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# MODIFIED DUALITY SCHEME FOR SOLVING MODEL CRACK PROBLEM IN MECHANICS

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Abstract. Duality methods based on modified Lagrangian functional for solving a model crack problem is considered. Without additional assumptions of regularity of the solution of an initial problem duality ratio is established for initial and dual problem.

## Introduction

The classical approach to the crack problem is characterized by the equality type boundary conditions considered at the crack faces, namely, they are assumed to be traction free. This approach cannot guarantee the natural nonpenetration condition between crack faces. Moreover, there are practical examples showing that interpenetration of crack faces may occur.

Recently crack models with nonlinear boundary conditions on crack faces are investigated. Suitable boundary conditions are written as inequalities which provide mutual nonpenetration between crack faces. From the standpoint of mechanics such models are more preferable than the linear classical models.

## 1. Model crack problem

The crack problem for the Poisson equation is considered with inequality type boundary conditions on the crack faces. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\Gamma$ and $\gamma \subset \Omega$ be a cut (crack) inside of $\Omega$. For simplicity we assume

$$\gamma = \{(x_1, x_2) \in \Omega \colon a < x_1 < b, x_2 = 0\}$$

and suppose that both end points $(a, 0)$ and $(b, 0)$ do not belong to the boundary $\Gamma$. Denote $\Omega_\gamma = \Omega \backslash \overline{\gamma}$.

Introduce the feasible displacement set

$$K = \{v \in H^1(\Omega_\gamma) \colon [v] \geq 0 \text{ on } \gamma, v = 0 \text{ on } \Gamma\}$$

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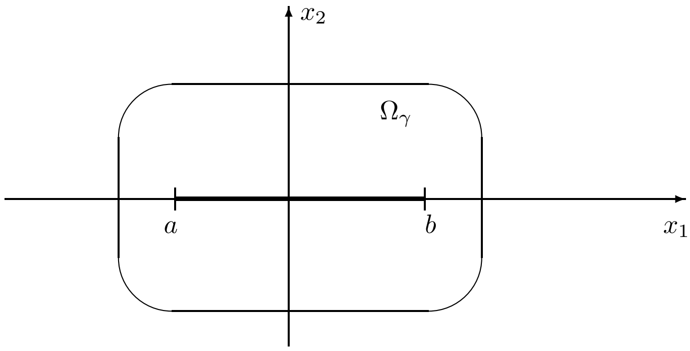


FIGURE 1. Domain with straight line cut

and consider the minimizing problem

$$\left\{ J(v) = \tfrac{1}{2} \int_{\Omega_\gamma} |\nabla v|^2 \, d\Omega - \int_{\Omega_\gamma} f v \, d\Omega \to \min_{v \in K} . \right. \tag{1}$$

Here $[v] = v^+ - v^-$ is the jump of $v$ across $\gamma$ ($v^+$ is a function value $v$ on upper crack face, $v^-$ is a function value $v$ on lower crack face, marks $\pm$ correspond to positive and negative directs of normal vector $n$ on cut $\gamma$); $f \in L_2(\Omega_\gamma)$ is a given function.

Problem (1) has a unique solution $u$, which is, simultaneously, a solution of variational inequality [3]

$$u \in K : \int_{\Omega_\gamma} \nabla u \nabla (v - u) \, d\Omega \geq \int_{\Omega_\gamma} f (v - u) \, d\Omega \qquad \forall v \in K. \tag{2}$$

It can be shown that $u$ is a solution (in the generalized sense) of such boundary value problem [3]

$$\begin{aligned} -\triangle u &= f \text{ in } \Omega_\gamma, \\ u &= 0 \text{ on } \Gamma, \\ [u] \geq 0, \quad \left[\frac{\partial u}{\partial x_2}\right] = 0, \quad \frac{\partial u}{\partial x_2} \leq 0, \quad \frac{\partial u}{\partial x_2} [u] &= 0 \text{ on } \gamma. \end{aligned} \tag{3}$$

In [3] questions of a regularity of the solution of problem (1) are considered, the behavior of the solution and it's derivatives in neighbourhood of crack tops is investigated.

