

## BIRATIONAL RIGIDITY IS NOT AN OPEN PROPERTY

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ABSTRACT. We show that birational rigidity of Mori fibre spaces is not open in moduli.

We assume that all varieties are normal, projective and defined over the field  $\mathbb{C}$ .

### 1. Introduction

In this paper we give a negative answer to the question that is closely related to the nature of birationally rigid Mori fibre spaces: whether birational rigidity is open in moduli.

**Definition 1.1.** A Mori fibre space is a surjective morphism  $\pi: X \rightarrow S$  such that

- the variety  $X$  has terminal and  $\mathbb{Q}$ -factorial singularities,
- the inequality  $\dim(S) < \dim(X)$  holds and  $\pi_*(\mathcal{O}_X) = \mathcal{O}_S$ ,
- the divisor  $-K_X$  is relatively ample for  $\pi$ ,
- the equality  $\text{rk Pic}(X) = \text{rk Pic}(S) + 1$  holds.

Let  $\pi: X \rightarrow S$  be a Mori fibre space such that  $\dim(X) = 3$ . Then

- either  $\dim(S) = 0$  and  $X$  is a Fano 3-fold,
- or  $\dim(S) = 1$  and  $\pi: X \rightarrow S$  is a del Pezzo fibration,
- or  $\dim(S) = 2$  and  $\pi: X \rightarrow S$  is a conic bundle.

**Definition 1.2.** The Mori fibre space  $\pi: X \rightarrow S$  is birationally rigid if, given any birational map  $\xi: X \dashrightarrow X'$  to another Mori fibre space  $\pi': X' \rightarrow S'$ , there exists a commutative diagram

$$\begin{array}{ccccc}
 X & \dashrightarrow^{\rho} & X & \dashrightarrow^{\xi} & X' \\
 \pi \downarrow & & & & \downarrow \pi' \\
 S & \dashrightarrow^{\sigma} & & & S'
 \end{array}$$

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for some birational maps  $\rho$  and  $\sigma$  such that the composition map  $\xi \circ \rho$  induces an isomorphism of the generic fibers of the Mori fibre spaces  $\pi$  and  $\pi'$ .

We say that  $X$  is birationally rigid if  $\dim(S) = 0$  and  $\pi: X \rightarrow S$  is birationally rigid.

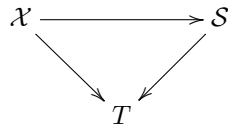
**Example 1.3.** Let  $X$  be a general complete intersection in  $\mathbb{P}^5$  of a quadric and a cubic. Then

$$-K_X \equiv \mathcal{O}_{\mathbb{P}^5}(1)|_X$$

and  $\text{Pic}(X) = \mathbb{Z}[-K_X]$ . The threefold  $X$  is birationally rigid (see [4] and [7, Chapter 2]).

The following conjecture is Conjecture 1.4 in [2].

**Conjecture 1.4.** For any scheme  $T$ , and a flat family of Mori fibre spaces parametrised by  $T$



the set of all  $t \in T$  such that the corresponding fibre  $\mathcal{X}_t \rightarrow \mathcal{S}_t$  is birationally rigid is open in  $T$ .

In this paper, we show that Conjecture 1.4 fails in general.

**Definition 1.5.** The Mori fibre space  $\pi: X \rightarrow S$  is square birationally equivalent to a Mori fiber space  $\pi': X' \rightarrow S'$  if there is a birational map  $\xi: X \dashrightarrow X'$  that fits a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\xi} & X' \\ \pi \downarrow & & \downarrow \pi' \\ S & \xrightarrow{\sigma} & S' \end{array}$$

for some birational map  $\sigma$  such that  $\xi$  induces an isomorphism of the generic fibers of  $\pi$  and  $\pi'$ .

The following definition is due to [3].

**Definition 1.6.** The *pliability* of a variety  $V$  is the set

$$\mathcal{P}(V) = \left\{ \text{Mori fibre space } \tau: Y \rightarrow T \mid Y \text{ is birational to } V \right\} / \approx,$$

where  $\approx :=$  square birational equivalence.

Let  $V_1$  be a complete intersection of a quadric  $Q_1 \subset \mathbb{P}^5$  and a cubic  $T_1 \subset \mathbb{P}^5$  such that  $V_1$  has singular point  $P$ , but  $Q_1$  is non-singular at the point  $P$ . Then  $Q_1$  can be given by the equation

$$y_5 h(y_0, y_1, y_2, y_3, y_4) = q_1(y_0, y_1, y_2, y_3, y_4)$$

in  $\text{Proj}(\mathbb{C}[y_0, y_1, y_2, y_3, y_4, y_5]) \cong \mathbb{P}^5$ , where  $h(y_0, y_1, y_2, y_3, y_4)$  and  $q_1(y_0, y_1, y_2, y_3, y_4)$  are homogeneous polynomials of degree 1 and 2, respectively, and the point  $P$  is given by the equations  $y_0 = y_1 = y_2 = y_3 = y_4 = 0$ .

Similarly, the cubic hypersurface  $T_1 \subset \mathbb{P}^5$  can be given by the equation

$$y_5 q_2(y_0, y_1, y_2, y_3, y_4) = t(y_0, y_1, y_2, y_3, y_4),$$

where  $t(y_0, y_1, y_2, y_3, y_4)$  is a homogeneous polynomial of degree 3, and  $q_2(y_0, y_1, y_2, y_3, y_4)$  is a homogeneous polynomial of degree 2.

Let  $V_2$  be a complete intersection in  $\mathbb{P}^5$  of a quadric  $Q_2$  and a cubic  $T_2$  such that  $Q_2$  is given by

$$y_5 h(y_0, y_1, y_2, y_3, y_4) = q_2(y_0, y_1, y_2, y_3, y_4),$$

and the cubic hypersurface  $T_2 \subset \mathbb{P}^5$  is given by

$$y_5 q_1(y_0, y_1, y_2, y_3, y_4) = t(y_0, y_1, y_2, y_3, y_4).$$

*Remark 1.7.* The threefold  $V_2$  is singular at the point  $P \in V_2$  as well.

Suppose that both  $V_1$  and  $V_2$  satisfy the following generality conditions:

- (A) the quadric hypersurface  $Q_i \subset \mathbb{P}^5$  is non-singular,
- (B) the threefold  $V_i = Q_i \cap T_i$  is smooth outside of the point  $P \in V_i$ ,
- (C) the point  $P$  is an ordinary double point of the threefold  $V_i \subset \mathbb{P}^5$ ,
- (D) the threefold  $V_i$  contains 12 lines that pass through the point  $P \in V_i$ .

*Remark 1.8.* The varieties  $V_1$  and  $V_2$  are  $\mathbb{Q}$ -factorial, and

$$\text{rk Pic}(V_1) = \text{rk Pic}(V_2) = 1 \text{ (see [8]).}$$

The threefolds  $V_1$  and  $V_2$  are birationally equivalent. Indeed, there is a commutative diagram

$$\begin{array}{ccccc}
 & \tilde{V}_1 & \overset{\gamma}{\dashrightarrow} & \tilde{V}_2 & \\
 \beta_1 \swarrow & & \alpha_1 \searrow & \alpha_2 \swarrow & \beta_2 \searrow \\
 V_1 & \overset{\phi_1}{\dashrightarrow} & Y & \overset{\phi_2}{\dashleftarrow} & V_2,
 \end{array}$$

where  $Y$  is a singular quartic hypersurface in  $\mathbb{P}^4$  that is given by the equation

$$h(y_0, \dots, y_4) t(y_0, \dots, y_4) = q_1(y_0, \dots, y_4) q_2(y_0, \dots, y_4)$$

in  $\text{Proj}(\mathbb{C}[y_0, \dots, y_4]) \cong \mathbb{P}^4$ , the maps  $\phi_1$  and  $\phi_2$  are projections from the point  $P$ , the morphisms  $\alpha_1$  and  $\alpha_2$  are flopping contractions, the morphisms  $\beta_1$  and  $\beta_2$  are blow ups of  $P$ , and  $\gamma$  is a flop in 12 smooth curves.

*Remark 1.9.* Suppose that  $h, t, q_1$  and  $q_2$  are general. Then it follows from [8, Remark 4.3] that  $Y$  does not have automorphism that swaps the quadric surfaces given by

$$h(y_0, \dots, y_4) = q_1(y_0, \dots, y_4) = 0$$

and  $h(y_0, \dots, y_4) = q_2(y_0, \dots, y_4) = 0$ . This implies that  $V_1 \not\cong V_2$ .

Consider the following additional generality conditions:

- (E) for any line  $L \subset V_i$ , and for any two-dimensional linear subspace  $\Pi \subset \mathbb{P}^5$  such that  $L \subset \Pi$ , the cycle  $V_i|_\Pi$  is reduced along the line  $L$ ,
- (F) for any two-dimensional linear subspace  $\Pi \subset \mathbb{P}^5$ , the intersection  $V_i \cap \Pi$  is not three lines with a common point, and if  $P \in \Pi$ , the intersection  $V_i \cap \Pi$  does not consist of three lines,
- (G) for any line  $L \subset V_i$  such that  $P \in L$ , and for any three-dimensional linear subspace

$$\Lambda \subset \mathbb{P}^5$$

such that the intersection  $Q_i \cap \Lambda$  consists of two different planes, the three-dimensional linear subspace  $\Lambda$  is not a tangent space to the three-fold  $V_i$  at any point of  $L \setminus P$ ,

- (H) for any line  $L \subset V_i$  such that  $P \in L$ , and for any point  $O \in L \setminus P$ , the complete intersection  $V_i \subset \mathbb{P}^5$  contains at most three lines that pass through  $O$ ,
- (I) for any lines  $L \subset V_i \supset L'$  such that  $L \ni P \notin L'$  and  $L \cap L' \neq \emptyset$ , and for any three-dimensional linear subspace  $\Lambda \subset \mathbb{P}^5$  such that  $L \subset \Lambda \supset L'$ , the inequality

$$\text{mult}_{L \cap L'} \left( V_i \Big|_\Lambda \right) \leq 4$$

holds in the case when the scheme  $V_i|_\Lambda$  is not reduced along the lines  $L$  and  $L'$ .

In this paper, we prove the following result.

**Theorem 1.10.** *Suppose that  $V_1$  and  $V_2$  satisfy the conditions A, B, C, D, E, F, G, H, I. Then*

$$\mathcal{P}(V_1) = \mathcal{P}(V_2) = \{V_1, V_2\}.$$

Let  $\mathcal{F}$  be the family of all complete intersections in  $\mathbb{P}^5$  that are constructed similar to  $V_1$  or  $V_2$ . In Section 8, we will show that general threefolds in  $\mathcal{F}$  satisfy A, B, C, D, E, F, G, H, I.

**Corollary 1.11.** *Let  $V$  be a general threefold in  $\mathcal{F}$ . Then  $|\mathcal{P}(V)| = 2$  and  $V$  is non-rational.*

Now we construct a subfamily  $\mathcal{R} \subsetneq \mathcal{F}$ . Let  $\iota \in \text{Aut}(\mathbb{P}^5)$  be an involution that is given by

$$y_0 \rightarrow -y_0, \quad y_1 \rightarrow y_1, \quad y_2 \rightarrow y_2, \quad y_3 \rightarrow y_3, \quad y_4 \rightarrow y_4 \quad y_5 \rightarrow y_5,$$

let  $U_1$  be a complete intersection in  $\mathbb{P}^5$  that is given by the equations

$$\begin{cases} y_5 f(y_1, y_2, y_3, y_4) = q(y_0, y_1, y_2, y_3, y_4), \\ y_5 \iota^* \left( q(y_0, y_1, y_2, y_3, y_4) \right) = g(y_0, y_1, y_2, y_3, y_4) \end{cases}$$

in  $\text{Proj}(\mathbb{C}[y_0, y_1, y_2, y_3, y_4, y_5]) \cong \mathbb{P}^5$ , and let  $U_2$  be a complete intersection in  $\mathbb{P}^5$  that is given by the equations

$$\begin{cases} y_5 f(y_1, y_2, y_3, y_4) = \iota^*(q(y_0, \dots, y_4)), \\ y_5 q(y_0, y_1, y_2, y_3, y_4) = g(y_0, y_1, y_2, y_3, y_4), \end{cases}$$

where  $f$ ,  $g$  and  $q$  are homogeneous forms of degree 1, 3 and 2, respectively. Suppose that

- the equality  $g(-y_0, y_1, y_2, y_3, y_4) = g(y_0, y_1, y_2, y_3, y_4)$  holds,
- the threefolds  $U_1$  and  $U_2$  satisfy the conditions A, B, C, D.

*Remark 1.12.* The threefolds  $U_1$  and  $U_2$  are isomorphic, because  $\iota(U_1) = U_2$ .

For a fixed biregular involution  $\iota \in \text{Aut}(\mathbb{P}^5)$ , let  $\mathcal{R}$  be a family of complete intersections that are constructed similar to  $U_1$  or  $U_2$ . Then  $\mathcal{R} \subsetneq \mathcal{F}$ . In this paper, we prove the following result.

**Theorem 1.13.** *A general threefold in  $\mathcal{R}$  satisfies the conditions A, B, C, D, E, F, G, H, I.*

**Corollary 1.14.** *Let  $U$  be a general threefold in  $\mathcal{R}$ . Then*

$$\mathcal{P}(U) = \{U\},$$

*i.e., the threefold  $U$  is birationally rigid, and in particular  $U$  is non-rational.*

**Corollary 1.15.** *Birational rigidity is not open in moduli.*

We organize the paper in the following way: we prove Theorem 1.10 in Section 2 omitting the proofs of Lemmas 2.2 and 2.6, we prove Lemma 2.2 in Section 3, we prove Lemma 2.6 in Section 4 omitting the proofs of Lemmas 4.1, 4.4 and 4.7, we prove Lemmas 4.1, 4.4 and 4.7 in Sections 5, 6 and 7, respectively, we prove Theorem 1.13 in Section 8.

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## 2. Pliability count

Let us use the assumptions and notation of Theorem 1.10.

*Remark 2.1.* It follows from Proposition 3.1.2 in [4] that the following conditions are equivalent:

- for any two-dimensional linear subspace  $\Pi \subset \mathbb{P}^5$  such that  $L \subset \Pi$ , the scheme-theoretical intersection  $V_i \cap \Pi$  is reduced along  $L$ ,
- the normal sheaf  $\mathcal{N}_{L/V_i}$  is isomorphic to  $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ ,

where  $L$  is any line in the threefold  $V_i$  such that  $P \notin L$ .

