w_0) = v_1 \otimes Fw_0 = 0.$$

Now, we assume that $X^iY^j = E^iF^j(v_1 \otimes w_0)$, where $i = 0$ or 1, and $j = 0$.

$$K_i(X^0Y^0) = 0,$$
$$E(X^0Y^0) = X^1Y^0 = E^1F^0(v_1 \otimes w_0) = E(v_1 \otimes w_0),$$
$$E(X^1Y^0) = E(E(v_1 \otimes w_0)) = 0,$$
$$F(X^0Y^0) = X^0Y^1 = E^0F^1(v_1 \otimes w_0) = F(v_1 \otimes w_0) = 0.$$

Therefore

$$V_{\lambda_1,\lambda_2,1}(1) \otimes W(0) \cong M(1,0).$$

Case 4. For $V_{\lambda_1,\lambda_2,0}(1) \otimes W(1)$, this means that $\overline{\lambda}_1\lambda_2 - \lambda_1\overline{\lambda}_2 \neq 0$. We have

$$K_i(v_i \otimes w_j) = 0,\ E(v_0 \otimes w_0) = 0,\ F(v_0 \otimes w_0) = v_0 \otimes w_1,$$
$$E(v_0 \otimes w_1) = 0,\ F(v_0 \otimes w_1) = 0,$$
$$E(v_1 \otimes w_0) = Ev_1 \otimes w_0 = \frac{\overline{\lambda}_1\lambda_2 - \lambda_1\overline{\lambda}_2}{q - q^{-1}} v_0 \otimes w_0,$$
$$F(v_1 \otimes w_0) = v_1 \otimes Fw_0 = v_1 \otimes w_1,\ F(v_1 \otimes w_1) = 0,$$
$$E(v_1 \otimes w_1) = Ev_1 \otimes w_1 = \frac{\overline{\lambda}_1\lambda_2 - \lambda_1\overline{\lambda}_2}{q - q^{-1}} v_0 \otimes w_1.$$

Let $X^iY^j = E^iF^j(v_1 \otimes w_0)$, where $i, j = 0$ or 1.

$$K_i(X^0Y^0) = 0,$$
$$E(X^0Y^0) = X^1Y^0 = E^1F^0(v_1 \otimes w_0) = E(v_1 \otimes w_0),$$
$$E(X^1Y^0) = E(E(v_1 \otimes w_0)) = 0,$$
$$E(X^0Y^1) = X^1Y^1 = E^1F^1(v_1 \otimes w_0) = E(v_1 \otimes w_1),$$
$$E(X^1Y^1) = E(E(v_1 \otimes w_1)) = 0,$$
$$F(X^0Y^0) = X^0Y^1 = E^0F^1(v_1 \otimes w_0) = F(v_1 \otimes w_0),$$
$$F(X^0Y^1) = F(F(v_1 \otimes w_0)) = 0,$$

$$F(X^1Y^0) = X^1Y^1 = E^1F^1(v_1 \otimes w_0) = E(v_1 \otimes w_1),$$
$$F(X^1Y^1) = F(F(v_1 \otimes w_0)) = 0.$$

Therefore

$$V_{\lambda_1,\lambda_2,0}(1) \otimes W(1) \cong M(1,1).$$

For $V_{\lambda_1,\lambda_2,1}(0) \otimes W(1)$ and $V_{\lambda_1,\lambda_2,1}(1) \otimes W(0)$, in a similar way we get

$$V_{\lambda_1,\lambda_2,1}(0) \otimes W(1) \cong W(1),$$
$$V_{\lambda_1,\lambda_2,1}(1) \otimes W(0) \cong W(0) \oplus W(0).$$

(5) Note that $E(W(m) \otimes V_{\lambda_1,\lambda_2,\delta}(n)) = 0$. We consider the action of F on $W(m) \otimes V_{\lambda_1,\lambda_2,\delta}(n)$.