## 2. Duality method for solving the model crack problem

For arbitrary $m \in L_2(\gamma)$ construct the set

$$K_m = \{ v \in H^1(\Omega_\gamma) \colon v = 0 \text{ on } \Gamma, \; -[v] \leq m \text{ a.e. on } \gamma \}.$$

It is easy to show that $K_m$ is a convex closed set in $H^1(\Omega_\gamma)$.

On space $L_2(\gamma)$ define the sensitivity functional

$$\chi(m) = \begin{cases} \inf_{v\in K_m} J(v), & \text{if } K_m \neq \varnothing, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4}$$

Taking into account that $J(v)$ is coercive in $H^1(\Omega_\gamma)$ we note that problem $\inf_{v\in K_m} J(v)$ is solvable under condition $K_m \neq \varnothing$. It is easy to see if $m$ is a lower bounded function, then $K_m$ is not empty. The set $K_m$ can be empty if $m \in L_2(\gamma)\backslash H^{1/2}(\gamma)$ is a lower unbounded function on $\gamma$ [3, 5].

Functional $\chi(m)$ is a proper convex functional on $L_2(\gamma)$, but it's effective domain $\operatorname{dom}\chi = \{m \in L_2(\gamma)\colon \chi(m) < +\infty\}$ does not coincide with $L_2(\gamma)$. Notice that $\operatorname{dom}\chi$ is a convex but not closed set. In this case, $\overline{\operatorname{dom}\chi} = L_2(\gamma)$.

We define the following functional on the space $H^1(\Omega_\gamma) \times L_2(\gamma) \times L_2(\gamma)$ [6, 8]

$$K(v,l,m) = \begin{cases} J(v) + \frac{1}{2r}\int_\gamma \left((l+rm)^2 - l^2\right) d\Gamma, & \text{in } -[v] \leq m \text{ a.e. on } \gamma, \\ +\infty, & \text{otherwise,} \end{cases}$$

and modified Lagrangian functional $M(v,l)$ on space $H^1(\Omega_\gamma) \times L_2(\gamma)$

$$M(v,l) = \inf_{m\in L_2(\gamma)} K(v,l,m) = J(v) + \frac{1}{2r}\int_\gamma \left(\left((l - r[v])^+\right)^2 - l^2\right) d\Gamma.$$

Here $r > 0$ is a constant, $(l - r[v])^+ = \max\{l - r[v], 0\}$.

Let us introduce the modified dual functional

$$\underline{M}(l) = \inf_{v\in H^1(\Omega_\gamma)} M(v,l) = \inf_{v\in H^1(\Omega_\gamma)} \left\{ J(v) + \frac{1}{2r}\int_\gamma \left(\left((l - r[v])^+\right)^2 - l^2\right) d\Gamma \right\}.$$

Since

$$\inf_{v\in H^1(\Omega_\gamma)} \inf_{m\in L_2(\gamma)} K(v,l,m) = \inf_{m\in L_2(\gamma)} \inf_{v\in H^1(\Omega_\gamma)} K(v,l,m),$$

then functional $\underline{M}(l)$ has the another presentation [8]

$$\underline{M}(l) = \inf_{m\in L_2(\gamma)} \left\{ \chi(m) + \int_\gamma lm\, d\Gamma + \frac{1}{2}\int_\gamma m^2\, d\Gamma \right\}.$$

It is easy to see that

$$\underline{M}(l) \leq \chi(0) = \inf_{v\in K} J(v) \qquad \forall l \in L_2(\gamma). \tag{5}$$

Let us consider the dual problem

$$\left\{ \underline{M}(l) \to \sup\nolimits_{l\in L_2(\gamma)} \right. . \tag{6}$$

Dual functional $\underline{M}(l)$ is a concave, but not strongly concave functional. Therefore the problem (6) can be unsolvable. The question of solvability of problem (6) is closely connected with the regularity of a solution $u$ of problem (1). It can be proved if $u \in H^2(\Omega_\gamma)$, then $-\frac{\partial u}{\partial x_2} \in H^{1/2}(\gamma)$ is a solution of problem (6) [3]. In our earlier publications we investigated the duality methods based on modified Lagrangian functional under assumption of $H^2$-regularity of a solution of initial problem. This assumption looks unnatural in crack problem. Let us consider the duality methods using $H^1$-regularity of the solution $u$ of problem (1).

