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PATTERN FORMATION IN A GENERAL DEGN-HARRISON REACTION MODEL

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Abstract. In this paper, we study the pattern formation to a general Degn-Harrison reaction model. We show Turing instability happens by analyzing the stability of the unique positive equilibrium with respect to the PDE model and the corresponding ODE model, which indicate the existence of the non-constant steady state solutions. We also show the existence periodic solutions of the PDE model and the ODE model by using Hopf bifurcation theory. Numerical simulations are presented to verify and illustrate the theoretical results.

1. Introduction

In the early 1950s, the British mathematician Turing [5] proposed a model that accounts for pattern formation in morphogenesis. Turing showed mathematically that a system of coupled reaction-diffusion equations could give rise to spatial concentration patterns of a fixed characteristic length from an arbitrary initial configuration due to so-called diffusion-driven instability, that is, diffusion could destabilize an otherwise stable equilibrium of the reaction-diffusion system and lead to non-uniform spatial patterns.

The so called Degn-Harrison reaction model is proposed by Degen and Harrison in [1] to explain the observed oscillatory behavior of respiration rate in the continuous cultures of the bacteria Klebsiella aerogenes, which follows the form of the three-step reaction scheme:

$$A \to Y,\ B \leftrightarrows X,\ X + Y \to P, \tag{1.1}$$

where X and Y represent oxygen and nutrient respectively, which are the intermediate reactants; A and B account for "sources"or external parameters whose

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concentrations are to kept at a constant level all over the reactor vessel; P is the final product whose concentration is also assumed to be constant. In the reaction process, the last step is considered to be inhibited by excess of oxygen in the reactor. The first and last steps are assumed to be irreversible whereas the second is reversible.

If the last step followed a nonlinear rate equation of the type $XY/(1+qX^2)$, where q measures the strength of the inhibitory law, the above Degn-Harrison reaction scheme is governed by the following coupled nonlinear space-time differential equation in a dimensionless form:

$$\begin{cases} X_t - D_X \Delta X = k_2 B - k_3 X - k_4 \left(\dfrac{X}{1+qX^2} \right) Y, \\ Y_t - D_Y \Delta Y = k_1 A - k_4 \left(\dfrac{X}{1+qX^2} \right) Y, \end{cases} \tag{1.2}$$

where A, B, X and Y denote dimensionless concentrations of the reactants; the four constant k_i, $i = 1, 2, 3, 4$, are reaction rates; D_X and D_Y, respectively, denote the Fickian molecular diffusion coefficients of X and Y, and they are assumed to be positive constant all over the reactor vessel. The rate and diffusion constants are parameters characteristic for a given system, and the concentrations A and B are variable parameters which can be controlled in the reaction process. Problem (1.2) with homogeneous Neumann boundary condition was studied in [3, 4], where the stability of the positive constant equilibrium, Turing instability, Hopf bifurcation and the existence or nonexistence of positive constant steady state were obtain.

In this paper, we generalized the last step of (1.1) by using $f(X)Y$ to replace $XY/(1+qX^2)$. After resealing, we get the following reaction-diffusion system:

$$\begin{cases} u_t - d_1 \Delta u = a - u - rf(u)v, & x \in \Omega,\ t > 0, \\ v_t - d_2 \Delta v = b - rf(u)v, & x \in \Omega,\ t > 0, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain with smooth boundary $\partial\Omega$, ν is the outward unit normal vector of the boundary $\partial\Omega$, a, b, r, d_1 and d_2 are positive constants and f is always assumed to satisfy

(H_f)**:** $f \in C^1(0,\infty) \cap C[0,\infty)$, $f(0) = 0$ and $f(s) > 0$ for $s > 0$.

Example 1.1. A typical choice is $f(u) = \frac{u^p}{1+\kappa u^q}$ with $p, q \geq 1$ and $\kappa > 0$. Then model (1.3) is a generalized model of (1.2).

Obviously, problem (1.3) admits a unique positive constant equilibrium:

$$(u,v) = \left(a-b, \frac{b}{rf(a-b)} \right) =: (u_*, v_*/r) \tag{1.4}$$

if and only if $a > b$.

In this paper, we always assume $a > b$ holds and consider the stability of $(u_*, v_*/r)$, Turing instability and Hopf bifurcation near $(u_*, v_*/r)$. Throughout this paper, $\mathcal{N}$ is the set of natural numbers and $\mathcal{N}_0 = \mathcal{N} \cup \{0\}$. The eigenvalues of operator $-\Delta$ with homogeneous Neumann boundary condition in Ω are denoted by $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq \cdots$, and the eigenfunction corresponding to μ_n is $\phi_n(x)$.