Let us prove Theorem 1.10. Suppose that there is a Mori fibre space  $\rho: U \rightarrow S$ , and a birational map  $\chi: U \dashrightarrow V_1$ . To prove Theorem 1.10, we must show that  $U \cong V_1$  or  $U \cong V_2$ .

Take a sufficiently big very ample divisor  $A$  on the variety  $S$ . Consider a linear system

$$\mathcal{M} = \left| -mK_U + \rho^*(A) \right|$$

for  $m \gg 0$ . Take any  $\sigma \in \text{Bir}(V_1)$ . Put  $\mathcal{D}_1^\sigma = \sigma \circ \chi(\mathcal{M})$  and  $\mathcal{D}_2^\sigma = \phi_2^{-1} \circ \phi_1(\mathcal{D}_1^\sigma)$ . Then

$$n_1^\sigma K_{V_1} + \mathcal{D}_1^\sigma \equiv 0 \equiv n_2^\sigma K_{V_2} + \mathcal{D}_2^\sigma$$

for some natural numbers  $n_1^\sigma$  and  $n_2^\sigma$ . Choose  $\sigma \in \text{Bir}(V_1)$  that minimizes  $\min(n_1^\sigma, n_2^\sigma)$ .

Without loss of generality, we may assume that  $n_1^\sigma \leq n_2^\sigma$ . Put  $\mathcal{D} = \mathcal{D}_1^\sigma$  and  $n = n_1^\sigma$ .

It follows from Theorem 2.4 in [2] that either  $\sigma \circ \chi$  is an isomorphism, or the singularities of the log pair  $(V_1, \frac{1}{n}\mathcal{D})$  are not canonical. Thus, we may assume that  $(V_1, \frac{1}{n}\mathcal{D})$  is not canonical.

**Lemma 2.2.** *Let  $C \subset V_1$  be an irreducible curve. Suppose that  $\text{mult}_C(\mathcal{D}) > n$ . Then*

- either the curve  $C$  is a line,
- or the curve  $C$  is a conic such that  $\langle C \rangle \subset Q_1$ ,
- or the curve  $C$  is a conic such that  $\langle C \rangle \not\subset Q_1$  and  $P \in C$ .

*Proof.* See Section 3. □

For a curve  $C \subset \mathbb{P}^5$ , we denote by  $\langle C \rangle$  the smallest linear subspace in  $\mathbb{P}^5$  containing  $C$ .

**Lemma 2.3.** *Let  $C \subset V_1$  be a conic such that  $\langle C \rangle \not\subset Q_1$  and  $P \in C$ . Then  $\text{mult}_C(\mathcal{D}) \leq n$ .*

*Proof.* Let  $\Lambda \subset \mathbb{P}^5$  be a general three-dimensional linear subspace such that  $\langle C \rangle \subset \Lambda$ . Then

$$V_1|_\Lambda = C + Z,$$

where  $Z$  is an elliptic curve such that  $P \in C \cap Z$ . There is a commutative diagram

$$\begin{array}{ccc} \tilde{V}_1 & \xleftarrow{\delta} & \bar{V} \\ \beta_1 \downarrow & & \downarrow \omega \\ V_1 & \dashrightarrow_{\psi} & \mathbb{P}^2, \end{array}$$

where  $\psi$  is the restriction of the projection  $\mathbb{P}^5 \dashrightarrow \mathbb{P}^2$  from  $\langle C \rangle$ , the morphism  $\beta_1$  is the blow-up of the singular point  $P$ , the morphism  $\delta$  is the blow up of the

proper transform of the curve  $C$  on the threefold  $\tilde{V}_1$ , and  $\omega$  is a rational map whose general fiber is a smooth elliptic curve.

Let  $\bar{E}_1$  be the proper transform of the  $\beta_1$ -exceptional divisor on the threefold  $\bar{V}$ . The map

$$\omega \Big|_{\bar{E}_1} : \bar{E}_1 \dashrightarrow \mathbb{P}^2$$

is birational, which simply means that  $\bar{E}_1$  is a section of a rational fibration  $\omega$ .

For a general point  $O \in \bar{V}_1$ , let  $\bar{Z}$  be the fiber of  $\omega$  that passes through  $O$ . Then  $\bar{Z}$  is a smooth elliptic curve such that  $\bar{E}_1 \cap \bar{Z}$  consists of a single point. Let  $O'$  be a point on  $\bar{Z}$  that is a usual reflection of the point  $O$  on the elliptic curve  $\bar{Z}$  with respect to the point  $\bar{E}_1 \cap \bar{Z}$ .

Let us define an involution  $\tau \in \text{Bir}(\bar{V})$  by putting  $\tau(O) = O'$ , which implies that  $\tau(\bar{E}_1) = \bar{E}_1$ , and  $\tau$  is an isomorphism in codimension one.

Let  $F$  be the  $\delta$ -exceptional divisor, and let  $\bar{\mathcal{D}}$  be the proper transform of  $\mathcal{D}$  on  $\bar{V}$ . Then

$$\bar{\mathcal{D}} \equiv (\beta_1 \circ \delta)^* \left( -nK_{V_1} \right) - \nu_0 \bar{E}_1 - \text{mult}_C(\mathcal{D})F,$$

where  $\nu_0$  is a natural number. It follows from Proposition 4.5 in [8] that

$$\tau(\bar{\mathcal{D}}) \equiv (\beta_1 \circ \delta)^* \left( -\left(15n - 14\text{mult}_C(\mathcal{D})\right)K_{V_1} \right) - \nu_0 \bar{E}_1 - \left(16n - 15\text{mult}_C(\mathcal{D})\right)F,$$

which immediately implies that the equivalence

$$\beta_1 \circ \delta \circ \tau \circ \delta^{-1} \circ \beta_1^{-1}(\bar{\mathcal{D}}) \equiv -\left(15n - 14\text{mult}_C(\mathcal{D})\right)K_{V_1}$$

holds. But  $\beta_1 \circ \delta \circ \tau \circ \delta^{-1} \in \text{Bir}(V_1)$ . But  $15n - 14\text{mult}_C(\mathcal{D}) \geq n$  by the minimality in the choice of the number  $n \in \mathbb{N}$ . Thus,  $\text{mult}_C(\mathcal{D}) \leq n$ .  $\square$

**Lemma 2.4.** *Let  $C \subset V_1$  be a conic such that  $\langle C \rangle \subset Q$  and  $P \in C$ . Then  $\text{mult}_C(\mathcal{D}) \leq n$ .*

*Proof.* Arguing as in [4], we construct an involution  $\zeta \in \text{Bir}(V_1)$  such that

$$\zeta(\mathcal{D}) \equiv -\left(13n - 12\text{mult}_C(\mathcal{D})\right)K_{V_1},$$

which implies that  $\text{mult}_C(\mathcal{D}) \leq n$  due to the minimality of the number  $n$ .  $\square$

**Lemma 2.5.** *Let  $C \subset V_1$  be a line. Then  $\text{mult}_C(\mathcal{D}) \leq n$ .*

*Proof.* Arguing as in [4], we construct an involution  $\zeta \in \text{Bir}(V_1)$  such that

$$\zeta(\mathcal{D}) \equiv -\left(4n - 3\text{mult}_C(\mathcal{D})\right)K_{V_1},$$

which implies that  $\text{mult}_C(\mathcal{D}) \leq n$  due to the minimality of the number  $n$ .  $\square$

Therefore, we see that the log pair  $(V_1, \frac{1}{n}\mathcal{D})$  is canonical outside of finitely many points.

**Lemma 2.6.** *Let  $O$  be a point in  $V_1 \setminus P$ . Then  $(V_1, \frac{1}{n}\mathcal{D})$  is canonical at  $O$ .*

*Proof.* See Section 4.  $\square$

Thus, the log pair  $(V_1, \frac{1}{n}\mathcal{D})$  is not canonical at the point  $P = \text{Sing}(V_1)$ .

Let  $E_1$  be the  $\beta_1$ -exceptional divisor, and let  $\tilde{\mathcal{D}}$  be the proper transform of  $\mathcal{D}$  on  $\tilde{V}_1$ . Then

$$\tilde{\mathcal{D}} \equiv \beta_1^* \left( -nK_{V_1} \right) - \nu_0 E_1,$$

where  $\nu_0$  is a natural number. It follows from Theorem 3.10 in [2] that  $\nu_0 > n$ .

Let  $E_2$  be the  $\beta_2$ -exceptional divisor. Then  $\gamma(E_1) \equiv \beta_2^*(-K_{V_2}) - 2E_2$  and

$$\gamma \left( -K_{V_1} \right) = \beta_2^* \left( -2K_{V_2} \right) - 3E_2$$

because  $\gamma(K_{\tilde{V}_1}) \equiv K_{\tilde{V}_2} \equiv \beta_2^*(K_{V_2}) + E_2$  and  $\gamma$  is an isomorphism in codimension one. But

$$\gamma(\tilde{\mathcal{D}}) \equiv \beta_2^* \left( - (2n - \nu_0) K_{V_2} \right) - (3n - 2\nu_0) E_2,$$

which implies that  $\mathcal{D}_2^\sigma \equiv -(2n - \nu_0)K_{V_2}$ . Then  $n_2^\sigma = 2n - \nu_0 < n = n_1^\sigma$ , which is a contradiction, because  $n_1^\sigma \leq n_2^\sigma$ . The assertion of Theorem 1.10 is proved.

### 3. Exclusion of curves

Let us use the assumptions and notation of Lemma 2.2.

*Remark 3.1.* Let  $\Lambda \subset \mathbb{P}^5$  be a three-dimensional linear subspace. Then  $V_1|_\Lambda$  is reduced along any curve that is not contained in two-dimensional linear subspace, because cubic surface does not intersect an irreducible quadric surface by a double twisted cubic.

Put  $\nu = \text{mult}_C(\mathcal{D})$ . Let  $\Omega$  be the smallest linear subspace in  $\mathbb{P}^5$  such that  $C \subseteq \Omega$ .

Suppose that  $\nu > n$ . To prove Lemma 2.2, we must show that

- either  $\Omega \subset Q_1$  and  $\text{deg}(C) \leq 2$ ,
- or  $P \in C$  and  $\text{deg}(C) = 2$ .

Arguing as in the proof of [4, Lemma 3.3.6], we see that  $\Omega \subset Q_1$  and  $\text{deg}(C) \leq 2$  in the case when  $P \notin \Omega$ . Therefore, to complete the proof of Lemma 2.2, we may assume that  $P \in \Omega$ .

**Lemma 3.2.** *Suppose that  $\text{deg}(C) = 2$ . Then  $P \in C$ .*

*Proof.* Suppose that  $\Omega \subset Q_1$ . Then  $T_1|_\Omega = C + L$ , where  $L$  is a line, which immediately implies that  $P \in C \cap L$ , because  $\text{mult}_P(T_1) = 2$  and  $P \in \Omega$ .  $\square$

Thus, to complete the proof of Lemma 2.2, we may assume that  $\text{deg}(C) \geq 3$ . Let us show that this assumption leads to a contradiction with the inequality  $\nu > n$ .

**Lemma 3.3.** *The inequality  $\text{deg}(C) \leq 5$  holds.*

*Proof.* Let  $D_1$  and  $D_2$  be general surfaces in  $\mathcal{D}$ . Then

$$6n^2 = D_1 \cdot D_2 \cdot H \geq \text{mult}_C(D_1 \cdot D_2) \text{deg}(C) > n^2 \text{deg}(C),$$



where  $D_1 \cdot D_2 \cdot H$  is a degree of the zero-cycle of the corresponding scheme-theoretic intersection, and  $H$  is a general hyperplane section of the threefold  $V_1 \subset \mathbb{P}^5$ . □

**Lemma 3.4.** *The inequality  $\dim(\Omega) \neq 2$  holds.*

*Proof.* Suppose that  $\dim(\Omega) = 2$ . Then  $\Omega \subset Q_1$  and  $\deg(C) = 3$ , because  $C \subseteq \text{Supp}(T_1|_\Omega)$ .

Let  $\Lambda$  be a sufficiently general three-dimensional linear subspace in  $\mathbb{P}^5$  that contains  $\Omega$ . Then

$$\Omega \cap V_1 = C \cup \bar{C},$$

where  $\bar{C}$  is a plane cubic. But  $C \cap \bar{C}$  consists of three distinct points different from  $P$ . Then

$$3n = D \cdot \bar{B} \geq 3\nu > 3n,$$

where  $D$  is a general surface in the linear system  $\mathcal{D}$ . □

**Lemma 3.5.** *The curve  $C$  is singular.*

*Proof.* Suppose that  $C$  is non-singular. Then we have the following cases:

- $\deg(C) = \dim(\Omega) \in \{4, 5\}$  and  $g(C) = 0$ ,
- $\deg(C) = 5$ ,  $\dim(\Omega) = 4$  and  $g(C) = 0$ ,
- $\deg(C) = 5$ ,  $\dim(\Omega) = 4$  and  $g(C) = 1$ ,

where  $g(C)$  is the genus of the curve  $C$ .

Put  $d = \deg(C)$ . Let  $m$  be a natural number such that the curve  $C$  is cut out on  $V_1$  by surfaces in  $| -mK_{V_1} |$  that pass through  $C$ , the scheme-theoretic intersection of two general surfaces in  $| -mK_{V_1} |$  that pass through the curve  $C$  is reduced in a general point of the curve  $C$ .

We have  $m \leq 3$ , and we can put  $m = 2$  unless  $\deg(C) = 5$ ,  $\dim(\Omega) = 4$  and  $g(C) = 0$ .

Let  $\delta: \bar{V} \rightarrow V_1$  be a terminal extraction with the center  $C$  and exceptional divisor  $E$ . Then

$$\left( \delta^* \left( -mK_{V_1} \right) - E \right) \cdot \left( \delta^* \left( -nK_{V_1} \right) - \nu E \right)^2 \geq 0,$$

because  $\delta^* \left( -mK_{V_1} \right) - E$  is nef. Thus, the inequality

$$6mn^2 - dm\nu^2 - 2d\nu n - n^2 \left( 2 - 2g(C) - d - \frac{\text{mult}_P(C)}{2} \right) \geq 0,$$

holds, which easily leads to a contradiction. □

Let  $\tilde{\mathcal{D}}$  be the proper transform of  $\mathcal{D}$  on  $\tilde{V}_1$ , and let  $E_1$  be the  $\beta_1$ -exceptional divisor. Then

$$\tilde{\mathcal{D}} \equiv \beta_1^* \left( -nK_{V_1} \right) - \nu_0 E_1$$

for some integer  $\nu_0 \geq 0$ . Then  $\nu_0 \geq \nu/2$  in the case when  $P \in C$  (see the proof of Lemma 4.12).