We investigate the sensitivity functional $\chi(m)$ and the related dual functional $\underline{M}(l)$.

**Theorem 1.** *Sensitivity functional $\chi(m)$ is weakly lower semicontinuous on $L_2(\gamma)$.*

*Proof.* Since $\chi(m)$ is a convex functional [1, 8], then it is sufficient to show that $\chi(m)$ is a lower semicontinuous (according to norm in $L_2(\gamma)$) functional.

Let $\{m_i\} \subset L_2(\gamma)$ be an arbitrary convergent sequence and $\overline{m} = \lim_{i\to\infty} m_i$. Sensitivity functional $\chi(m)$ will be lower semicontinuous if $\lim_{i\to\infty} \chi(m_i) = +\infty$ under $\overline{m} \notin \operatorname{dom}\chi$ and $\underline{\lim}_{i\to\infty} \chi(m_i) \geq \chi(\overline{m})$ under $\overline{m} \in \operatorname{dom}\chi$.

1. Let $\overline{m} \notin \operatorname{dom}\chi$. We can assume that $m_i \in \operatorname{dom}\chi$. Let us show that $\lim \|u_i\|_{H^1(\Omega_\gamma)} = \infty$, where $u_i = \operatorname{argmin}_{v\in K_{m_i}} J(v)$, $i = 1, 2, \dots$. Suppose the contrary, that is $\{u_i\}$ has a bounded subsequence. Without loss of generality we suppose that $\{u_i\}$ is a bounded sequence in $H^1(\Omega_\gamma)$. From the trace theorem [3, p. 12] follows that $\|[u_i]\|_{H^{1/2}(\gamma)} \leq C$, where $C > 0$ is a constant. Then $\{[u_i]\}$ is a compact sequence in $L_2(\gamma)$. Let $t \in H^{1/2}(\gamma)$ be a weak limit point of this sequence. Without loss of generality we can assume that $t$ is a weak limit of $\{[u_i]\}$ in $H^{1/2}(\gamma)$. Then $\{[u_i]\}$ strongly (that is, in the norm) converges to $t$ in $L_2(\gamma)$. Since $-[u_i] \leq m_i$ on $\gamma$, then $-t \leq \overline{m}$, which implies that $K_{\overline{m}} \neq \varnothing$. This contradiction proves that $\lim \|u_i\|_{H^1(\Omega_\gamma)} = \infty$.

Because of $J(v)$ is a coercive functional it follows that

$$\lim_{i\to\infty} \chi(m_i) = \lim_{i\to\infty} J(u_i) = +\infty.$$

2. Let $\overline{m} \in \operatorname{dom}\chi$. As above we can assume that $m_i \in \operatorname{dom}\chi$. From the sequence $\{m^i\}$, we extract a subsequence $\{m^{i_j}\}$ for which

$$\lim_{j\to\infty} \chi(m_{i_j}) = \underline{\lim}_{i\to\infty} \chi(m_i).$$

Consider the subsequence $\{u_{i_j}\}$, where $u_{i_j} = \operatorname{argmin}_{v\in K_{m_i}} J(v)$. We can suppose that $\{u_{i_j}\}$ is a bounded sequence in $H^1(\Omega_\gamma)$ (otherwise, $\lim_{j\to\infty} \chi(m_{i_j}) = +\infty$ and, together with part 1, it means that Theorem 1 is proved). From trace theorem [3, p. 12] follows, that sequence $\{[u_{i_j}]\}$ is bounded in $H^{1/2}(\gamma)$. Therefore $\{[u_{i_j}]\}$ is weakly compact in $H^{1/2}(\gamma)$. Let $t \in H^{1/2}(\gamma)$ be a weak limit point of this sequence. Without loss of generality we can suppose that $\{[u_{i_j}]\}$ is a weakly convergent sequence, that is $t$ is a weak limit of $\{[u_{i_j}]\}$ in $H^{1/2}(\gamma)$. Since $H^{1/2}(\gamma)$ is compactly embedded in $L_2(\gamma)$ and $L_2(\gamma) \subset H^{-1/2}(\gamma)$, then $\{[u_{i_j}]\}$ converges to $t$ in $L_2(\gamma)$. We have $m_{i_j} \longrightarrow \overline{m}$ in $L_2(\gamma)$, $\{[u_{i_j}]\} \longrightarrow t$ in $L_2(\gamma)$, and $-[u_{i_j}] \leq m_{i_j}$ on $\gamma$. Then $-t \leq \overline{m}$ on $\gamma$.