2. Analysis of the local system

For system (1.3), the local system is an ordinary differential equation in the form of

$$\begin{cases} \dfrac{du}{dt} = a - u - rf(u)v, & t > 0, \\ \dfrac{dv}{dt} = b - rf(u)v, & t > 0. \end{cases} \tag{2.1}$$

By (1.4), $(u_*, v_*/r)$ is the unique positive equilibrium of (2.1). In the following we fix the parameters $a > b > 0$, *i.e.*, u_* and v_* are fixed, and use r as the main bifurcation parameter. The Jacobian matrix of (2.1) at $(u_*, v_*/r)$ is

$$L_0(r) = \begin{pmatrix} -1 - f'(u_*)v_* & -f(u_*)r \\ -f'(u_*)v_* & -f(u_*)r \end{pmatrix}. \tag{2.2}$$

The characteristic equation of $L_0(r)$ is

$$\xi^2 - T(r)\xi + D(r) = 0, \tag{2.3}$$

where

$$T(r) = -1 - f'(u_*)v_* - f(u_*)r, \ D(r) = f(u_*)r > 0.$$

Since $D(r) > 0$, then $(u_*, v_*/r)$ is locally asymptotically stable if $T(r) < 0$ and it is unstable if $T(r) > 0$. Let

$$r_0 := -\frac{1 + f'(u_*)v_*}{f(u_*)}, \tag{2.4}$$

then $r = r_0$ is the only root of $T(r) = 0$. The equilibrium (u_*, v_*) is stable if $r > r_0$, and it is unstable if $r < r_0$.

When $r = r_0$, the characteristic equation has a pair of imaginary roots $\pm i\sqrt{D(r_0)}$. Let $\xi = \alpha(r) \pm i\beta(r)$ be the roots of the characteristic equation. Then

$$\alpha(r) = \frac{T(r)}{2}, \ \beta(r) = \frac{\sqrt{4D(r) - T(r)^2}}{2} \text{ and } \alpha'(r_0) = \frac{1}{2}T'(r_0) = -f(u_*) < 0.$$

This shows that the transverse condition holds. By Poincaré-Andronov-Hopf Bifurcation Theorem [7, Theorem 3.1.3], $r = r_0$ is the unique Hopf bifurcation point for (2.1). From Poincaré-Bendixson theory, the system (2.1) possess a periodic orbit when $r < r_0$.

Theorem 2.1. *Let $(u_*, v_*/r)$ be the unique positive equilibrium of* (2.1) *defined in* (1.4)*, and r_0 be the constant defined in* (2.4)*. Then*

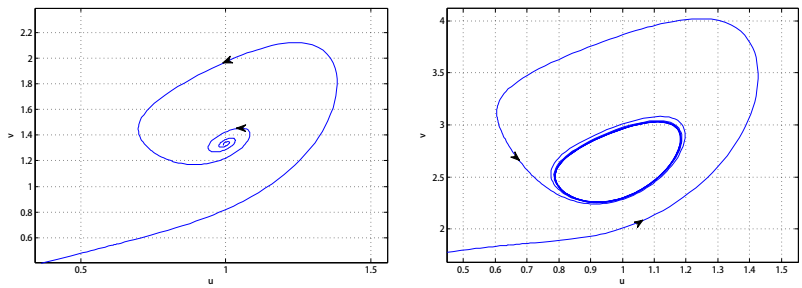

FIGURE 1. When $r = 1.5 > r_0 = 1$, the solution trajectories spiral toward the positive equilibrium $(1, 4/3)$ (left). When $r = 0.8 < r_0 = 1$, there is a limit cycle surrounding the positive equilibrium $(1, 2.5)$ (right).

(1) *The equilibrium* $(u_*, v_*/r)$ *is locally asymptotically stable if* $r > r_0$;
(2) *The equilibrium* (u_*, v_*) *is unstable if* $r < r_0$;
(3) $r = r_0$ *is the unique Hopf bifurcation value for system* (2.1)*, and system* (2.1) *possess a periodic orbit when* $r < r_0$.