**Lemma 3.6.** *The equality  $\dim(\Omega) = 3$  holds.*

*Proof.* Suppose that  $\dim(\Omega) \neq 3$ . Then  $\dim(\Omega) = 4$  and  $\deg(C) = 5$  by Lemma 3.5.

Suppose that  $P \in \text{Sing}(C)$ . Let  $\tilde{C}$  be the proper transform of  $C$  on the threefold  $\tilde{V}_1$ . Then

$$c_1\left(\mathcal{N}_{\tilde{C}/\tilde{V}_1}\right) = -2 - K_{\tilde{V}_1} \cdot \tilde{C} = -2 - (\beta_1^*(K_{V_1}) + E_1) \cdot \tilde{C} = -2 - K_{V_1} \cdot C - E_1 \cdot \tilde{C} = 1,$$

because  $\tilde{C} \cong \mathbb{P}^1$  and  $\text{mult}_P(C) = 2$ . Let  $\delta: \bar{V} \rightarrow \tilde{V}_1$  be a blow up of the curve  $\tilde{C}$ . Then

$$\left| (\beta_1 \circ \delta)^* \left( -nK_{V_1} \right) - \nu_0 \delta^*(E_1) - \nu F \right|$$

does not have fixed components, where  $F$  is the exceptional divisor of the blow up  $\delta$ . But

$$\left| (\beta_1 \circ \delta)^* \left( -3K_{V_1} \right) - \delta^*(E_1) - F \right|$$

does not have base curves. Thus, we have

$$\left( (\beta_1 \circ \delta)^* \left( -nK_{V_1} \right) - \nu_0 \delta^*(E_1) - \nu F \right)^2 \left( (\beta_1 \circ \delta)^* \left( -3K_{V_1} \right) - \delta^*(E_1) - F \right) \geq 0,$$

which leads to a contradiction, because  $\nu > n$  and

$$0 \leq 18n^2 - 10n\nu - 12\nu^2 + 4\nu\nu_0 - \nu_0^2 = \left( 18n^2 - 10n\nu - 8\nu^2 \right) - (2\nu - \nu_0)^2 < 0.$$

Therefore, there is a point  $O \in V_1$  such that  $P \neq O$  and  $\text{Sing}(C) = O$ .

Let  $\nu: \check{V} \rightarrow \tilde{V}_1$  be the blow up of the point that dominates  $O$ , let  $\check{C}$  be the proper transform of the curve  $C$  on the threefold  $\check{V}$ , and let  $\zeta: \check{V} \rightarrow \check{V}$  be the blow up of the curve  $\check{C}$ . Then

$$\left| (\beta_1 \circ \delta \circ \zeta)^* \left( -nK_{V_1} \right) - \nu_0 (\delta \circ \zeta)^*(E_1) - \text{mult}_O(\mathcal{D}) \zeta^*(F) - \nu G \right|$$

has no fixed components, where  $F$  and  $G$  are exceptional divisors of  $\nu$  and  $\zeta$ , respectively.

Suppose that  $P \in C$ . Then  $\text{mult}_O(\mathcal{D}) \geq \nu$  and  $\nu_0 > \nu/2$ . But the linear system

$$\left| (\beta_1 \circ \delta \circ \zeta)^* \left( -3K_{V_1} \right) - (\delta \circ \zeta)^*(E_1) - \zeta^*(F) - G \right|$$

does not have base curves. Arguing as in the proof of Lemma 3.6, we see that

$$18n^2 - 10n\nu - 14\nu^2 + 2\nu\nu_0 - \nu_0^2 + 4\nu \text{mult}_O(\mathcal{D}) - \text{mult}_O^2(\mathcal{D}) \geq 0$$

which leads to a contradiction. Thus, we see that  $P \notin C$ . Then the linear system

$$\left| (\beta_1 \circ \delta \circ \zeta)^* \left( -3K_{V_1} \right) - \zeta^*(F) - G \right|$$

does not have base curves. Arguing as in the case  $P \in C$ , we obtain a contradiction.  $\square$

Thus, we proved that  $\Omega \cong \mathbb{P}^3$  and  $\deg(C) \geq 3$ . Then

$$V_1|_{\Omega} = C + \sum_{i=1}^r m_i C_i,$$

where  $C_i$  is an irreducible curve, and  $m_i \in \mathbb{N}$ . Then

$$\deg(C) + \sum_{i=1}^r m_i \deg(C_i) = 6,$$

and  $C_i \neq C$  for every  $i = 1, \dots, r$  by Remark 3.1.

*Remark 3.7.* The quadric  $Q_1 \cap \Omega$  is irreducible, because  $\dim(\Omega) = 3$ .

Let  $H$  be a general hyperplane section of  $V_1$  such that  $C \subset H$ . Then  $P \in H$  is a singularity of type  $A_k$ . Let  $L$  be a fiber of a natural projection  $E_1 \cong \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ .

**Lemma 3.8.** *Let  $Z_1$  and  $Z_2$  be lines on the threefold  $V_1$  such that  $Z_1 \cap Z_2 = P$ . Then*

$$\tilde{Z}_1 \cap L \neq \emptyset \Rightarrow \tilde{Z}_2 \cap L = \emptyset,$$

where  $\tilde{Z}_1$  and  $\tilde{Z}_2$  be the proper transform of the lines  $Z_1$  and  $Z_2$  on the threefold  $\tilde{V}_1$ , respectively.

*Proof.* The surface  $\alpha_1(E_1)$  is a quadric surface in  $\mathbb{P}^4$  that is given by the equations

$$h(y_0, y_1, y_2, y_3, y_4) = q_2(y_0, y_1, y_2, y_3, y_4) = 0$$

in  $\text{Proj}(\mathbb{C}[y_0, y_1, y_2, y_3, y_4]) \cong \mathbb{P}^4$ , the curve  $\alpha_1(L)$  is a line, and  $\alpha_1(\tilde{Z}_1)$  and  $\alpha_1(\tilde{Z}_2)$  are singular points of the quartic  $Y \subset \mathbb{P}^4$ .

Let  $\Pi$  be the two-dimensional linear subspace in  $\mathbb{P}^5$  that contains  $Z_1$  and  $Z_2$ , and let

$$\phi: \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$$

be a projection from the point  $P$ . Then  $\phi(\Pi)$  is a line in  $\mathbb{P}^4$  such that

$$\alpha_1(\tilde{Z}_1) \in \phi(\Pi) \ni \alpha_1(\tilde{Z}_2).$$

Suppose that  $\tilde{Z}_1 \cap L \neq \emptyset$  and  $\tilde{Z}_2 \cap L \neq \emptyset$ . Then  $\phi(\Pi) = \alpha_1(L)$ , which implies that  $\Pi \subset Q_2$ .

The linear subspace  $\Pi \subset \mathbb{P}^5$  contains two lines  $Z'_1$  and  $Z'_2$  such that

$$P \in Z'_1 \subset V_2 \supset Z'_2 \ni P$$

and  $\phi_2(Z'_1) = \alpha_1(\tilde{Z}_1)$  and  $\phi_2(Z'_2) = \alpha_1(\tilde{Z}_2)$ . Then  $Z'_1 \subseteq V_2 \cap \Pi \supseteq Z'_2$  and  $\Pi \subset Q_2$ , which is impossible, because  $V_2$  satisfies the conditions E and F.  $\square$

Let  $\Upsilon \subset \mathbb{P}^5$  be a hyperplane that is tangent to the quadric  $Q_1$  at the point  $P \in V_1$ . Then

$$\text{mult}_P(C) + \sum_{i=1}^r m_i \text{mult}_P(C_i) \geq 4$$

in the case when  $\Omega \subset \Upsilon$ . Let  $\tilde{H}$  be a proper transform of  $H$  on the threefold  $\tilde{V}_1$ .

**Lemma 3.9.** *Suppose that  $\Omega \not\subset \Upsilon$ . Then  $k \leq 2$ .*

*Proof.* Suppose that  $\Omega \not\subset \Upsilon$  and  $k \geq 3$ . Let us show that this assumption leads to a contradiction.

Let  $\mathcal{H}$  be a linear subsystem in  $|-K_X|$  consisting of surfaces passing through the curve  $C$ , and let  $\tilde{\mathcal{H}}$  be the proper transform of the linear system  $\mathcal{H}$  on the threefold  $\tilde{V}_1$ . Then

$$\tilde{\mathcal{H}}|_{E_1} = L_1 + |L_2|,$$

where  $L_1$  and  $L_2$  are fibers of two different projections  $\mathbb{P}^1 \times \mathbb{P}^1 \cong E_1 \rightarrow \mathbb{P}^1$ .

The surface  $\tilde{H}$  is a general surface in the linear system  $\tilde{\mathcal{H}}$  and

$$E_1 \cap \tilde{H} = L_1 \cup L_{\tilde{H}},$$

where  $L_{\tilde{H}} \in |L_2|$ . Put  $O = L_1 \cap L_{\tilde{H}}$ . Then  $\tilde{H}$  is singular at the point  $O$ .

Let  $\tilde{H}'$  be another sufficiently general surface in the linear system  $\tilde{\mathcal{H}}$ . Then

$$\tilde{H}'|_{\tilde{H}} = mL_1 + L_{\tilde{H}'} + \tilde{C} + \sum_{i=1}^r m_i \tilde{C}_i$$

for some curve  $L_{\tilde{H}'} \in |L_2|$  and for some natural number  $m$ , where  $\tilde{C}$  and  $\tilde{C}_i$  are proper transforms of the irreducible curves  $C$  and  $C_i$  on the surface  $\tilde{H}$ , respectively. Then  $m \geq \text{mult}_O(\tilde{H}) \geq 2$ .

Put  $\check{H} = \alpha_1(\tilde{H})$ . Then  $\check{H}$  is a general hyperplane section of the quartic  $Y \subset \mathbb{P}^4$  given by

$$\begin{aligned} h(y_0, \dots, y_4)t(y_0, \dots, y_4) &= q_1(y_0, \dots, y_4)q_2(y_0, \dots, y_4) \\ &\subset \text{Proj}(\mathbb{C}[y_0, \dots, y_4]) \cong \mathbb{P}^4 \end{aligned}$$

such that  $\alpha_1(L_1) \subset \check{H}$ . Similarly, we have  $\alpha_1(L_1) \subset \alpha_1(\tilde{H}')$ . Then

$$\alpha_1(\tilde{H}')|_{\tilde{H}} = m\alpha_1(L_1) + \alpha_1(L_{\tilde{H}'}) + \alpha_1(\tilde{C}) + \sum_{i=1}^r m_i \alpha_1(\tilde{C}_i),$$

which implies that the cubic  $t(y_0, \dots, y_4) = 0$  contains the line  $\alpha_1(L_1)$ , because  $m \geq 2$ .

The quartic  $Y$  has 12 different singular points that are given by the equations

$$\begin{aligned} h(y_0, y_1, y_2, y_3, y_4) &= t(y_0, y_1, y_2, y_3, y_4) \\ &= q_1(y_0, y_1, y_2, y_3, y_4) \end{aligned}$$

$$= q_2(y_0, y_1, y_2, y_3, y_4) = 0,$$

which implies that  $|\alpha_1(L_1) \cap \text{Sing}(Y)| = 2$ . The latter is impossible by Lemma 3.8.  $\square$

For a given point  $O \in V_1 \setminus P$ , the inequality

$$\text{mult}_O(C) + \sum_{i=1}^r m_i \text{mult}_O(C_i) \geq 4$$

holds in the case when  $\Omega$  is the tangent linear subspace to  $V_1$  at the point  $O$ .

**Lemma 3.10.** *Suppose that the subspace  $\Omega \subset \mathbb{P}^5$  is a tangent linear subspace to the complete intersection  $V_1$  at some point  $O \in V_1 \setminus P$ . Then  $O$  is an ordinary double point of the surface  $H$ .*

*Proof.* An affine part of the complete intersection  $V_1 \subset \mathbb{P}^5$  can be given by the equations

$$\begin{aligned} \mathbb{C}^5 &\cong \text{Spec}(\mathbb{C}[x, y, z, t, w]) \\ &\supset \begin{cases} w = xh_1(x, y, z, t) + y^2 + z^2 + t^2, \\ x = xh_2(x, y, z, t) + g_2(y, z, t) + g_3(x, y, z, t, w) \end{cases} \end{aligned}$$

such that  $O$  is given by  $x = y = z = t = w = 0$ , where  $h_i$  and  $g_i$  are homogeneous polynomials of degree  $i$ . Then  $\Omega$  is given by  $w = x = 0$ , and the surface  $H$  is given by

$$\begin{cases} x = \lambda w, \\ w = xh_1(x, y, z, t) + y^2 + z^2 + t^2, \\ x = xh_2(x, y, z, t) + g_2(y, z, t) + g_3(x, y, z, t, w) \end{cases}$$

for some general  $\lambda \in \mathbb{C}$ . We can consider monomials  $y, z, t$  as local coordinates on the quadric

$$Q = \begin{cases} x = \lambda w, \\ w = xh_1(x, y, z, t) + y^2 + z^2 + t^2 \end{cases}$$

in a neighbourhood of the point  $O$ . Then the surface  $H \subset Q$  is locally given by

$$g_2(y, z, t) - \lambda(y^2 + z^2 + t^2) + \text{higher degree terms},$$

which implies that  $O$  is an ordinary double point of the surface  $H$ , because  $\lambda$  is general.  $\square$

The subspace  $\Omega \subset \mathbb{P}^5$  can be a tangent linear subspace to  $V_1$  at no more than one point, because the quadric  $Q_1|_\Omega$  is irreducible and reduced (see Lemma 3.4.1 in [4]).