Let $\widetilde{t} = \operatorname{argmin}_{[u]=t \text{ on } \gamma} J(u)$. We have

$$\begin{aligned} J(u_{i_j}) - J(\widetilde{t}) &= \int_{\Omega_\gamma} \nabla\widetilde{t}\nabla(u_{i_j} - \widetilde{t})\, d\Omega - \int_{\Omega_\gamma} f(u_{i_j} - \widetilde{t})\, d\Omega \\ &\quad + \frac{1}{2}\int_{\Omega_\gamma} |\nabla(u_{i_j} - \widetilde{t})|^2\, d\Omega \\ &= \langle \mu, [u_{i_j} - \widetilde{t}]\rangle + \frac{1}{2}\int_{\Omega_\gamma} |\nabla(u_{i_j} - \widetilde{t})|^2\, d\Omega, \end{aligned}$$

where $\langle \mu, [v]\rangle = \int_{\Omega_\gamma} \nabla\widetilde{t}\nabla v\, d\Omega - \int_{\Omega_\gamma} fv\, d\Omega$ and $\mu \in H^{-1/2}(\gamma)$ [3, 4].

Since $\{[u_{i_j}]\}$ weakly converges to $t$ in $H^{1/2}(\gamma)$, we have

$$\lim_{j\to\infty} \langle \mu, [u_{i_j} - \widetilde{t}]\rangle = 0.$$

From this relation and from the convergence of $\{[u_{i_j}]\}$ to $t$ in $L_2(\gamma)$, we conclude that

$$\lim_{j\to\infty} \chi(m_{i_j}) = \lim_{j\to\infty} J(u_{i_j}) \geq J(\widetilde{t}) \geq \chi(\overline{m})$$

or

$$\underline{\lim}_{i\to\infty} \chi(m_i) \geq \chi(\overline{m}).$$

Parts 1 and 2 imply that the functional $\chi(m)$ is weakly lower semicontinuous on $L_2(\gamma)$ and theorem has been proved. □

For an arbitrary $l \in L_2(\gamma)$, we consider the functional

$$F_l(m) = \chi(m) + \int_\gamma lm\, d\Gamma + \frac{r}{2}\int_\gamma m^2\, d\Gamma,$$

where $r > 0$ is a constant. Then the dual functional $\underline{M}(l)$ has the form

$$\underline{M}(l) = \inf_{m\in L_2(\gamma)} F_l(m).$$

From Theorem 1 follows that $F_l(m)$ is a weakly lower semicontinuous functional on $L_2(\gamma)$.

Since $\chi(m)$ is a lower semicontinuous functional in $L_2(\gamma)$, then the epigraph of sensitivity functional

$$\operatorname{epi}\chi \equiv \{(m, a) \in L_2(\gamma) \times R, R = (-\infty, +\infty)\colon \chi(m) \leq a\}$$

is a convex closed set in $L_2(\gamma) \times R$. According Mazur separation theorem [2, p. 164] there are $\psi \in L_2(\gamma)$ and $d \in R$, such that

$$\chi(m) + \int_\gamma \psi m\, d\Gamma + d \geq 0 \qquad \forall m \in \operatorname{dom}\chi.$$

Hence $F_l(m) \geq -\int_\gamma \psi m\, d\Gamma + \int_\gamma lm\, d\Gamma + \frac{r}{2}\int_\gamma m^2\, d\Gamma - d \qquad \forall m \in L_2(\gamma)$.