Example 2.2. Let $f(u)$ be the function defined in Example 1.1 with $\kappa = p = 1$, $q = 5$, $a = 2$ and $b = 1$. Then

$$r_0 = -\frac{\kappa(a-b)^{q+1} - \kappa b(q-p)(a-b)^q + (a-b) + bp}{(a-b)^{p+1}} = 1,$$

and we get the following system:

$$\begin{cases} \dfrac{du}{dt} = 2 - u - \dfrac{ruv}{1+u^5}, & t > 0, \\ \dfrac{dv}{dt} = 1 - \dfrac{ruv}{1+u^5}, & t > 0. \end{cases} \tag{2.5}$$

It follows from Theorem 2.1 that the positive equilibrium $(u_*, v_*/r) = (1, 2/r)$ is locally asymptotically stable when $r > 1$ and it is unstable when $r < 1$. Moreover when r passes through 1 from the right side of 1, $(u_*, v_*/r)$ will lose its stability and Hopf bifurcation occurs, that is, a family of periodic solutions bifurcate from $(u_*, v_*/r)$. Numerical simulations are presented in Figure 1. The left of Figure 1 shows the stable behavior when $r > r_0$. The right of Figure 1 is the phase portrait of the problem (2.5) which depicts the limit cycle arising out of Hopf bifurcation around $(1, 2.5)$.

3. Analysis of the PDE model (1.3)

In this section, we consider the PDE model (1.3). The main studies are stability, Turing instability and Hopf bifurcation from the unique positive constant equilibrium $(u_*, v_*/r)$ defined by (1.4).

3.1. Stability

The stability of $(u_*, v_*/r)$ is determined by the following eigenvalue problem, which is got by linearizing the steady state of the system (1.3) about $(u_*, v_*/r)$:

$$\begin{cases} L(r)\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \mu \begin{pmatrix} \phi \\ \psi \end{pmatrix}, & x \in \Omega, \\ \partial_\nu \phi = \partial_\nu \psi = 0, & x \in \partial\Omega, \end{cases} \tag{3.1}$$

where

$$L(r) = \begin{pmatrix} d_1\Delta - 1 - f'(u_*)v_* & -f(u_*)r \\ -f'(u_*)v_* & d_2\Delta - f(u_*)r \end{pmatrix}. \tag{3.2}$$

For each $n \in \mathcal{N}_0$, we define a 2×2 matrix

$$L_n(r) = \begin{pmatrix} -d_1\mu_n - 1 - f'(u_*)v_* & -f(u_*)r \\ -f'(u_*)v_* & -d_2\mu_n - f(u_*)r \end{pmatrix}. \tag{3.3}$$

The following statements hold true by using Fourier decomposition:

(1) If μ is an eigenvalue of (3.1), then there exists $n \in \mathcal{N}_0$ such that μ is an eigenvalue of $L_n(r)$.
(2) The constant equilibrium $(u_*, v_*/r)$ is locally asymptotically stable with respect to (1.3) if and only if for every $n \in \mathcal{N}_0$, all eigenvalues of $L_n(r)$ have negative real part.
(3) The constant equilibrium $(u_*, v_*/r)$ is unstable with respect to (1.3) if there exists an $n \in \mathcal{N}_0$ such that $L_n(r)$ has at least one eigenvalue with positive real part.

The characteristic equation of $L_n(r)$ is

$$\mu^2 - T_n(r)\mu + D_n(r), \tag{3.4}$$

where

$$T_n(r) = -(d_1 + d_2)\mu_n - f(u_*)(r - r_0), \tag{3.5}$$

$$D_n(r) = d_1 d_2 \mu_n^2 + f(u_*)(d_1 r - d_2 r_0)\mu_n + f(u_*)r. \tag{3.6}$$

Here r_0 is defined by (2.4). Then $(u_*, v_*/r)$ is locally asymptotically stable if $T_n(r) < 0$ and $D_n(r) > 0$ for all $n \in \mathcal{N}_0$, and it is unstable if there exists $n \in \mathcal{N}_0$ such that $T_n(r) > 0$ or $D_n(r) < 0$.

If $r_0 \le 0$, then $(u_*, v_*/r)$ is locally asymptotically stable.