**Lemma 3.11.** *The intersection form of  $C_1, \dots, C_r$  on  $H$  is not semi-negative definite.*

*Proof.* Suppose that the intersection form of  $C_1, \dots, C_r$  is semi-negative definite. Then

$$\mathcal{D}|_H = \nu C + \sum_{i=1}^r \nu_i C_i + \mathcal{B} \equiv nC + \sum_{i=1}^r nm_i C_i,$$

where  $\nu_i$  is a non-negative integer, and  $\mathcal{B}$  is a linear system that does not have fixed curves. Then

$$\begin{aligned} & \left( (\nu - n)C + \mathcal{B} + \sum_{\nu_i \geq nm_i} (\nu_i - nm_i)C_i \right) \left( \sum_{nm_i \geq \nu_i} (nm_i - \nu_i)C_i \right) \\ &= \left( \sum_{nm_i > \nu_i} (nm_i - \nu_i)C_i \right)^2, \end{aligned}$$

which implies that  $nm_i \leq \nu_i$  for every  $i$ , because  $\nu > n$  and  $C \cap C_i \neq \emptyset$  for every  $i$ . Then

$$(\nu - n)C + \sum_{i=1}^r (nm_i - \nu_i)C_i + \mathcal{B} \equiv 0,$$

which is a contradiction, because  $\nu > n$ . □

It follows from [1] that the intersection form of  $C_1, \dots, C_r$  on the surface  $H$  is negative definite if and only if they can be contracted on the surface  $H$  to an isolated singular point.

**Lemma 3.12.** *The inequality  $\deg(C) \neq 5$  holds.*

*Proof.* Suppose that  $\deg(C) = 5$ . Then  $r = m_1 = \deg(C_1) = 1$ , and  $H$  is smooth outside of

$$\text{Sing}(C) \cup (C \cap C_1),$$

where  $H$  has singularity of type  $\mathbb{A}_k$  at the point  $P$ .

Suppose that  $\Omega \not\subset \Upsilon$ . Then  $k \leq 2$  by Lemma 3.9. Therefore, the set  $\text{Sing}(H) \setminus P$  contains at most one point that must be ordinary double point of the surface  $H$  by Lemma 3.10. Then

$$C_1 \cdot C_1 \leq -2 + \frac{1}{2} + \frac{k}{k+1} < 0,$$

which is impossible by Lemma 3.11. Thus, we see that  $\Omega \subset \Upsilon$ . Then

$$\text{mult}_P(C) + 1 = \text{mult}_P(C) + \text{mult}_P(C_1) \geq 4,$$

which implies that  $P = \text{Sing}(C) \in C \cap C_1$  and  $\text{mult}_P(C) = 3$ . We see that  $\text{Sing}(H) = \{P\}$ .

The inequality  $k \geq 3$  holds, because it follows from the subadjunction formula that

$$C_1 \cdot C_1 = -2 + \frac{k}{k+1}$$

if  $k \leq 2$ . But  $C_1 \cdot C_1 > 0$  by Lemma 3.11. Then  $E_1|_{\tilde{H}} = \tilde{L}_1 + \tilde{L}_k$ , where  $\tilde{L}_1$  and  $\tilde{L}_2$  are irreducible curves. Put  $\pi = \beta_1|_{\tilde{H}}: \tilde{H} \rightarrow H$  and  $O = \tilde{L}_1 \cap \tilde{L}_k$ . Then  $\tilde{H}$  has singularity of type  $\mathbb{A}_{k-2}$  at  $O$ , and  $\pi$  contracts  $\tilde{L}_1$  and  $\tilde{L}_k$ . Let  $\tilde{C}_1$  be the proper transform of  $C_1$  on  $\tilde{H}$ . Then

$$\tilde{C}_1 \cap (\tilde{L}_1 \cup \tilde{L}_k) = O,$$

because  $C_1 \cdot C_1 = -2 + \frac{k}{k+1}$  in the case when  $O \notin \tilde{C}_1$ . But  $C_1 \cdot C_1 > 0$  by Lemma 3.11.

Let  $\eta: \tilde{H} \rightarrow \bar{H}$  be the minimal resolution of singularities of  $\tilde{H}$ , let  $\bar{L}_1$  and  $\bar{L}_k$  be the proper transforms of the curves  $\tilde{L}_1$  and  $\tilde{L}_k$  on the surface  $\bar{H}$ , respectively. Then  $\eta$  contracts a chain of smooth rational curves  $\bar{L}_2, \dots, \bar{L}_{k-1}$  to the point  $O$  such that

$$\bar{L}_1 \cdot \bar{L}_1 = \dots = \bar{L}_k \cdot \bar{L}_k = -2, \bar{L}_1 \cdot \bar{L}_2 = \bar{L}_2 \cdot \bar{L}_3 = \dots = \bar{L}_{k-2} \cdot \bar{L}_{k-1} = \bar{L}_{k-1} \cdot \bar{L}_k = 1.$$

Let  $\bar{C}_1$  be the proper transform of the curve  $\tilde{C}_1$  on the surface  $\bar{H}$ . Then  $\bar{C}_1 \cdot \bar{C}_1 = -2$  and

$$\bar{C}_1 \cap \bar{L}_1 = \bar{C}_1 \cap \bar{L}_k = \emptyset,$$

where  $\bar{C}_1$  intersects only one curve among the curves  $\bar{L}_1, \bar{L}_2, \bar{L}_3, \dots, \bar{L}_{k-1}, \bar{L}_k$ .

Arguing as in the proof of Lemma 3.9, we easily see that the morphism  $\alpha_1|_{\bar{H}}: \bar{H} \rightarrow \alpha_1(\bar{H})$  contracts the curve  $\bar{C}_1$  to a singular point of the surface  $\alpha_1(\bar{H})$  of type  $\mathbb{A}_s$ . Then the curves

$$\bar{C}_1, \bar{L}_2, \bar{L}_3, \dots, \bar{L}_{k-1}$$

must form a chain and  $s = k - 1$ , where either  $\bar{C}_1 \cap \bar{L}_2 \neq \emptyset$ , or  $\bar{C}_1 \cap \bar{L}_{k-2} \neq \emptyset$ .

Therefore, the curves  $\bar{C}_1, \bar{L}_1, \bar{L}_2, \bar{L}_3, \dots, \bar{L}_{k-1}, \bar{L}_k$  form a graph of type  $\mathbb{D}_{k+1}$ , which implies that their intersection form must be negative definite. Therefore, the inequality  $C_1 \cdot C_1 < 0$  holds, which is impossible by Lemma 3.11.  $\square$

Recall that the singular point  $P \in V_1$  is a singular point of type  $\mathbb{A}_k$  of the surface  $H$ .

**Lemma 3.13.** *The inequality  $\deg(C) \neq 4$  holds.*

*Proof.* Suppose that  $\deg(C) = 4$ . Then either  $C$  is smooth, or  $C$  has one double point.

Suppose that  $r = m_1 = 1$ . Then  $C_1$  is a smooth conic. Then  $H$  is smooth, which implies that  $C_1 \cdot C_1 = -2$ . But the inequality  $C_1 \cdot C_1 > 0$  holds by Lemma 3.11.

Suppose that  $r = 2$  and  $m_1 = m_2 = 1$ . Then  $C_1$  and  $C_2$  are lines in  $\mathbb{P}^5$ . The equalities

$$C_1 \cdot C_1 = -2, C_2 \cdot C_2 = -2, C_1 \cdot C_2 \leq 1$$

hold in the case when  $C_1 \cap C_2 \in H \setminus \text{Sing}(H)$ , which is impossible by Lemma 3.11. Then

$$C_1 \cdot C_1 = -3/2, C_2 \cdot C_2 = -3/2, C_1 \cdot C_2 = 1/2$$

by Lemma 3.10 in the case when  $P \neq C_1 \cap C_2 \in \text{Sing}(H)$ . Thus, we see that

$$C_1 \cap C_2 = P = \text{Sing}(H)$$

by Lemma 3.11. Similarly, we see that  $P$  is not an ordinary double point of the surface  $H$ .

Thus, the intersection  $\tilde{H} \cap E_1$  consists of two smooth irreducible curves  $\tilde{L}_1$  and  $\tilde{L}_2$ .

Let  $\tilde{C}_1$  and  $\tilde{C}_2$  be the proper transforms of  $C_1$  and  $C_2$  on the surface  $\tilde{H}$ , respectively. Then

$$\tilde{C}_1 \cap \tilde{L}_i \neq \emptyset \Rightarrow \tilde{C}_2 \cap \tilde{L}_i = \emptyset$$

for  $i = 1$  and  $i = 2$  by Lemma 3.8. The curves  $\tilde{C}_1, \tilde{C}_2, \tilde{L}_1, \tilde{L}_2$  on the surface  $\tilde{H}$  can be contracted to an isolated singular point of type  $\mathbb{A}_{k+2}$ , which is impossible by Lemma 3.11.

Therefore, we proved that  $r = 1$  and  $m_1 = 2$ . Then  $C_1$  is a line, and

$$C_1 \cdot C_1 = \frac{1 - C \cdot C_1}{2}$$

on the surface  $H$ . So, the inequality  $C \cdot C_1 < 1$  holds, because  $C_1 \cdot C_1 > 0$  by Lemma 3.11. Then

$$C \cap C_1 \subset \text{Sing}(H),$$

because  $C \cap C_1 \neq \emptyset$ . To complete the proof, we must show that  $C \cdot C_1 \geq 1$ .

Suppose that  $P \neq C \cap C_1$ . There is a point  $O \in C \cap C_1$  such that  $O \in \text{Sing}(H)$ . Then

$$\text{mult}_O(C) + 2\text{mult}_O(C_1) \geq 4,$$

because  $\Omega$  must be the tangent linear subspace to  $V_1$  at the point  $O$ . Then  $\text{mult}_O(C) = 2$  and

$$1 > C \cdot C_1 \geq \frac{\text{mult}_O(C)}{2} = 1,$$

because  $O$  is an ordinary double point of the surface  $H$  by Lemma 3.10.

Thus, we see that  $C \cap C_1 = P$ . Put  $\bar{Q} = Q_1|_\Omega$  and  $\bar{T} = T_1|_\Omega$ .

Suppose that  $\bar{Q}$  is a cone. Then  $C_1$  is its rulings. Either the cubic  $\bar{T}$  is singular along  $C_1$ , or the cubic  $\bar{T}$  is tangent to the cone  $\bar{Q}$  along the line  $C_1$ . Hence, there is a two-dimensional linear subspace  $\Pi \subset \Omega$  that is tangent to both  $\bar{T}$  and  $\bar{Q}$  along the line  $C_1$ . The sub-scheme  $V_1|_\Pi$  is not reduced along the line  $C_1$ , which is impossible, because  $V_1$  satisfies the condition E.

We see that the surface  $\bar{Q}$  is smooth. Then  $\bar{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$ , where  $C$  and  $C_1$  are divisors of bi-degree  $(3, 1)$  and  $(0, 1)$  on the quadric  $\bar{Q}$ , respectively.

It follows from  $C \cap C_1 = P$  that  $C_1$  is tangent to the curve  $C$  at the point  $P$  with multiplicity 3, because the equality  $C_1 \cdot C = 3$  holds on the quadric surface  $\bar{Q}$ .

Let  $\tilde{H}$  be a proper transform of  $H$  on the threefold  $\tilde{V}$ , let  $\tilde{C}$  and  $\tilde{C}_1$  be the proper transforms of the curves  $C$  and  $C_1$  on the surface  $\tilde{H}$ , respectively. Then



$\tilde{H}$  is smooth by Lemma 3.9, and

$$C \cdot C_1 = \begin{cases} 3/2 & \text{in the case when } k = 1, \\ 4/3 & \text{in the case when } k = 2, \end{cases}$$

because  $\tilde{C} \cdot \tilde{C}_1 = 2$  on the surface  $\tilde{H}$ . But  $C \cdot C_1 < 1$ . □

Thus, the curve  $C$  is a smooth rational curve of degree 3.

**Lemma 3.14.** *Suppose that  $r = 1$ . Then  $m_1 \neq 3$ .*

*Proof.* Suppose that  $m_1 = 3$ . Then  $C_1$  is a line. Put  $\bar{Q} = Q_1|_\Omega$  and  $\bar{T}_1 = T|_\Omega$ . Then

$$\bar{Q} \cdot \bar{T}_1 = C + 3C_1$$

in  $\Omega \cong \mathbb{P}^3$ . But the quadric surface  $\bar{Q}$  is irreducible.

Suppose that  $\bar{Q}$  is smooth. Then  $C$  is a divisor of type  $(3, 0)$  on the surface  $\bar{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$ , which is impossible, because the curve  $C$  is irreducible and reduced. Thus, the quadric  $\bar{Q}$  is a cone.

The line  $C_1$  is a ruling of the cone  $\bar{Q}$ . Then either  $\bar{T}_1$  is singular along  $C_1$ , or  $\bar{T}_1$  is tangent to the quadric  $\bar{Q}$  along  $C_1$ . Then there is a two-dimensional linear subspace  $\Pi \subset \Omega$  that is tangent to  $\bar{T}_1$  and  $\bar{Q}$  along  $C_1$ . Then  $V_1|_\Pi$  is not reduced along  $C_1$ , which contradicts the condition E. □

**Lemma 3.15.** *The inequality  $r \neq 1$  holds.*

*Proof.* Suppose that  $r = 1$ . Then  $m_1 = 1$  by Lemma 3.14, and  $V_1|_\Omega = C + C_1$ , where  $C_1$  is an irreducible reduced cubic curve. Then the curve  $C_1$  is not contained in any two-dimensional linear subspace in  $\Omega \cong \mathbb{P}^3$ , because  $Q_1|_\Omega$  is irreducible. Then  $C_1$  is a smooth rational cubic curve.

It follows from Lemmas 3.10 and 3.9 that  $H$  is smooth outside of  $P$ , and either  $P$  is an ordinary double point of the surface  $H$ , or  $P$  is a singular point of the surface  $H$  of type  $A_2$ . Then

$$C_1 \cdot C_1 = \begin{cases} -3/2 & \text{in the case when } k = 1, \\ -4/3 & \text{in the case when } k = 2, \end{cases}$$

on the surface  $H$ . But  $C_1 \cdot C_1 > 0$  by Lemma 3.11. □

Let  $\pi: S \rightarrow H$  be a minimal resolution of singularities. Then  $\pi$  contracts a chain of smooth rational curves  $L_1, \dots, L_k$  to the point  $P$  such that  $L_i^2 = -2$  on the surface  $S$  for all  $i$ , and

$$L_1 \cdot L_2 = L_2 \cdot L_3 = \dots = L_{k-2} \cdot L_{k-1} = L_k \cdot L_{k-1} = 1$$

on the surface  $S$ . Then  $L_1 \cdot L_i = L_k \cdot L_j = 0$  for all  $i \neq 2$  and  $j \neq k - 1$ .