Therefore $F_l(m) \longrightarrow +\infty$ under $\|m\|_{L_2(\gamma)} \longrightarrow \infty$, that is $F_l(m)$ is a coercive functional in $L_2(\gamma)$.

From weakly lower semicontinuity and coercivity of $F_l(m)$ follows, that the problem

$$\Big\{F_l(m) \to \min_{m\in L_2(\gamma)}$$

has a solution $m(l) \quad \forall l \in L_2(\gamma)$. It is easy to see, that for every $l \in L_2(\gamma)$ element $m(l)$ is unique.

We will formulate for dual functional $\underline{M}(l)$ some statements which proofs is repeated by proofs of corresponding theorems in [7].

**Statement 1.** *Dual functional $\underline{M}(l)$ is continuous is $L_2(\gamma)$ (see Theorem 4 in* [7]*).*

**Statement 2.** *The dual functional $\underline{M}(l)$ is Gateaux differentiable in $L_2(\gamma)$ and it's derivative $\nabla\underline{M}(l)$ satisfies the Lipschitz condition with the constant $\frac{1}{r}$; that is, for all $l_1, l_2 \in L_2(\gamma)$, it holds that*

$$\|\nabla\underline{M}(l_1) - \nabla\underline{M}(l_2)\|_{L_2(\gamma)} \le \frac{1}{r}\|l_1 - l_2\|_{L_2(\gamma)}$$

*and, moreover, $\nabla\underline{M}(l) = m(l) \quad \forall l \in L_2(\gamma)$ (see Theorem 5 in* [7]*).*

Since the gradient of the functional $\underline{M}(l)$ satisfies the Lipschitz condition, the dual problem (6) can be solved by using the gradient method for maximizing a functional [7]

$$l^{s+1} = l^s + rm(l^s), \qquad s = 0, 1, 2, \dots, \tag{7}$$

where $l^0 \in L_2(\gamma)$ is an arbitrary initial value, $r > 0$ is a constant and

$$m(l^s) = \nabla\underline{M}(l^s) = \operatorname*{argmin}_{m\in L_2(\gamma)} \Big\{\chi(m) + \int_\gamma l^s m\, d\Gamma + \frac{r}{2}\int_\gamma m^2\, d\Gamma\Big\}.$$

**Statement 3.** *The sequence $\{l^s\}$ constructed by the gradient method* (7) *satisfies the limit equality*

$$\lim_{s\to\infty} \|m(l^s)\|_{L_2(\gamma)} = 0$$

*(see Theorem 7 in* [7]*).*

The gradient method (7) can be written in following way (see [6])

(i) $u^{s+1} = \operatorname{argmin}_{v\in H^1(\Omega_\gamma)} \Big\{J(v) + \frac{1}{2r}\int_\gamma \big(((l^s - r[v])^+)^2 - (l^s)^2\big)\, d\Gamma\Big\},$

(ii) $l^{s+1} = l^s + r\max\Big\{-[u^{s+1}], -\frac{l^s}{r}\Big\},$

where $l^0 \in L_2(\gamma)$.

From inequality (6) follows, that

$$\underline{M}(l^s) \le \chi(0) = \inf_{v\in K} J(v), \qquad s = 1, 2, 3, \dots,$$

or

$$\chi(m(l^s)) + \int_\gamma l^s m(l^s)\, d\Gamma + \frac{r}{2}\int_\gamma m^2(l^s)\, d\Gamma \le \chi(0), \quad s = 1, 2, 3, \dots,$$

$$\int_\gamma l^s m(l^s)\, d\Gamma \le \chi(0) - \chi(m(l^s)) - \frac{r}{2}\int_\gamma m^2(l^s)\, d\Gamma, \quad s = 1, 2, 3, \dots .$$

Since $\underline{\lim}_{s\to\infty} \chi(m(l^s)) \ge \chi(0)$, then $\overline{\lim}_{s\to\infty} \int_\gamma l^s m(l^s)\, d\Gamma \le 0$.