Next, we consider the case that $r_0 > 0$. We define

$$T(r, \mu) := -(d_1 + d_2)\mu - f(u_*)(r - r_0), \tag{3.7}$$

$$D(r, \mu) := d_1 d_2 \mu^2 + f(u_*)(d_1 r - d_2 r_0)\mu + f(u_*)r, \tag{3.8}$$

and

$$H = \{(r, \mu) \in (0, \infty) \times [0, \infty) : T(r, \mu) = 0\}, \tag{3.9}$$

$$S = \{(r, \mu) \in (0, \infty) \times [0, \infty) : D(r, \mu) = 0\}. \tag{3.10}$$

Then H is the Hopf bifurcation curve and S is the state bifurcation curve. Furthermore, the sets of H and S are graphs of functions defined as follows:

$$r_H(\mu) = -\frac{d_1 + d_2}{f(u_*)}\mu + r_0, \tag{3.11}$$

$$r_S(\mu) = \frac{-d_1 d_2 \mu^2 + d_2 f(u_*) r_0 \mu}{(d_1\mu + 1) f(u_*)}. \tag{3.12}$$

We have the following properties of the functions $r_H(\mu)$ and $r_S(\mu)$:

(1) The function $r_H(\mu)$ is strictly decreasing for $\mu \geq 0$ such $r_H(0) = r_H(\mu_{H0}) = 0$,

$$\mu_{H0} := \frac{f(u_*) r_0}{d_1 + d_2}. \tag{3.13}$$

(2) Let

$$\mu_c := \frac{\sqrt{1 + f(u_*) r_0} - 1}{d_1} < \frac{f(u_*) r_0}{d_1} =: \mu_{S0}. \tag{3.14}$$

Then $\mu = \mu_c$ is the unique critical value of $r_S(\mu)$, the function $r_S(\mu)$ is strictly increasing for $\mu \in (0, \mu_c)$, and $r_S(\mu)$ is strictly decreasing for $\mu > \mu^*$. $r_S(0) = r_S(\mu_{S0}) = 0$, $r^* = \sup_{\mu \in (0,+\infty)} r_S(\mu)$, where

$$r^* = r_S(\mu_c) = \frac{d_2}{d_1}\left[\frac{\left(\sqrt{1 + f(u_*) r_0} - 1\right)^2}{f(u_*)}\right]. \tag{3.15}$$

Furthermore, $r^* > (=, <) r_0$ if and only if $d_2/d_1 > (=, <)\chi$, where

$$\chi := \frac{f(u_*) r_0}{\left(\sqrt{1 + f(u_*) r_0} - 1\right)^2}. \tag{3.16}$$

(3) $r_H(\mu)$ and $r_S(\mu)$ cross at only one point $(\mu_{cro}, \lambda_H(\mu_{cro}))$, where

$$\mu_{cro} := \frac{-[(d_2 - d_1) f(u_*) r_0 + d_1 + d_2] + \sqrt{[(d_2 - d_1) f(u_*) r_0 + d_1 + d_2]^2 + 4 d_1^2 f(u_*) r_0}}{2 d_1^2}. \tag{3.17}$$

Now we can give a stability result regarding to the constant equilibrium $(u_*, v_*/r)$ by the analysis above. We define

$$\overline{r} := \max_{n \in \mathcal{N}} r_S(\mu_n) \leq r^*, \tag{3.18}$$

and we get $T_n(r) < 0$ and $D_n(r) > 0$ for all $n \in \mathcal{N}_0$ if

$$r > \max\{r_0, \overline{r}\}. \tag{3.19}$$

On the other hand, if

$$r < \max\{r_0, \overline{r}\}, \tag{3.20}$$

then there exists $n \in \mathcal{N}_0$ such that $T_n(r) > 0$ or $D_n(r) < 0$. The above discussions lead to the following theorem:

Theorem 3.1. *Assume a, b, d_1, d_2 are fixed. Let r_0, r^* and $\overline{r}$ be the constants defined by* (2.4), (3.15) *and* (3.18) *respectively. Then*

(1) *the constant equilibrium $(u_*, v_*/r)$ is locally asymptotically stable with respect to* (1.3) *if*
   (a) $r_0 \le 0$; *or*
   (b) $r_0 > 0$ *and* (3.19) *holds. In particular* (3.19) *holds if*
$$r > \max\{r_0, r^*\};$$
(2) *the constant equilibrium $(u_*, v_*/r)$ is unstable with respect to* (1.3) *if $r_0 > 0$ and* (3.20) *holds.*