**Lemma 3.16.** *The inequality  $r \neq 3$  holds.*

*Proof.* Suppose that  $r = 3$ . Then the curves  $C_1, C_2$  and  $C_3$  are distinct lines.

The intersection  $C_1 \cap C_2 \cap C_3$  contains no smooth points of the surface  $H$ , because otherwise there is a two-dimensional linear subspace  $\Pi \subset \mathbb{P}^5$  that contains  $C_1, C_2, C_2$ , which is impossible, because  $\dim(\Omega) = 3$  and the quadric surface  $Q_1|_\Omega$  is irreducible and reduced.

To complete the proof, we must consider the following possible cases:

- the intersection  $C_1 \cap C_2 \cap C_3$  consists of the point  $P$ ,
- the intersection  $C_1 \cap C_2 \cap C_3$  consists of a point in  $\text{Sing}(H) \setminus P$ ,
- the intersection  $C_1 \cap C_2 \cap C_3$  is empty.

Suppose that  $C_1 \cap C_2 \cap C_3 = P$ . So, the surface  $H$  must be smooth outside of the point  $P$ , and it follows from Lemma 3.8 that  $k = 1$ . Hence, we can contract the curves  $C_1, C_2$  and  $C_3$  to an isolated singular points of type  $\mathbb{D}_4$ , which is a contradiction.

Suppose that  $C_1 \cap C_2 \cap C_3$  consists of a point  $O \in \text{Sing}(H) \setminus P$ . Then  $O$  is an ordinary double point of the surface  $H$  by Lemma 3.10. But  $P \in C_1 \cup C_2 \cup C_3$ , and  $k \leq 2$  by Lemma 3.9, which implies that  $C_1, C_2$  and  $C_3$  can be contracted to a points of type  $\mathbb{D}_{k+4}$ , which is a contradiction.

Thus, we see that  $C_1 \cap C_2 \cap C_3 = \emptyset$ . Then  $Q_1|_\Omega$  is smooth. Thus, we may assume that

$$C_1 \cap C_2 \neq \emptyset, C_2 \cap C_3 \neq \emptyset, C_1 \cap C_3 = \emptyset.$$

The surface  $H$  has singularity of type  $\mathbb{A}_k$  at the point  $P = \text{Sing}(H)$ , and  $k \leq 2$  by Lemma 3.9. Moreover, the blow up  $\beta_1: \tilde{V}_1 \rightarrow V_1$  induces a partial resolution of singularities of the surface  $H$ . Thus, it follows from Lemma 3.8 that  $C_1, C_2, C_3$  can be contracted to the following points:

- a point of type  $\mathbb{D}_{k+3}$  in the case when  $P \in C_2$  and  $C_1 \not\ni P \notin C_3$ ,
- a point of type  $\mathbb{A}_{k+3}$  in the case when  $P \in C_1$  or  $P \in C_3$ ,

which is a contradiction. The obtained contradiction completes the proof.  $\square$

Hence, we see that  $r = 2$  by Lemmas 3.14 and 3.16.

**Lemma 3.17.** *Either  $m_1 = 2$ , or  $m_2 = 2$ .*

*Proof.* Suppose that  $m_1 = m_2 = 1$ . We may assume that  $C_1$  is a line, and  $C_2$  is a conic, which implies that  $\text{Sing}(H) = P$  and  $k \leq 2$  by Lemma 3.9. Then  $C_1$  and  $C_2$  can be contracted on the surface  $H$  to the following points:

- a singular point of type  $\mathbb{A}_{k+2}$  in the case when  $P \notin C_2 \cap C_2$ ,
- a singular point of type  $\mathbb{A}_{k+2}$  or  $\mathbb{D}_{k+2}$  in the case when  $P \in C_1 \cap C_2$ ,

because  $C_1 \cup C_2$  is not contained in a two-dimensional linear subspace of  $\mathbb{P}^5$ .  $\square$

We see that  $r = 2$ , the curves  $C_1$  and  $C_2$  are lines. We may assume that  $m_1 = 1$  and  $m_2 = 2$ .

**Lemma 3.18.** *Let  $O$  be a point in  $V_1 \setminus P$ . Then  $\Omega$  is not a tangent subspace to  $V_1$  at the point  $O$ .*

*Proof.* Suppose that  $\Omega$  is a tangent linear subspace to  $V_1$  at the point  $O$ . Then

$$\text{mult}_O(C) + \text{mult}_O(C_1) + 2\text{mult}_O(C_2) \geq 4,$$

which implies that  $O = C \cap C_1 \cap C_2$ . Put  $\bar{Q} = Q_1|_\Omega$  and  $\bar{T}_1 = T|_\Omega$ . Then

$$\bar{Q} \cdot \bar{T} = C + C_1 + 2C_2$$

in  $\Omega \cong \mathbb{P}^3$ , where  $\bar{Q}$  is an irreducible quadric cone, whose vertex is  $O$ .

The line  $C_2$  is a ruling of the cone  $\bar{Q}$ . Then

- either the cubic  $\bar{T}$  is singular along  $C_2$ ,
- or the cubic  $\bar{T}$  is tangent to the quadric  $\bar{Q}$  along  $C_2$ .

There is a two-dimensional linear subspace  $\Pi \subset \Omega$  that is tangent to  $\bar{T}$  and  $\bar{Q}$  along  $C_2$ , which implies that  $V_1|_\Pi$  is not reduced along the line  $C_2$ . The latter contradicts the condition E.  $\square$

Arguing as in the proof of Lemma 3.18, we see that  $Q_1|_\Omega$  is smooth and  $\Omega \not\subset \Upsilon$ .

*Remark 3.19.* It follows from  $Q_1|_\Omega \cong \mathbb{P}^1 \times \mathbb{P}^1$  that  $C, C_1, C_2$  form the following configuration:

- the curve  $C$  intersects the curve  $C_1$  transversally in one point,
- the curve  $C_1$  intersects the curve  $C_2$  transversally in one point,
- either  $C$  intersects  $C_2$  transversally in two points, or  $C$  is tangent to  $C_2$  in one point.

The subspace  $\Omega \subset \mathbb{P}^3$  is not a tangent linear subspace to  $V_1$  at any smooth point of  $V_1 \subset \mathbb{P}^5$ .

*Remark 3.20.* The surface  $H$  is smooth outside of the set  $C_2 \cup P$ . Moreover, we have

$$\text{Sing}(H) \subsetneq P \cup \left( C_2 \setminus \left( (C_2 \cap C_1) \cup (C_2 \cap C_1) \right) \right),$$

and  $H$  has singularity of type  $\mathbb{A}_k$  at the point  $P \in C_2 \cup (C_1 \cap C)$ , where  $k \leq 2$  by Lemma 3.9.

The equivalence  $K_H \sim 0$  holds. Thus, it follows from the adjunction formula that

$$C \cdot C - \frac{k}{k+1} \text{mult}_P(C) = C_1 \cdot C_1 - \frac{k}{k+1} \text{mult}_P(C_1) = -2,$$

because  $C$  and  $C_1$  are smooth rational curves. It follows from  $C + C_1 + 2C_2 \equiv \mathcal{O}_{\mathbb{P}^5}(1)|_H$  that

$$(C + C_1 + 2C_2) \cdot C_1 = (C + C_1 + 2C_2) \cdot C_2 = 1.$$

**Lemma 3.21.** *The equality  $k = 2$  holds.*

*Proof.* Suppose that  $k = 1$ . In the case when  $P = C \cap C_1 \cap C_2$ , we have

$$C \cdot C_1 = 1/2 = C_2 \cdot C_1 = 1/2,$$

which implies that  $C_1 \cdot C_1 = -1/2$ . But  $C_1 \cdot C_1 = -3/2$  by the adjunction formula.

Therefore, we see that  $P \neq C \cap C_1 \cap C_2$ . Then

- in the case when  $P \in C_2$  and  $P \notin C \cup C_1$ , we have

$$C \cdot C_1 = 1, C \cdot C_2 = 2, C_1 \cdot C_2 = 1, C_1 \cdot C_1 = -2, C_2 \cdot C_2 = -1,$$

- in the case when  $P \in C \cap C_2$  and  $P \neq C \cap C_1$ , we have

$$C \cdot C_1 = 1, C \cdot C_2 = 3/2, C_1 \cdot C_2 = 1, C_1 \cdot C_1 = -2, C_2 \cdot C_2 = -3/4,$$

- in the case when  $P = C \cap C_1$  and  $P \notin C_2$ , we have

$$C \cdot C_1 = 1/2, C \cdot C_2 = 2, C_1 \cdot C_2 = 1, C_1 \cdot C_1 = -3/2, C_2 \cdot C_2 = -1,$$

which is impossible by Lemma 3.11. Thus, we see that  $P = C_1 \cap C_2 \neq C \cap C_1$ . Then

$$2 + C_1 \cdot C_1 = 1 + C_1 \cdot C_1 + 2C_2 \cdot C_1 = (C + C_1 + 2C_2) \cdot C_1 = 1,$$

which implies that  $C_1 \cdot C_1 = -1$ . But  $C_1 \cdot C_1 = -3/2$  by the adjunction formula.  $\square$

Hence, we see that  $H$  has singularity of type  $\mathbb{A}_2$  at the point  $P = \text{Sing}(V_1)$ .

**Lemma 3.22.** *The case  $P \notin C \cup C_1$  is impossible.*

*Proof.* Suppose that  $P \notin C \cup C_1$ . Then  $P \in C_2$  and

$$C \cdot C_1 = 1, C \cdot C_2 = 2, C_1 \cdot C_2 = 1, C_1 \cdot C_1 = -2, C_2 \cdot C_2 = -1,$$

which is impossible by Lemma 3.11.  $\square$

**Lemma 3.23.** *The case  $P \notin C_2$  is impossible.*

*Proof.* Suppose that  $P \notin C_2$ . Then  $P = C \cap C_1$ . Therefore, we have

$$C \cdot C_2 = 2, C_1 \cdot C_2 = 1, C_1 \cdot C_1 = -4/3,$$

which immediately implies that  $C \cdot C_1 = 1/3$  and  $C_2 \cdot C_2 = -1$ . Thus, we see that the intersection form of the curves  $C_1$  and  $C_2$  is negative definite, which is impossible by Lemma 3.11.  $\square$

Therefore, we see that  $P \in C_2 \cap (C_1 \cup C)$ .

**Lemma 3.24.** *The case  $C_1 \not\ni P \in C \cap C_2$  is impossible.*

*Proof.* Suppose that  $C_1 \not\ni P \in C \cap C_2$ . Then  $C \cdot C = -4/3$  by the adjunction formula. But

$$C \cdot C + C_1 \cdot C + 2C_2 \cdot C = 3,$$

which implies that  $C \cdot C_2 = 5/3$ , because  $C \cdot C_1 = 1$ . Thus, we have

$$C_1 \cdot C_2 = 1, C_1 \cdot C_1 = -2, C_2 \cdot C_2 = -5/6,$$

which is impossible by Lemma 3.11.  $\square$

**Lemma 3.25.** *The case  $C \not\ni P = C_1 \cap C_2$  is impossible.*

*Proof.* Suppose that  $C \not\cong P = C_1 \cap C_2$ . Then it follows from the adjunction formula that

$$C_1 \cdot C_1 = C_2 \cdot C_2 = -4/3,$$

but  $C_1 \cdot C_2 = 1/3$  by Lemma 3.8, which is impossible by Lemma 3.11.  $\square$

Hence, we see that  $P = C_1 \cap C \cap C_2$ . Then

$$C \cdot C = C_1 \cdot C_1 = C_2 \cdot C_2 = -4/3$$

by the adjunction formula. But it follows from Lemma 3.8 that  $C_1 \cdot C_2 = 1/3$ . Then

$$C \cdot C_1 + C_1 \cdot C_1 + 2C_2 \cdot C_1 = \text{deg}(C_1) = 1,$$

which implies that  $C \cdot C_1 = 5/3$ . But  $C \cdot C_1 \leq 2/3$ , because the curves  $C$  and  $C_1$  intersect transversally in the point  $P$ . The obtained contradiction completes the proof of Lemma 2.2.

#### 4. Exclusion of points

Let us use the assumptions and notation of Lemma 2.6.

**Lemma 4.1.** *The inequality  $\text{mult}_O(\mathcal{D}) \leq 2n$  holds.*

*Proof.* See Section 5.  $\square$

Thus, it follows from Theorem 1.1 in [5] that there is a sequence of blow ups

$$\begin{aligned} X_N &\xrightarrow{\pi_N} X_{N-1} \xrightarrow{\pi_{N-1}} \cdots \xrightarrow{\pi_{L+1}} X_{L+1} \xrightarrow{\pi_L} X_L \\ &\xrightarrow{\pi_{L-1}} X_{L-1} \xrightarrow{\pi_{L-2}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} V_1 \end{aligned}$$

such that  $1 \leq L < N$  and the following conditions are satisfied:

- the morphism  $\pi_1$  is a blow up of the point  $O \in V_1$ ,
- for  $i \geq 2$ , the morphism  $\pi_i$  is a blow up of a smooth subvariety  $B_{i-1} \subset X_{i-1}$ ,
- let  $E_i \subset X_i$  be an exceptional divisor of the blow up  $\pi_i$ , then  $B_i \subset E_i$ ,
- for  $s > i$ , let  $E_i^s$  be a proper transform of  $E_i$  on the threefold  $V_s$ , then  $B_s \not\subset E_i^s \subset X_s$ ,
- for  $i \leq L - 1$ , the subvariety  $B_i \subset E_i$  is a point,
- for  $i \geq L$ , the subvariety  $B_i \subset E_i$  is a smooth curve such that
  - the curve  $B_L \subset E_L$  is a line in  $E_L \cong \mathbb{P}^2$ ,
  - for  $i > L$ , the curve  $B_i \subset E_i$  is a section of the  $\mathbb{P}^1$ -bundle  $\pi_i|_{E_i}: E_i \rightarrow B_{i-1}$ ,
- let  $\mathcal{D}_i$  be the proper transform of the linear system  $\mathcal{D}$  on the threefold  $X_i$ , then

$$K_{X_s} + \frac{1}{n} \mathcal{D}_s \equiv \sum_{i=1}^L \left( \frac{\nu_1 + \cdots + \nu_i}{n} - 2i \right) E_i^s + \sum_{i=L+1}^s \left( \frac{\nu_1 + \cdots + \nu_i}{n} - L - s \right) E_i^s$$

- for  $s > L$ , where  $\nu_{i+1} = \text{mult}_{B_i}(\mathcal{D}_i)$  and  $\nu_1 = \text{mult}_O(\mathcal{D})$ ,
- the inequality  $\sum_{i=1}^s \nu_i \leq n(L + s)$  holds for every  $s > L$  such that  $s \neq N$ , but

$$(4.2) \quad \nu_1 + \dots + \nu_N > n(L + N).$$

Let  $\Lambda$  be the three-dimensional linear subspace in  $\mathbb{P}^5$  that is tangent to the threefold  $V_1$  at the point  $O$ . Arguing as in the proof of [4, Proposition 3.3.1], we see that  $P \in \Lambda$ .