Let us show, that $\overline{\lim}_{s\to\infty} \int_\gamma l^s m(l^s)\, d\Gamma = 0$. We suppose the contrary, that is $\overline{\lim}_{s\to\infty} \int_\gamma l^s m(l^s)\, d\Gamma = \delta < 0$. Then we can take such $\delta_1$, $\delta < \delta_1 < 0$ and number $N$, that $\int_\gamma l^s m(l^s)\, d\Gamma < \delta_1 \quad \forall s > N$. We have

$$\begin{aligned}\|l^{s+1}\|^2_{L_2(\gamma)} &= \|l^s + rm(l^s)\|^2_{L_2(\gamma)}\\ &= \|l^s\|^2_{L_2(\gamma)} + 2r\int_\gamma l^s m(l^s)\, d\Gamma + r^2\|m(l^s)\|^2_{L_2(\gamma)}\\ &\le \|l^s\|^2_{L_2(\gamma)} + 2r\delta_1 + r^2\|m(l^s)\|^2_{L_2(\gamma)}.\end{aligned}$$

Now from Statement 3 follows that under sufficiently large $s$ the inequality $\|l^{s+1}\|^2_{L_2(\gamma)} \le \|l^s\|^2_{L_2(\gamma)}$ is correct. Then

$$\overline{\lim_{s\to\infty}} \int_\gamma l^s m(l^s)\, d\Gamma = \lim_{s\to\infty} \int_\gamma l^s m(l^s)\, d\Gamma \;= 0.$$

It contradicts our assumption. It means that $\overline{\lim}_{s\to\infty} \int_\gamma l^s m(l^s)\, d\Gamma = 0$.

Let $\{l^{s_j}\}$ be a subsequence of $\{l^s\}$, such that

$$\overline{\lim_{s\to\infty}} \int_\gamma l^s m(l^s)\, d\Gamma = \lim_{j\to\infty} \int_\gamma l^{s_j} m(l^{s_j})\, d\Gamma.$$

As the sequence $\{\underline{M}(l^{s_j})\}$ is monotonously increasing [6] and limited below by value $\chi(0)$, then

$$\begin{aligned}\chi(0) &\ge \lim_{s\to\infty}\left(\chi(m(l^s)) + \int_\gamma l^s m(l^s)\, d\Gamma + \frac{r}{2}\int_\gamma m^2(l^s)\, d\Gamma\right)\\ &= \lim_{j\to\infty}\left(\chi(m(l^{s_j})) + \int_\gamma l^{s_j} m(l^{s_j})\, d\Gamma + \frac{r}{2}\int_\gamma m^2(l^{s_j})\, d\Gamma\right)\\ &= \lim_{j\to\infty} \chi(m(l^{s_j})) \ge \chi(0) = \inf_{v\in K} J(v).\end{aligned}$$

Thus for method (i), (ii) such limit equality is established

$$\lim_{s\to\infty} \underline{M}(l^s) = J(u) = \inf_{v\in K} J(v). \tag{8}$$

From here the duality ratio follows for initial and dual problem

$$\sup_{l\in L_2(\gamma)} \underline{M}(l) = \inf_{v\in K} J(v).$$

## 3. Conclusion and discussion

The limit equality (8) is established under natural for crack problem (1) condition that solution $u$ belongs to space $H^1(\Omega_\gamma)$ only. In this case the dual problem (6) may not have a solution, for example, if $\lim_{s\to\infty} \|l^s\|_{L_2(\gamma)} = +\infty$.

However, if the dual problem (6) has a solution, then $\{l^s\}$ is a bounded sequence in $L_2(\gamma)$ [6, 7]. Together with statement 3 it means that

$$\lim_{s\to\infty}\int_\gamma l^s m(l^s)\,d\Gamma = 0.$$

From here at once follows that method (i), (ii) converges according the initial functional $J(v)$, that is

$$\lim_{s\to\infty}\chi(m(l^s)) = \lim_{s\to\infty} J(u^{s+1}) = J(u).$$

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