Next we derive conditions for the Turing instability with respect to the positive equilibrium $(u_*, v_*/r)$, which means that $(u_*, v_*/r)$ is locally asymptotically stable with respect to the ODE (2.1) and unstable with respect to the PDE (1.3). In view of Theorems 2.1 and 3.1, Turing instability happens if r satisfies

$$r_0 < r < \overline{r}. \tag{3.21}$$

Since $\overline{r} \le r^*$, in view the properties of $r_S(\mu)$, we need to assume $d_2/d_1 > \chi$ holds, then the equation $r_S(\mu) = \lambda_0$ has two different positive roots μ_l and μ_r, where

$$\begin{aligned} \mu_l &:= \frac{\left(\frac{d_2}{d_1} - 1\right) f(u_*) r_0 - \sqrt{\left(\frac{d_2}{d_1} - 1\right)^2 (f(u_*) r_0)^2 - 4 f(u_*) r_0 \frac{d_2}{d_1}}}{2 d_2}, \\ \mu_r &:= \frac{\left(\frac{d_2}{d_1} - 1\right) f(u_*) r_0 + \sqrt{\left(\frac{d_2}{d_1} - 1\right)^2 (f(u_*) r_0)^2 - 4 f(u_*) r_0 \frac{d_2}{d_1}}}{2 d_2}. \end{aligned} \tag{3.22}$$

In view of above analysis and the properties of $r_S(\mu)$, we know that (3.21) holds if there exists $n \in \mathcal{N}$ such that

$$\mu_n \in (\mu_l, \mu_r). \tag{3.23}$$

The above discussions lead to the following theorem:

Theorem 3.2. *Assume a, b, d_1, d_2 are fixed such that $r_0 > 0$ and $d_2/d_1 > \chi$, where χ is defined in* (3.16). *Let r_0, $\overline{r}$, μ_l and μ_r be the constants defined by* (2.4), (3.18), *and* (3.22) *respectively. Then Turing instability happens if there exists $n \in \mathcal{N}$ such that* (3.23) *holds and r satisfies* (3.21).

Example 3.3. We give an numerical example to show Turing instability. Consider the following PDE model related the ODE given in (2.5):

$$\begin{cases} u_t - d_1 \Delta u = 2 - u - \dfrac{ruv}{1+u^5}, & x \in (0, 3\pi),\ t > 0, \\ v_t - d_2 \Delta v = 1 - \dfrac{ruv}{1+u^5}, & x \in (0, 3\pi),\ t > 0, \\ u_x(0) = u_x(3\pi) = v_x(0) = v_x(3\pi) = 0. \end{cases} \tag{3.24}$$

Then $\mu_i = i^2/9$, $i \in \mathcal{N}_0$. We can compute $r_0 = 1$, $u_* = 1$, $v_* = 2$, $\chi = 1/(5 - 2\sqrt{6}) \approx 9.9$, then we choose $d_2 = 1.5$, $d_1 = 0.1$ such that $d_2/d_1 > \chi$, and then we can compute $\mu_l = (7 - \sqrt{19})/3 \approx 0.88$, $\mu_r = (7 + \sqrt{19})/3 \approx 3.79$. Finally we find that

$$\mu_2 = \frac{4}{9} \approx 0.44 < \mu_l < \mu_3 = 1 < \mu_4 \approx 1.78 < \mu_5 \approx 2.78 < \mu_r < \mu_6 = 4,$$

then

$$\overline{r} = \max\{r_S(\mu_3), r_S(\mu_4), r_S(\mu_5)\} = r_S(\mu_4) \approx 1.46.$$

Figure 2 shows the graphs of $r = r_S(\mu)$ and $r = r_H(\mu)$ in this case. By Theorem

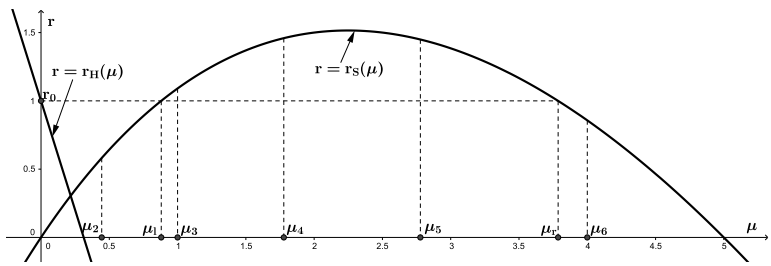

FIGURE 2. The graphs of $r = r_S(\mu)$ and $r = r_H(\mu)$.