Let  $L_1, \dots, L_r$  be lines in  $V_1$  that pass through the point  $O$ . Then  $P \in \cup_{i=1}^r L_i$ , and we may assume that  $P \in L_1$ . Let  $D_1$  and  $D_2$  be general surfaces in  $\mathcal{D}$ . Put

$$D_1 \cdot D_2 = \alpha_t L_t + C_t,$$

where  $D_1 \cdot D_2$  is an effective one-cycle that corresponds to the scheme-theoretic intersection of the divisors  $D_1$  and  $D_2$ , and  $C_t$  is an effective one-cycle on  $V_1$  such that  $L_t \not\subset \text{Supp}(C_t)$ . Then

$$6n^2 = -K_{V_1} \cdot (\alpha_t L_t + C_t) = \alpha_t + \text{deg}(C_t) \geq \alpha_t.$$

Let  $L_t^s$  be a proper transform of the line  $L_t$  on the threefold  $X_s$ . Put

$$k_t = \max \left\{ s \leq L \mid B_{s-1} \in L_t^{s-1} \right\}$$

whenever  $B_1 \in L_t^1$ . In the case when  $B_1 \notin L_t^1$ , we put  $k_t = 1$ . Then either  $B_{k_t} \notin L_t^{k_t}$  or  $k_t = L$ .

Let  $C_t^s$  be a proper transform of the one-cycle  $C_t$  on the threefold  $X_s$ . Then

$$(4.3) \quad k_t \alpha_t + \text{mult}_O(C_t) + \sum_{i=1}^{L-1} \text{mult}_{B_i}(C_t^i) \geq \sum_{i=1}^N \nu_i^2 > \frac{(N + L)^2}{N} n^2$$

by Theorem 7.5 in [6], because  $\sum_{i=1}^N \nu_i > n(L + N)$ . It should be pointed out that

$$\text{mult}_O(C_t) \geq \text{mult}_{B_1}(C_t^1) \geq \dots \geq \text{mult}_{B_{L-1}}(C_t^{L-1}).$$

**Lemma 4.4.** *The inequality  $L \neq 1$  holds.*

*Proof.* See Section 6. □

The inequalities  $3 \geq r \geq 1$  hold, because  $V_1$  satisfies the condition H.

**Lemma 4.5.** *The inequality  $k_1 \dots k_r > 1$  holds.*

*Proof.* Suppose that  $k_1 = \dots = k_r = 1$ . Then the linear system

$$\left| (\pi_1 \circ \pi_2)^* \left( -K_{V_1} \right) - \pi_2^*(E_1) - E_2 \right|$$

does not have base curves. Therefore, we have

$$\begin{aligned} & 6n^2 - \alpha_t - \text{mult}_O(C_t) - \text{mult}_{B_1}(C_t^1) \\ &= \left( (\pi_1 \circ \pi_2)^* \left( -K_{V_1} \right) - \pi_2^*(E_1) - E_2 \right) \cdot C_t^2 \geq 0, \end{aligned}$$

but it follows from  $\text{mult}_O(C_t) \geq \text{mult}_{B_1}(C_t^1) \geq \dots \geq \text{mult}_{B_{L-1}}(C_t^{L-1})$  that the inequality

$$\text{mult}_O(C_t) + \sum_{i=1}^{L-1} \text{mult}_{B_i}(C_t^i) \leq \frac{(\text{mult}_O(C_t) + \text{mult}_{B_1}(C_t^1))L}{2}$$

holds. Thus, we have

$$\begin{aligned} 3n^2L &\geq \alpha_t + \left(3n^2 - \frac{\alpha_t}{2}\right)L \geq \alpha_t + \text{mult}_O(C_t) + \sum_{i=1}^{L-1} \text{mult}_{B_i}(C_t^i) \\ &\geq \sum_{i=1}^N \nu_i^2 > \frac{(N+L)^2}{N}n^2 > 4Ln^2, \end{aligned}$$

which is a contradiction. □

Let  $H_t$  be a proper transform on the threefold  $V_{k_t}$  of a sufficiently general hyperplane section of the complete intersection  $V_1 \subset \mathbb{P}^5$  that passes through the line  $L_t$ . Then

$$H_t \sim (\pi_1 \circ \pi_2 \circ \dots \circ \pi_{k_t})^* (-K_{V_1}) - (\pi_2 \circ \dots \circ \pi_{k_t})^*(E_1) - \dots - E_{k_t}$$

and  $L_t^{k_t}$  is the only curve on  $V_{k_t}$  that has negative intersection with  $H_t$ . Then

$$0 \leq H_t \cdot C_t^{k_t} \leq 6n^2 - \alpha_t - \text{mult}_O(C_t) - \sum_{i=1}^{k_t-1} \text{mult}_{B_i}(C_t^i),$$

and it follows from the inequality (4.3) that

$$(4.6) \quad (k_t - 1)\alpha_t + 6n^2 \frac{L}{k_t} \geq (k_t - 1)\alpha_t + 6n^2 \frac{L}{k_t} + \alpha_t \left(1 - \frac{L}{k_t}\right) > \frac{(N+L)^2}{N}n^2,$$

because  $\text{mult}_O(C_t) \geq \text{mult}_{B_1}(C_t^1) \geq \dots \geq \text{mult}_{B_{L-1}}(C_t^{L-1})$  and  $L \geq k_t$ .

**Lemma 4.7.** *The inequality  $k_1 \neq 1$  holds.*

*Proof.* See Section 7. □

Put  $k = k_1$  and  $\alpha = \alpha_1$  and  $\mu = \text{mult}_{L_1}(\mathcal{D})$ .

*Remark 4.8.* The inequality  $\mu \leq n$  holds by Lemma 2.5.

Let  $v_k: \tilde{X}_k \rightarrow X_k$  be the blow up of the point dominating  $P$ , let  $\omega_k: \bar{X}_k \rightarrow \tilde{X}_k$  be the blow up of the proper transform of  $L_1$ , let  $F_k$  and  $G_k$  be the exceptional divisor of  $v_k$  and  $\omega_k$ , respectively.

**Lemma 4.9.** *The isomorphisms  $F_k \cong G_k \cong \mathbb{P}^1 \times \mathbb{P}^1$  hold.*

*Proof.* The isomorphism  $F_k \cong \mathbb{P}^1 \times \mathbb{P}^1$  is obvious. There is a commutative diagram

$$\begin{array}{ccccccc}
 \bar{X}_k & \xrightarrow{\bar{\pi}_k} & \bar{X}_{k-1} & \xrightarrow{\bar{\pi}_{k-1}} & \cdots & \xrightarrow{\bar{\pi}_2} & \bar{X}_1 & \xrightarrow{\bar{\pi}_1} & \bar{V}_1 \\
 \downarrow \omega_k & & \downarrow \omega_{k-1} & & & & \downarrow \omega_1 & & \downarrow \delta_1 \\
 \tilde{X}_k & \xrightarrow{\tilde{\pi}_k} & \tilde{X}_{k-1} & \xrightarrow{\tilde{\pi}_{k-1}} & \cdots & \xrightarrow{\tilde{\pi}_2} & \tilde{X}_1 & \xrightarrow{\tilde{\pi}_1} & \tilde{V}_1 \\
 \downarrow \nu_k & & \downarrow \nu_{k-1} & & & & \downarrow \nu_1 & & \downarrow \beta_1 \\
 X_k & \xrightarrow{\pi_k} & X_{k-1} & \xrightarrow{\pi_{k-1}} & \cdots & \xrightarrow{\pi_2} & X_1 & \xrightarrow{\pi_1} & V_1,
 \end{array}$$

where  $\tilde{\pi}_i$  and  $\bar{\pi}_i$  are birational morphisms,  $\nu_i$  is the blow up of the point that dominates  $P \in V_1$ , the morphism  $\omega_i$  is the blow up of the proper transform of the curve  $L_1$ , and  $\delta_1$  is the blow up of the proper transform of the line  $L_1$  on the threefold  $\tilde{V}_1$ ,

Let  $\tilde{O}$  be the point in  $\tilde{V}_1$  that dominates  $O \in V_1$ . Then  $\tilde{\pi}_1$  is the blow up of the point  $\tilde{O}$ .

Let  $G$  be the exceptional divisor of  $\delta_1$ . Then  $G \cong \mathbb{P}^1 \times \mathbb{P}^1$ , because  $V_1$  satisfies the generality condition D. But  $G_1 \cong G \cong \mathbb{P}^1 \times \mathbb{P}^1$ , because  $\bar{\pi}_1$  is a blow up of a smooth curve in  $G$ .

Let  $G_i$  be the exceptional divisor of  $\omega_i$ . Arguing as above, we see that

$$G_k \cong G_{k-1} \cong \cdots \cong G_1 \cong G \cong \mathbb{P}^1 \times \mathbb{P}^1,$$

which completes the proof. □

Let  $Z_1$  and  $Z_1$  be curves on  $G_k$  such that  $Z_1 \cdot Z_1 = Z_2 \cdot Z_2 = 0$ ,  $Z_1 \cdot Z_2 = 1$  and  $Z_2$  is contracted by the morphism  $\omega_k$  to a point in  $\tilde{X}_k$ .

**Lemma 4.10.** *The equivalence  $-G_k|_{G_k} \sim Z_1 + (k + 1)Z_2$  holds.*

*Proof.* There is an integer  $\epsilon$  such that  $-G_k|_{G_k} \sim Z_1 + \epsilon Z_2$ . Then

$$2\epsilon = (Z_1 + \epsilon Z_2) \cdot (Z_1 + \epsilon Z_2) = G_k^3 = -c_1(\mathcal{N}_{\tilde{L}_1^k/\tilde{X}_k}) = 2 + K_{\tilde{X}_k} \cdot \tilde{L}_1^k = 2 + 2k,$$

where  $\tilde{L}_1^k$  is the proper transform of the line  $L_1$  on the threefold  $\tilde{X}_k$ . □

Let  $\bar{\mathcal{D}}_k$  be the proper transform of the linear system  $\mathcal{D}$  on the threefold  $\bar{X}_k$ . Then

$$\begin{aligned}
 \bar{\mathcal{D}}_k \sim & (\nu_k \circ \omega_k)^* \left( (\pi_1 \circ \cdots \circ \pi_k)^* \left( -nK_{V_1} \right) - (\pi_2 \circ \cdots \circ \pi_k)^* (\nu_1 E_1) \right. \\
 & \left. - \cdots - \nu_k E_k \right) - \omega_k^* (\nu_0 F_k) - \mu G_k,
 \end{aligned}$$

where  $\nu_0$  is an integer number. Therefore, it follows from Lemma 4.10 that

$$\bar{\mathcal{D}}_k|_{G_k} \sim \mu Z_1 + (n + (k + 1)\mu - \nu_1 - \cdots - \nu_k - \nu_0) Z_2,$$

which implies that  $\sum_{i=1}^k \nu_i \leq n + \mu(k + 1) - \nu_0$ , because  $\bar{\mathcal{D}}_k|_{G_k}$  is effective.



**Lemma 4.11.** *The inequality*

$$\sum_{i=1}^s \nu_i N/s > (N + L)n$$

holds for every  $1 \leq s \leq N$ .

*Proof.* The inequality  $\sum_{i=1}^N \nu_i > n(N + L)$  implies that

$$\left( \sum_{i=1}^s \nu_i \right) \frac{N}{s} > n(N + L)$$

for every  $1 \leq s \leq N$ , because  $\nu_1 \geq \nu_2 \geq \dots \geq \nu_N$ . □

Put  $\theta = \sum_{i=1}^k \nu_i/k$ . Then  $n + \mu(k + 1) - \nu_0 \geq k\theta$ .

**Lemma 4.12.** *The inequality  $\nu_0 \geq \mu/2$  holds.*

*Proof.* Let  $\tilde{D}_k$  be the proper transform on the threefold  $\tilde{X}_k$  of a general surface in  $\mathcal{D}$ . Then

$$\tilde{D}_k|_{F_k} \equiv -\nu_0 F_k|_{F_k},$$

which implies that  $\tilde{D}_k|_{F_k}$  is an effective divisor of bi-degree  $(\nu_0, \nu_0)$  on  $F_k \cong \mathbb{P}^1 \times \mathbb{P}^1$ .

Let  $\tilde{L}_1^k$  be the proper transform of the line  $L_1$  on the threefold  $\tilde{X}_k$ . Then

$$\text{mult}_Q \left( \tilde{D}_k|_{F_k} \right) \geq \text{mult}_Q(\tilde{D}_k) \geq \text{mult}_{\tilde{L}_1^k}(\tilde{D}_k) = \mu,$$

where  $Q = \tilde{L}_1^k \cap F_k$ . Thus, we see that  $\nu_0 \geq \mu/2$ . □

**Corollary 4.13.** *The inequality  $(k + 1/2)\mu + n \geq k\theta$  holds.*

Let  $\bar{\mathcal{H}}$  be the proper transform on  $\bar{X}_k$  of the linear system that is cut out on threefold  $V_1 \subset \mathbb{P}^5$  by hyperplanes that pass through the line  $L_1$ . Then  $\bar{\mathcal{H}}$  has no base curves and

$$\begin{aligned} \bar{\mathcal{H}} = & \left| (v_k \circ \omega_k)^* \left( (\pi_1 \circ \dots \circ \pi_k)^* \left( -K_{V_1} \right) - (\pi_2 \circ \dots \circ \pi_k)^* (E_1) - \dots - E_k \right) \right. \\ & \left. - \omega_k^*(F_k) - G_k \right|, \end{aligned}$$

which implies that the equivalence  $\bar{\mathcal{H}}|_{G_k} \sim Z_1 + Z_2$  holds by Lemma 4.12.