Case 1. Considering $W(0) \otimes V_{\lambda_1,\lambda_2,\delta}(0)$, we have

$$K_i(w_0 \otimes v_0) = 0,\ F(w_0 \otimes v_0) = 0,$$

hence

$$W(0) \otimes V_{\lambda_1,\lambda_2,\delta}(0) \cong W(0).$$

Case 2. For $W(0) \otimes V_{\lambda_1,\lambda_2,\delta}(1)$, it is easy to see that

$$K_i(w_0 \otimes v_0) = 0,$$
$$F(w_0 \otimes v_0) = w_0 \otimes Fv_0 = w_0 \otimes v_1,\ F(w_0 \otimes v_1) = 0.$$

Therefore

$$W(0) \otimes V_{\lambda_1,\lambda_2,\delta}(1) \cong W(1).$$

Case 3. For $W(1) \otimes V_{\lambda_1,\lambda_2,\delta}(0)$, note that $\overline{\lambda}_1\lambda_2 \neq 0$, and we get

$$K_i(w_0 \otimes v_0) = 0,$$
$$F(w_0 \otimes v_0) = w_0 \otimes Fv_0 + Fw_0 \otimes \overline{K}_1K_2v_0 = \overline{\lambda}_1\lambda_2 w_1 \otimes v_0 \neq 0,$$
$$F(\overline{\lambda}_1\lambda_2 w_1 \otimes v_0) = \overline{\lambda}_1\lambda_2 w_1 \otimes Fv_0 = 0.$$

Thus

$$W(1) \otimes V_{\lambda_1,\lambda_2,\delta}(0) \cong W(1).$$

Case 4. Considering the case $W(1) \otimes V_{\lambda_1,\lambda_2,\delta}(1)$, we have

$$K_i(w_0 \otimes v_0) = 0,\ F(w_0 \otimes v_0) = w_0 \otimes v_1 + \overline{\lambda}_1\lambda_2 w_1 \otimes v_0,$$

$$\begin{aligned} F(w_0 \otimes v_1 + \overline{\lambda}_1\lambda_2 w_1 \otimes v_0) &= F(w_0 \otimes v_1) + F(\overline{\lambda}_1\lambda_2 w_1 \otimes v_0) \\ &= Fw_0 \otimes \overline{K}_1K_2v_1 + w_1 \otimes F\overline{\lambda}_1\lambda_2 v_0 = 0. \end{aligned}$$

This means that

$$\mathfrak{w}X_q(A_1)(w_0 \otimes v_0) \cong W(1).$$

Assume that $w = aw_0 \otimes v_1 + bw_1 \otimes v_0,\ b \neq a\overline{\lambda}_1\lambda_2$,

$$K_iw = K_i(aw_0 \otimes v_1 + bw_1 \otimes v_0) = 0,$$
$$Fw = aF(w_0 \otimes v_1) + bF(w_1 \otimes v_0) = (b - a\overline{\lambda}_1\lambda_2)w_1 \otimes v_1 \neq 0,$$
$$F(F(w)) = 0.$$

It follows that $\mathfrak{w}X_q(A_1)w \cong W(1)$. Hence

$$W(1) \otimes V_{\lambda_1,\lambda_2,\delta}(1) = \mathfrak{w}X_q(A_1)(w_0 \otimes v_0) \oplus \mathfrak{w}X_q(A_1)w \cong W(1) \oplus W(1).$$

(6) It is easy to see that $E(W(m) \otimes W(n)) = 0$. Consider the action of F on $W(m) \otimes W(n)$.

Case 1. For $W(0) \otimes W(0)$, $F(w_0 \otimes w_0') = 0$, hence

$$W(0) \otimes W(0) \cong W(0).$$

Case 2. For $W(0) \otimes W(1)$, we have

$$K_i(w_0 \otimes w_0') = 0,$$

$$F(w_0 \otimes w_0') = w_0 \otimes Fw_0' = w_0 \otimes w_1', \ F(w_0 \otimes w_1') = 0.$$

So

$$W(0) \otimes W(1) \cong W(1).$$

Case 3. For $W(1) \otimes W(0)$, we get

$$K_i(w_0 \otimes w_0') = 0, \ F(w_0 \otimes w_0') = w_0 \otimes Fw_0' = 0,$$

$$K_i(w_1 \otimes w_0') = 0, \ F(w_1 \otimes w_0') = w_1 \otimes Fw_0' = 0.$$

Consequently

$$W(1) \otimes W(0) \cong W(0) \oplus W(0).$$

Case 4. For $W(1) \otimes W(1)$, we get

$$K_i(w_0 \otimes w_0') = 0, \ K_i(w_1 \otimes w_0') = 0,$$

$$F(w_0 \otimes w_0') = w_0 \otimes Fw_0' = w_0 \otimes w_1', \ F(w_0 \otimes w_1') = 0,$$
$$F(w_1 \otimes w_0') = w_1 \otimes Fw_0' = w_1 \otimes w_1', \ F(w_1 \otimes w_1') = 0.$$