3.2, Turing instability happens when $r_0 = 1 < r < \overline{r} \approx 1.46$. We choose $r = 1.3$, $u_0(x) = 1 + 0.1\cos x$, $v_0(x) = 20/13 + 0.1\cos x$. The solution trajectories of the corresponding ODE spiral toward the positive equilibrium $(1, 20/13)$, while the solution of the PDE converges to a spatially nonhomogeneous steady state solution (see Figure 3).

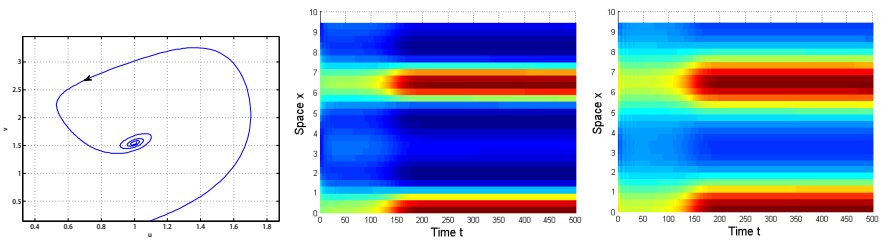

FIGURE 3. Numerical simulation of problem (3.24). The solution trajectories of the corresponding ODE with $r = 1.3$ spiral toward the positive equilibrium $(1, 20/13)$ (see left). The solution of the PDE with $d_1 = 0.1$, $d_2 = 1.5$, $r = 1.3$, $u_0(x) = 1 + 0.1\cos x$, $v_0(x) = 20/13 + 0.1\cos x$, converges to a spatially nonhomogeneous steady state solution (see middle for u and right for v).

3.2. Hopf bifurcation

In this part, we study the Hopf bifurcation from the constant equilibrium $(u_*, v_*/r)$ under the assumption $r_0 \geq 0$ since $(u_*, v_*/r)$ is stable and there is no change of stability for $r_0 \leq 0$. We assume that all eigenvalues μ_i are simple, and denote the corresponding eigenfunction by $\phi_i(x)$, $i \in \mathcal{N}_0$. We use r as the main bifurcation parameter. To identify possible Hopf bifurcation value r_H, we recall the following necessary and sufficient condition from [2, 8, 9]:

(HS) There exists $n \in \mathcal{N}_0$ such that

$$\begin{aligned} &T_n(r_H) = 0,\ D_n(r_H) > 0 \text{ and} \\ &T_m(r_H) \neq 0,\ D_m(r_H) \neq 0 \text{ for } m \in \mathcal{N}_0 \setminus \{n\}, \end{aligned} \tag{3.25}$$

where $T_n(r)$ and $D_n(r)$ are given in (3.5) and (3.6) respectively, and for the unique pair of complex eigenvalues $A(r) \pm iB(r)$ near the imaginary axis,

$$A'(r_H) \neq 0 \text{ and } B(r_H) > 0. \tag{3.26}$$

For $n \in \mathcal{N}_0$, we define

$$r_{n,H} = r_H(\mu_n), \tag{3.27}$$

where the function $r_H(\mu)$ is given in (3.11). Then $T_n(r_{n,H}) = 0$ and $T_m(r_{m,H}) \neq 0$ for $m \neq n$. By (3.25), we need $D_n(r_{n,H}) > 0$ to make $r_{n,H}$ as a possible bifurcation value, which means $\mu_n < \mu_{cro}$ by the properties of $r_H(\mu)$ and $r_S(\mu)$, where μ_{cro} is given in (3.17). Let $n_0 \in \mathcal{N}_0$ such that $\mu_{n_0} < \mu_{cro} \leq \mu_{n_0+1}$, then we can see $D_m(r_{n,H}) > 0$ holds $m, n \in \{0, \ldots, n_0\}$. On the other hand, it is also possible that

$$D_m(r_{n,H}) = 0 \text{ for some } n \in \{0, \ldots, n_0\} \text{ and for some } m \in \mathcal{N} \text{ and } m > n_0. \tag{3.28}$$

Then (3.25) does not holds for such n. However from an argument in [9], for $N = 1$ and $\Omega = (0, \ell\pi)$, there are only countably many ℓ, such that (3.28) occurs. For general bounded domains in $\mathbb{R}^N$, one can also show that (3.28) does not occur for generic domains [6].