**Lemma 4.14.** *The inequality  $\alpha \leq 6n^2 - 2n\theta + n^2/k$  holds.*

*Proof.* Let  $\bar{D}_1$  and  $\bar{D}_2$  be general surfaces in  $\bar{\mathcal{D}}_k$ , and let  $\bar{H}$  be general surface in  $\bar{\mathcal{H}}$ . Then

$$\bar{D}_1 \cdot \bar{D}_2 \cdot \bar{H} = 6n^2 - \sum_{i=1}^k \nu_i^2 - 2\mu \left( n - \sum_{i=1}^k \nu_i \right) - \nu_0^2 - (\mu - \nu_0)^2 - (k + 1)\mu^2,$$

but  $\alpha \leq \mu^2 + \bar{D}_1 \cdot \bar{D}_2 \cdot \bar{H}$ , because  $\bar{H}|_{G_k}$  is ample. Thus, we have

$$\alpha \leq 6n^2 - k\theta^2 - 2\mu(n - k\theta) - k\mu^2,$$

because the inequality  $\sum_{i=1}^k \nu_i^2 \geq k\theta^2$  holds. Put  $\phi(\mu) = 6n^2 - k\theta^2 - 2\mu(n - k\theta) - k\mu^2$ . Then

$$\phi(\mu) \leq \phi\left(\frac{k\theta - n}{k}\right) = 6n^2 - 2n\theta + \frac{n^2}{k},$$

which completes the proof.  $\square$

The inequality  $\theta N > n(N + L)$  holds by Lemma 4.11.

**Lemma 4.15.** *The inequality  $\theta > 5n/4$  holds.*

*Proof.* Suppose that  $\theta \leq 5n/4$ . Then  $N > 4L$  by Lemma 4.11. It follows from the inequalities (4.3) that  $\alpha > 6n^2$ . On the other hand, we know that  $\alpha \leq 6n^2$ .  $\square$

Therefore, the inequalities  $\alpha \leq 6n^2 - 2n\theta + n^2/k \leq (7/2 + 1/k)n^2 \leq 4n^2$  hold.

**Lemma 4.16.** *The equality  $L = k$  holds.*

*Proof.* Suppose that  $L > k \geq 3$ . Then it follows from Lemma 4.14 that the inequality

$$\alpha \leq 6n^2 - 2n\theta + \frac{n^2}{k} < \frac{7n^2}{2} + \frac{n^2}{3} \leq \frac{23n^2}{6}$$

holds, because  $\theta > 5n/4$ . But  $\mu \leq n$ . Therefore, it follows from Corollary 4.13 that

$$\theta \leq \frac{(k + 1/2)\mu + n}{k} \leq 3n/2,$$

which implies that  $N > 2L$  by Lemma 4.11. Then it follows from the inequalities (4.6) that

$$\left(\frac{23(k-1)}{6} + \frac{6L}{k}\right)n^2 \geq (k-1)\alpha + 6n^2\frac{L}{k} > \frac{(N+L)^2}{N}n^2 > \frac{9}{2}Ln^2,$$

which implies that  $k \leq 1$ . Therefore, the inequalities  $L > k \geq 3$  are inconsistent.

To complete the proof, we may assume that  $L > k = 2$ . Then

$$\alpha \leq 6n^2 - 2n\theta + \frac{n^2}{k} < 4n^2$$

by Lemma 4.14, because  $\theta > 5n/4$ . On the other hand, it follows from Corollary 4.13 that

$$\theta \leq \frac{(k + 1/2)\mu + n}{k} \leq 7n/4,$$

which implies that  $3N > 4L$  by Lemma 4.11. Then it follows from the inequalities (4.6) that

$$3Ln^2 + \alpha(2 - L/2) = (k-1)\alpha + 6n^2\frac{L}{k} + \alpha\left(1 - \frac{L}{k}\right) > \frac{(N+L)^2}{N}n^2 > \frac{49}{12}Ln^2,$$

which implies that  $L \leq 2$ , because  $\alpha \leq 4n^2$ . But  $L > k = 2$ . □

Therefore, we have  $L = k \geq 2$ .

**Lemma 4.17.** *The inequality  $\theta > 4n/3$  holds.*

*Proof.* Suppose that  $\theta \leq 4n/3$ . Then  $N > 3L$ . Now it follows from the inequalities (4.6) that

$$(k - 1)\alpha + 6n^2 > \frac{(N + L)^2}{N}n^2 > \frac{16}{3}Ln^2,$$

which implies that  $L < 3/2$ , because  $\alpha \leq 4n^2$  by Lemma 4.14. But  $L \neq 1$  by Lemma 4.4. □

It follows from Lemma 4.14 that  $\alpha \leq 6n^2 - 2n\theta + n^2/k < (10/3 + 1/k)n^2 \leq 23n^2/6$ . But

$$(k + 3/2)n \leq (k + 1/2)\mu + n \geq k\theta > 4k/3$$

by Corollary 4.13. In particular, we have  $L = k \leq 4$ .

**Lemma 4.18.** *The inequality  $L \neq 4$  holds.*

*Proof.* Suppose that  $L = 4$ . Then  $\theta \leq 11n/8$  by Corollary 4.13, which implies that

$$\frac{35}{2}n^2 > 3\alpha + 6n^2 > \frac{(N + L)^2}{N}n^2 > \frac{98}{5}n^2,$$

because  $N \geq 10$  by Lemma 4.11. The obtained contradiction completes the proof. □

Therefore, we proved that either  $L = k = 2$ , or  $L = k = 3$ .

**Lemma 4.19.** *The equality  $L = 2$  holds.*

*Proof.* Suppose that  $L = 3$ . Then  $\theta \leq n(L + 3/2)/L = 3n/2$  by Corollary 4.13. But

$$\alpha \leq 6n^2 - 2n\theta + \frac{n^2}{3} < \frac{11}{3}n^2$$

by Lemma 4.14, because  $\theta > 4n/3$ . Thus, it follows from the inequalities (4.6) that

$$\frac{40}{3}n^2 > 2\alpha + 6n^2 > \frac{(N + L)^2}{N}n^2 > \frac{27}{2}n^2,$$

because  $N > 2L = 6$  by Lemma 4.11. The obtained contradiction completes the proof. □

Therefore, it follows from the inequalities (4.6) that

$$\frac{59}{6}n^2 > \alpha + 6n^2 > \frac{(N + 2)^2}{N}n^2,$$

because  $\alpha < 23n^2/6$  by Lemmas 4.14 and 4.17. Thus, the inequality  $N \leq 5$  holds.

**Lemma 4.20.** *The inequality  $N \neq 5$  holds.*

*Proof.* Suppose that  $N = 5$ . Then  $5\theta > 7n$  by Lemma 4.11. Then

$$\alpha \leq 6n^2 - 2n\theta + \frac{n^2}{2} < \frac{37}{10}n^2$$

by Lemma 4.14. Thus, it follows from the inequalities (4.6) that

$$\frac{97}{10}n^2 > \alpha + 6n^2 > \frac{(N+L)^2}{N}n^2 = \frac{49}{5}n^2,$$

which is a contradiction. □

Let  $\bar{B}_2$  be a proper transform of the curve  $B_2$  on the threefold  $\bar{X}_2$ .

**Lemma 4.21.** *The inequality  $G_2 \cap \bar{B}_2 \neq \emptyset$  holds.*

*Proof.* Suppose that  $G_2 \cap \bar{B}_2 = \emptyset$ . Then  $L_1^2 \cap B_2$  consists of a point  $Q \notin B_2$ .

Recall that  $B_2$  is a line in  $E_2 \cong \mathbb{P}^2$ . Let  $\Gamma$  be a general line in  $E_2 \cong \mathbb{P}^2$  that passes through the point  $Q$ . Then  $\nu_1 \geq \nu_2 = \mathcal{D}_2 \cdot \Gamma \geq \mu + \nu_3 > \mu + n$ . Then  $5\mu/2 + n > 2(\mu + n)$  by Corollary 4.13, which implies that  $\mu > 2n$ . But  $\mu \leq n$ . □

Let  $\Omega \subset G_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$  be a general curve in  $|Z_1 + Z_2|$  that contains the point  $G_2 \cap \bar{B}_2$ . Then

$$4\mu + n - \nu_1 - \nu_2 - \nu_0 = \bar{\mathcal{D}}_2 \Big|_{G_2} \cdot \Omega \geq \text{mult}_{\bar{B}_2}(\bar{\mathcal{D}}_2) = \nu_3,$$

which gives  $\nu_1 + \nu_2 + \nu_3 \leq 9n/2$ , because  $\mu \leq n$  by Lemma 2.5 and  $\nu_0 \geq \mu/2$  by Lemma 4.12. But

$$\nu_1 + \nu_2 + \nu_3 > \frac{3(N+2)}{N}n \geq \frac{9}{2}n$$

by Lemma 4.11, because  $N \leq 4$  by Lemma 4.20. The assertion of Lemma 2.6 is proved.

### 5. Exclusion of non-infinitely close points

We use the assumptions and notation of Lemma 4.1. Suppose that  $\text{mult}_O(\mathcal{D}) > 2n$ , and let us show that this assumption leads to a contradiction. Let  $\Lambda$  be the three-dimensional linear subspace in  $\mathbb{P}^5$  that is tangent to the threefold  $V_1$  at the point  $O$ . Put  $\nu = \text{mult}_O(\mathcal{D})$ .

*Remark 5.1.* The quadric  $Q_1|_\Lambda$  and the cubic  $T_1|_\Lambda$  are both singular at the point  $O$ .

Arguing as in the proof of [4, Proposition 3.3.1], we see that  $P \in \Lambda$ . Let  $L_1$  be a line in  $\mathbb{P}^5$  such that  $P \in L_1 \ni O$ . Then  $L_1 \subset V_1 \cap \Lambda$ , and  $\Lambda$  is not contained in the hyperplane in  $\mathbb{P}^5$  that is tangent to  $Q_1$  at the point  $P$ .

*Remark 5.2.* The quadric  $Q_1|_\Lambda$  is irreducible and reduced, because  $V_1$  satisfies the condition G.

Let  $H$  be a general hyperplane section of the threefold  $V_1$  such that  $\Lambda \cap V_1 \subset H$ .

**Lemma 5.3.** *The point  $O$  is an isolated ordinary double point of the surface  $H$ .*

*Proof.* Arguing as in the proof of Lemma 3.10, we see that the point  $O$  is an isolated ordinary double point of the surface  $H$ , because the quadric  $Q_1|_\Lambda$  is irreducible and reduced.  $\square$

Arguing as in the proof of the Lemma 3.14, we see that  $V_1|_\Lambda$  is reduced.

**Corollary 5.4.** *The surface  $H$  is smooth outside of the points  $P$  and  $O$ .*

The surface  $H$  has singularity of type  $\mathbb{A}_k$  at the point  $P$ , where  $k \leq 2$  by Lemma 3.9. Put

$$V_1|_\Lambda = L_1 + \sum_{i=1}^r C_i,$$

where  $C_i$  is an irreducible reduced curve such that  $C_i \neq C_j \iff i \neq j$  and  $C_i \neq L_1$  for all  $i$ . Then

$$\left(\deg(C_i), \text{mult}_O(C_i)\right) \notin \{(3, 2), (2, 1)\}$$

and  $\deg(C_i) = 1 \implies O \in C_i$  for all  $i$ , because  $Q_1|_\Lambda$  is a quadric cone whose vertex is the point  $O$ .

*Remark 5.5.* The inequality  $r \leq 3$  holds, because  $V_1$  satisfies the condition H. Then

$$O = L_1 \cap C_1 \cap \dots \cap C_r.$$

Let  $\pi: \bar{V}_1 \rightarrow V_1$  be a blow up of  $O$ , let  $E$  be the exceptional of  $\pi$ , and let  $\bar{H}$  be the proper transforms of the surface  $H$  on the threefold  $\bar{V}_1$ . Then  $\bar{H} \cap E$  is an irreducible conic in  $E \cong \mathbb{P}^2$ .

**Lemma 5.6.** *The equality  $\sum_{i=1}^r \text{mult}_O(C_i) = 3$  holds.*

*Proof.* Let  $H'$  be a general hyperplane section of  $V_1$  such that  $\Lambda \cap V_1 \subset H'$ . Then

$$H \cdot H' = L_1 + \sum_{i=1}^r C_i,$$

and the inequality  $\sum_{i=1}^r \text{mult}_O(C_i) \geq 3$  holds by construction.

Let  $\bar{H}'$  be the proper transforms of the surface  $H'$  on the threefold  $\bar{V}_1$ . Then either

$$1 + \sum_{i=1}^r \text{mult}_O(C_i) = \text{mult}_O(L_1) + \sum_{i=1}^r \text{mult}_O(C_i) = \text{mult}_O(H) \text{mult}_O(H') = 4,$$

or we have  $H \cap E = H' \cap E$ . But in the latter case, we have

$$\text{mult}_O(L_1) + \sum_{i=1}^r \text{mult}_O(C_i) \geq \text{mult}_O(H)\text{mult}_O(H') + 2 = 6,$$

which implies that  $r = 5$ . But  $r \leq 3$ . □

Let  $\bar{L}_1$  and  $\bar{C}_i$  be the proper transforms of  $L_1$  and  $C_i$  on the threefold  $V_1$ , respectively.

**Lemma 5.7.** *The intersection form of the curves*

$$\bar{L}_1, \bar{C}_1, \dots, \bar{C}_r$$

*on the normal surface  $\bar{H}$  is not semi-negative definite.*

*Proof.* Let  $\mathcal{B}$  be the proper transform of the linear system  $\mathcal{D}$  on the threefold  $\bar{V}_1$ . Then

$$\mathcal{B}|_{\bar{H}} = \epsilon L_1 + \sum_{i=1}^r \nu_i \bar{C}_i + \mathcal{R} \equiv n \bar{L}_1 + \sum_{i=1}^r n \bar{C}_i + (2n - \nu) E|_{\bar{H}},$$

where  $\epsilon$  and  $\nu_i$  are non-negative integers, and  $\mathcal{R}$  is a linear system that has no fixed curves.

The inequalities  $\epsilon \leq n$  and  $\nu_i \leq n$  hold for every  $i \in \{1, \dots, r\}$  by Lemmas 2.2, 2.3, 2.4, 2.5.