Therefore

$$W(1) \otimes W(n) \cong W(n) \oplus W(n) = 2W(n).$$

The proof is finished. □

Theorem 5.1 for $\mathfrak{w}X_q(A_1)$ of other types d can be discussed in a similar way. It is noted that if E (resp. F) is of type II, for two $\mathfrak{w}X_q(A_1)$-module V, W, we have to define the $\mathfrak{w}X_q(A_1)$-module on $V \otimes W$ by

$$E(v \otimes w) = K_1\overline{K}_2 v \otimes Ew + Ev \otimes Jw,$$

$$(\text{resp. } F(v \otimes w) = Jv \otimes Fw + Fv \otimes \overline{K}_1 K_2 w).$$

Theorem 5.1 should be restated. Explicitly,

- If $\mathfrak{w}X_q(A_1)$ is of $d = (0|1)$, Theorem 5.1(4) is replaced by
  (4′) $V_{\lambda_1,\lambda_2,\delta}(m) \otimes W(n) \cong (m+1)W(n)$.
- If $\mathfrak{w}X_q(A_1)$ is of $d = (1|0)$, Theorem 5.1(4)(5)(6) are respectively replaced by
  (4′) $V_{\lambda_1,\lambda_2,0}(0) \otimes W(0) \cong W(0)$, $V_{\lambda_1,\lambda_2,\delta}(1) \otimes W(0) \cong M(1,0)$,
  (5′) $W(0) \otimes V_{\lambda_1,\lambda_2,\delta}(n) \cong (n+1)W(0)$,
  (6′) $W(0) \otimes W(0) \cong W(0)$.

- If $\mathfrak{w}X_q(A_1)$ is of $d = (0|0)$, Theorem 5.1(4)(5)(6) are respectively replaced by
  (4′) $V_{\lambda_1,\lambda_2,\delta}(m) \otimes W(0) \cong (m+1)W(0)$,
  (5′) $W(0) \otimes V_{\lambda_1,\lambda_2,\delta}(n) \cong (n+1)W(0)$,
  (6′) $W(0) \otimes W(0) \cong W(0)$.

References

[1] A. Aghamohammadi, V. Karimipour, and S. Rouhani, *The multiparametric non-standard deformation of A_{n-1}*, J. Phys. A: Math. Gen. **26** (1993), no. 3, 75–82.

[2] M. Ge, G. Liu, and K. Xue, *New solutions of Yang-Baxter equations*: *Birman-Wenzl algebra and quantum group structures*, J. Phys. A: Math. Gen. **24** (1991), no. 12, 2679–2690.

[3] J. C. Jantzen, *Lectures on Guantum Groups*, Grad. Stud. Math. 6, Amer. Math. Soc., Providence. RI, 1996.

[4] N. Jing, M. Ge, and Y. Wu, *A new quantum group associated with a "nonstandard" braid group representation*, Lett. Math. Phys. **21** (1991), no. 3, 193–203.

[5] C. Kassel, *Quantum Groups*, GTM 155. Springer-Verlag, 1995.

[6] F. Li, *Weak Hopf algebras and new solutions of Yang-Baxter equation*, J. Algebra **208** (1998), no. 1, 72–100.

[7] F. Li and S. Duplij, *Weak Hopf algebras and singular solutions of quantum Yang-Baxter equation*, Comm. Math. Phys. **225** (2002), no. 1, 191–217.

[8] H. Wang and S. Yang, *The Isomorphisms and the center of weak quantum algebras* $\omega sl_q(2)$, Tamkang J. Math. **36** (2005), no. 4, 365–376.

[9] S. Yang, *Weak Hopf algebras corresponding to Cartan matrices*, J. Math. Phys. **46** (2005), no. 7, 073502, 18 pp.

[10] ———, *The Clebsch-Gordan decomposition for quantum algebra* $\mathfrak{w}sl_q(2)$, Advances in ring theory, 307–316, World Sci. Publ., Hackensack, NJ, 2005.

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