Finally, we consider the conditions in (3.26). Let the eigenvalues close to the pure imaginary one at $r = r_{n,H}$ be $A(r) \pm iB(r)$. Then

$$\begin{aligned} A'(r_{n,H}) &= \frac{T_n'(r_{n,H})}{2} = -f(u_*) < 0 \text{ and} \\ B(r_{n,H}) &= \sqrt{D_n(r_{n,H})} > 0 \text{ for } n = 0, \ldots, n_0. \end{aligned}$$

Then all conditions in (HS) are satisfied if $i \in \{0, \ldots, n_0\}$. Now by using the Hopf bifurcation theorem in [9], we have:

Theorem 3.4. *Assume a, b, d_1, d_2 are fixed such that $r_0 > 0$, where r_0 is defined by* (2.4)*. Let Ω be a smooth domain so that all eigenvalues μ_n, $n \in \mathcal{N}_0$, are simple, then there exists a $n_0 \in \mathcal{N}_0$ such that $\mu_{n_0} < \mu_{cro} \leq \mu_{n_0+1}$, where μ_{cro} is given in* (3.17)*. If* (3.28) *does not hold for some $n \in \{0, \ldots, n_0\}$, then $r_{n,H}$, defined in* (3.27)*, is a Hopf bifurcation value. At such $r_{n,H}$, the system*

(1.3) *undergoes a Hopf bifurcation, and the bifurcation periodic orbits near* $(r, u, v) = (r_{n,H}, u_*, v_*/r_{n,H})$ *can be parameterized as* $(r_n(\tau), u_n(\tau), v_n(\tau))$*, so that* $r_n(\tau)$ *is the form of* $r_n(\tau) = r_{n,H} + o(\tau)$ *for* $\tau \in (0, \rho)$ *for some constant* $\rho > 0$*, and*

$$u_n(\tau)(x,t) = u_* + \tau a_n \cos(\omega(r_{n,H})t)\phi_n(x) + o(\tau),$$
$$v_n(\tau)(x,t) = \frac{v_*}{r_{n,H}} + \tau b_n \cos(\omega(r_{n,H})t)\phi_n(x) + o(\tau),$$

where $\omega(r_{n,H}) = \sqrt{D_n(r_{n,H})}$ *with* $D_n(r)$ *given in* (3.6) *is the corresponding time frequency,* $\phi_n(x)$ *is the corresponding spatial eigenfunction, and* (a_n, b_n) *is the corresponding eigenvector, i.e.,*

$$(L(r_{n,H}) - i\omega(b_{n,H})I)\begin{pmatrix} a_n\phi_n(x) \\ b_n\phi_n(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $L(r)$ *is given in* (3.2)*. Moreover,*

(1) *The bifurcation periodic orbit from* $r_{0,H} = r_0$ *is spatially homogeneous*;
(2) *The bifurcation periodic orbits from* $r_{n,H}$*,* $n \in \{1, \ldots, n_0\}$*, are spatially nonhomogeneous.*

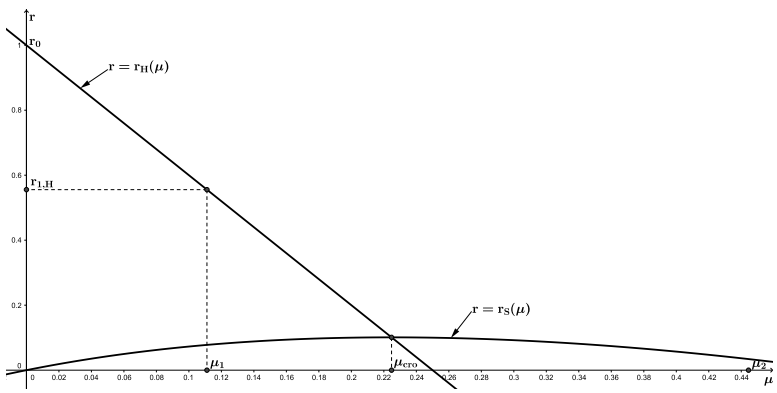

FIGURE 4. The graphs of $r_H(\mu)$ and $r_S(\mu)$.

Example 3.5. We give an numerical example to show Hopf bifurcation. Consider the problem (3.24) again. In this case, we choose $d_2 = d_1 = 1$ such that $d_2/d_1 = 1 < \chi \approx 9.9$. Then we can compute $\mu_{cro} = (\sqrt{6} - 2)/2 \approx 0.221$ and find that

$$\mu_0 = 0 < \mu_1 = \frac{1}{9} \approx 0.11 < \mu_{cro} < \mu_2 = \frac{4}{9} \approx 0.44.$$

Then $r_{1,H} \approx 0.56$, since $D_m(r_0) < 0$ and $D_m(r_{1,H}) < 0$ for $m > 1$, (3.28) does not hold. By Theorem 3.4, r_0 and $r_{1,H}$ are Hopf bifurcation values. Since the largest Turing bifurcation value is much smaller than r_0 and $r_{1,H}$ (see Figure 4), when b decreases, the first bifurcation point encountered is r_0, and a Hopf

bifurcation occurs. We choose $r = 0.65$, $u_0(x) = 1 + 0.1\cos x$, $v_0(x) = 2/0.65 + 0.1\cos x$. The solution converges to a spatially nonhomogeneous periodic orbit (see Figure 5).

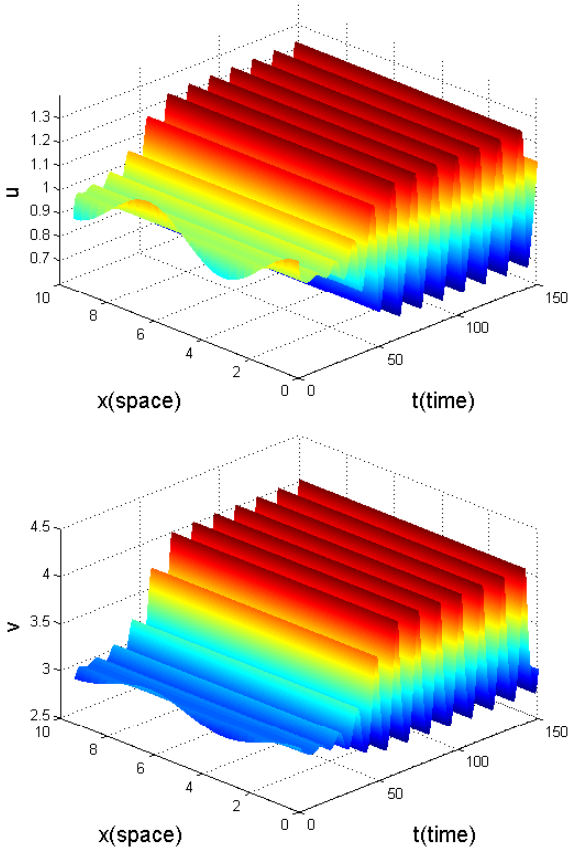

FIGURE 5. Numerical simulation of problem (3.24) with $d_1 = d_2 = 1$, $r = 0.65$, $u_0(x) = 1 + 0.1\cos x$, $v_0(x) = 2/0.65 + 0.1\cos x$. The solution converges to a spatially nonhomogeneous periodic orbit.

References

[1] H. Degn and D. E. F. Harrisson, *Theory of oscillations of respiration rate in continuous culture of klebsiella aerogenes*, J. Theoret. Biol. **22** (1969), no. 22, 238–248.

[2] B. D. Hassard, N. D. Kazarinoff, and Y. H. Wan, *Theory and Applications of Hopf Bifurcation*, volume 41 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1981.

[3] S. Li, J. Wu, and Y. Dong, *Turing patterns in a reaction-diffusion model with the Degn-Harrison reaction scheme*, J. Differential Equations **259** (2015), no. 5, 1990–2029.

[4] R. Peng, F. Q. Yi, and X. Q. Zhao, *Spatiotemporal patterns in a reaction-diffusion model with the Degn-Harrison reaction scheme*, J. Differential Equations **254** (2013), no. 6, 2465–2498.

[5] A. M. Turing, *The chemical basis of morphogenesis*, Philos. Trans. Roy. Soc. London Ser. B **237** (1952), no. 641, 37–72.

[6] J. Wang, J. Shi, and J. Wei, *Dynamics and pattern formation in a diffusive predator-prey system with strong Allee effect in prey*, J. Differential Equations **251** (2011), no. 4-5, 1276–1304.

[7] S. Wiggins and M. Golubitsky, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer, 1990.

[8] F. Yi, J. Wei, and J. Shi, *Diffusion-driven instability and bifurcation in the Lengyel-Epstein system*, Nonlinear Anal. Real World Appl. **9** (2008), no. 3, 1038–1051.

[9] ———, *Bifurcation and spatiotemporal patterns in a homogeneous diffusive predator-prey system*, J. Differential Equations **246** (2009), no. 5, 1944–1977.

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