Suppose that the intersection form of  $\bar{L}_1, \bar{C}_1, \dots, \bar{C}_r$  is semi-negative definite. Then

$$\begin{aligned} 0 &\leq \left( (\nu - 2n) E|_{\bar{H}} + \mathcal{R} \right) \cdot \left( (n - \epsilon) L_1 + \sum_{i=1}^r (n - \nu_i) \bar{C}_i \right) \\ &= \left( (n - \epsilon) L_1 + \sum_{i=1}^r (n - \nu_i) \bar{C}_i \right)^2 \leq 0, \end{aligned}$$

which gives  $\epsilon = \nu_1 = \dots = \nu_r = n$ , because  $O = L_1 \cap C_1 \cap \dots \cap C_r$ . Then  $\nu = 2n$ . But  $\nu > 2n$ . □

The equality  $(\bar{L}_1 + \sum_{i=1}^r \bar{C}_i) \cdot \bar{L}_1 = -1$  holds on the surface  $\bar{H}$ , because the equivalences

$$\bar{L}_1 + \sum_{i=1}^r \bar{C}_i \equiv -K_{\bar{V}_1}|_{\bar{H}} \equiv \left( \pi^* (-K_{V_1}) - 2E \right)|_{\bar{H}}$$

hold on the surface  $\bar{H}$ . Similarly, the equality

$$\left( \bar{L}_1 + \sum_{i=1}^r \bar{C}_i \right) \cdot \bar{C}_t = \text{deg}(C_t) - 2\text{mult}_O(C_t)$$

holds for every  $t \in \{1, \dots, r\}$ . Therefore, it follows from Lemma 5.7 and [1] that

$$\left(\bar{L}_1 + \sum_{i=1}^r \bar{C}_i\right) \cdot \bar{C}_s = \deg(C_k) - 2\text{mult}_O(C_s) > 0$$

for some  $s \in \{1, \dots, r\}$ . We may assume that  $s = r$ .

The equalities  $\sum_{i=1}^r \deg(C_i) = 5$  and  $\sum_{i=1}^r \text{mult}_O(C_i) = 3$  hold. Then  $r = \deg(C_3) = 3$  and

$$\deg(C_1) = \deg(C_2) = \text{mult}_O(C_1) = \text{mult}_O(C_2) = \text{mult}_O(C_3) = 1,$$

which implies, in particular, that  $\text{mult}_P(L_1) + \sum_{i=1}^r \text{mult}_P(C_i) = 2$ .

**Corollary 5.8.** *The point  $P$  is an ordinary double point of the surface  $H$ .*

On the surface  $H$ , the curves  $\bar{L}_1, \bar{C}_1, \bar{C}_2, \bar{C}_3$  can be contracted to an isolated singular point of type  $\mathbb{D}_5$ . So, their intersection form is negative definite by [1], which is impossible by Lemma 5.7.

The obtained contradiction completes the proof of Lemma 4.1.

### 6. Infinitely close points

Let us use the assumptions and notation of Lemma 4.4. Suppose that  $L = 1$ .

**Lemma 6.1.** *The inequality  $N \leq 3$  holds.*

*Proof.* The linear system  $|\pi_1^*(-K_{V_1}) - E_1|$  has no base points. Then

$$6n^2 - \alpha_t - \text{mult}_O(C_t) = \left(\pi_1^*(-K_{V_1}) - E_1\right) \cdot C_t^1 \geq 0,$$

which implies that  $\text{mult}_O(C_t) \leq 6n^2 - \alpha_t$ . Thus, we have

$$6n^2 \geq \alpha_t + \text{mult}_O(C_t) \geq \sum_{i=1}^N \nu_i^2 > n^2 \frac{(N+1)^2}{N},$$

which implies that  $N \leq 3$ . □

Let  $\mathcal{R}$  be a proper transform of the linear system

$$\left| (\pi_1 \circ \pi_2)^* \left( -K_{V_1} \right) - \pi_2^*(E_1) - E_2 \right|$$

on  $V_1$ . There is a two-dimensional linear subspace  $\Pi \subset \Lambda$  such that  $\mathcal{R}$  is cut out on  $V_1$  by hyperplanes that pass through  $\Pi$ , and  $\text{Bs}(\mathcal{R}) = \Pi \cap V_1$ .

Arguing as in the proof of [4, Proposition 3.3.1], we see that  $P \in \Pi$ . Let  $\Upsilon$  be a hyperplane in  $\mathbb{P}^5$  that is tangent to the quadric  $Q_1$  at the point  $P$ . Then

$$L_1 \subset \Pi \subset \Lambda \not\subset \Upsilon \subset \mathbb{P}^5,$$

because  $V_1$  satisfies the condition G. Let  $H$  be a sufficiently general surface in  $\mathcal{R}$ .

Arguing as in the proof of [4, Lemma 3.5.3], we see that  $H$  is smooth outside of  $P$ .

The surface  $H$  has singularity of type  $\mathbb{A}_k$  at the point  $P$ . Then  $k \leq 2$  by Lemma 3.9, and the point  $P$  is an ordinary double point of the surface  $H$  in the case when  $\Pi \not\subset \Upsilon$ . But

$$\Pi \subset \Upsilon \iff Q_1 \cap \Pi = L_1.$$

Let  $S$  be a proper transform of  $H$  on the threefold  $X_2$ , and let  $\pi: S \rightarrow H$  be a birational morphism induced by the composition  $\pi_1 \circ \pi_2$ . Then  $\pi$  is a blow up of the point  $O$  that contracts an irreducible smooth curve  $E \subset S$  such that  $E = E_2 \cap S$ , where  $E_2 \cong \mathbb{F}_2$  and  $B_2 \subset E_2$ .

Let  $C$  be a section of the natural projection  $E_2 \rightarrow \mathbb{P}^1$  such that  $C^2 = -2$ , and let  $F$  be a fiber of the natural projection  $E_2 \rightarrow \mathbb{P}^1$ . Then  $E \sim C + 2F$ . It follows from [5] that  $B_2 \sim C + 2F$  in the case when  $N = 3$ , where  $E \neq B_2$  due to generality in the choice of the surface  $H \in \mathcal{R}$ .

Let  $\bar{L}_1$  be a proper transform of  $L_1$  on the surface  $S$ . Then

$$\bar{L}_1 \cdot \bar{L}_1 = -3 + k/(k + 1).$$

**Lemma 6.2.** *We inequality  $\Pi \cap V_1 \neq L_1$  holds.*

*Proof.* Suppose that  $\Pi \cap V_1 = L_1$ . Then  $V_1|_\Pi = L_1$  and

$$\mathcal{D}_2|_S = \text{mult}_{L_1}(\mathcal{D})\bar{L}_1 + \mathcal{M} \equiv \pi^*\left(\mathcal{O}_{\mathbb{P}^5}(n)\Big|_H\right) - (\nu_1 + \nu_2)E,$$

where  $\mathcal{M}$  is a linear system on  $S$  that has no fixed curves. Then

$$\begin{aligned} M_1 \cdot M_2 &= 6n^2 - 2\text{mult}_{L_1}(\mathcal{D}) - \text{mult}_{L_1}^2(\mathcal{D})\bar{L}_1 \cdot \bar{L}_1 \\ &\quad + 2\text{mult}_{L_1}(\mathcal{D})(\nu_1 + \nu_2) - (\nu_1 + \nu_2)^2, \end{aligned}$$

where  $M_1$  and  $M_2$  are general curves in  $\mathcal{M}$ . But  $M_1 \cdot M_2 \geq 0$  and

$$\nu_1 + \nu_2 + \dots + \nu_N > (N + 1)n,$$

which implies that  $N = 3$ , because  $\bar{L}_1 \cdot \bar{L}_1 \leq -7/3$ . Let  $Q$  be a point in  $E \cap B_2 \neq \emptyset$ . Then

$$\text{mult}_{L_1}(\mathcal{D})\text{mult}_Q(\bar{L}_1) + \text{mult}_Q(\mathcal{M}) \geq \nu_3,$$

which implies that the inequality

$$M_1 \cdot M_2 \geq \left(\nu_3 - \text{mult}_{L_1}(\mathcal{D})\right)^2$$

holds. Therefore, we see that

$$\begin{aligned} &6n^2 - 2\text{mult}_{L_1}(\mathcal{D}) - \text{mult}_{L_1}^2(\mathcal{D})\bar{L}_1 \cdot \bar{L}_1 + 2\text{mult}_{L_1}(\mathcal{D})(\nu_1 + \nu_2) - (\nu_1 + \nu_2)^2 \\ &\geq \left(\nu_3 - \text{mult}_{L_1}(\mathcal{D})\right)^2, \end{aligned}$$

where  $\nu_1 + \nu_2 + \nu_3 > 4n$ . The obtained inequalities are inconsistent. We see that  $\Pi \cap V_1 \neq L_1$ . □



The quadric  $Q_1|_\Lambda$  is irreducible. Then  $\Pi \not\subset Q_1$ , because  $\Pi \subset \Lambda$ . So, we may assume that

$$\Pi \cap Q_1 = L_1 + L_2,$$

where  $L_2$  is a line on  $V_1$  such that  $O \in L_2 \neq L_1$ . Then  $k = 1$  and  $\bar{L}_1 \cdot \bar{L}_1 = -5/2$ .

Let  $\bar{L}_2$  be a proper transform of the line  $L_2$  on the surface  $S$ . Then

$$\mathcal{D}_2|_S = \text{mult}_{L_1}(\mathcal{D})\bar{L}_1 + \text{mult}_{L_2}(\mathcal{D})\bar{L}_2 + \mathcal{T} \equiv \pi^*\left(\mathcal{O}_{\mathbb{P}^5}(n)\Big|_H\right) - (\nu_1 + \nu_2)E,$$

where  $\mathcal{T}$  is a linear system on  $S$  that has no fixed curves. Then  $\nu_1 + \dots + \nu_N > (N + 1)n$  and

$$\begin{aligned} T_1 \cdot T_2 &= 6n^2 - \sum_{i=1}^2 2\text{mult}_{L_i}(\mathcal{D}) - \frac{5\text{mult}_{L_1}^2(\mathcal{D})}{2} - 3\text{mult}_{L_1}^2(\mathcal{D}) \\ &\quad + \sum_{i=1}^2 2(\nu_1 + \nu_2)\text{mult}_{L_i}(\mathcal{D}) - (\nu_1 + \nu_2)^2, \end{aligned}$$

where  $T_1$  and  $T_2$  are general curves in  $\mathcal{L}$ . But  $T_1 \cdot T_2 \geq 0$ , which implies that  $N = 3$ .

**Lemma 6.3.** *The equality  $|E \cap B_2| = 2$  holds.*

*Proof.* The equality  $|E \cap B_2| = 2$  holds, because the restriction of the linear system

$$\left| (\pi_1 \circ \pi_2)^* \left( -K_{V_1} \right) - \pi_2^*(E_1) - E_2 \right|$$

to the surface  $E_2$  is a pencil in  $|C + 2F|$ , whose base locus consists of  $\bar{L}_1 \cap E_2$  and  $\bar{L}_2 \cap E_2$ .  $\square$

Let  $Q_1$  and  $Q_2$  be two points in  $E \cap B_2$  such that  $Q_1 \neq Q_2$ . Then

$$\text{mult}_{L_1}(\mathcal{D})\text{mult}_{Q_1}(\bar{L}_1) + \text{mult}_{L_2}(\mathcal{D})\text{mult}_{Q_1}(\bar{L}_2) + \text{mult}_{Q_1}(\mathcal{M}) \geq \nu_3.$$

**Lemma 6.4.** *Either  $Q_1 \in \bar{L}_1 \cup \bar{L}_2$ , or  $Q_2 \in \bar{L}_1 \cup \bar{L}_2$ .*

*Proof.* Suppose that  $Q_1 \notin \bar{L}_1 \cup \bar{L}_2 \neq Q_2$ . Then  $T_1 \cdot T_2 \geq 2\nu_3^2$ . Therefore, we have

$$\begin{aligned} &6n^2 - \sum_{i=1}^2 2\text{mult}_{L_i}(\mathcal{D}) - \frac{5\text{mult}_{L_1}^2(\mathcal{D})}{2} - 3\text{mult}_{L_1}^2(\mathcal{D}) \\ &\quad + \sum_{i=1}^2 2(\nu_1 + \nu_2)\text{mult}_{L_i}(\mathcal{D}) - (\nu_1 + \nu_2)^2 \\ &\geq 2\nu_3^2, \end{aligned}$$

which is impossible, because  $\nu_1 + \nu_2 + \nu_3 > 4n$ .  $\square$

We may assume that  $Q_1 \in \bar{L}_1 \cup \bar{L}_2$ .

**Lemma 6.5.** *The set  $\bar{L}_1 \cup \bar{L}_2$  contains  $Q_2$ .*

*Proof.* Suppose that  $Q_1 \notin \bar{L}_1 \cup \bar{L}_2$ . Then

$$T_1 \cdot T_2 \geq \nu_3^2 + \left( \nu_3 - \text{mult}_{L_1}(\mathcal{D}) \text{mult}_{Q_1}(\bar{L}_1) - \text{mult}_{L_2}(\mathcal{D}) \text{mult}_{Q_1}(\bar{L}_2) \right)^2,$$

which leads to a contradiction, because  $\nu_1 + \nu_2 + \nu_3 > 4n$ . □

We may assume that  $Q_1 \in \bar{L}_1$  and  $Q_2 \in \bar{L}_1$ . Put

$$\mathcal{B} = \left| (\pi_1 \circ \pi_2 \circ \pi_3)^* \left( -K_{V_1} \right) - (\pi_2 \circ \pi_3)^*(E_1) - \pi_3^*(E_2) - E_3 \right|$$

and  $\mathcal{P} = \pi_1 \circ \pi_2 \circ \pi_3(\mathcal{B})$ . Then there is a three-dimensional linear subspace  $\Sigma \subset \mathbb{P}^5$  such that the system  $\mathcal{P}$  is cut out on  $V_1 \subset \mathbb{P}^5$  by hyperplanes in  $\mathbb{P}^5$  that pass through  $\Sigma$ . Then  $\Pi \subset \Sigma$ .

**Lemma 6.6.** *The inequality  $\Sigma \neq \Lambda$  holds.*

*Proof.* Suppose that  $\Sigma = \Lambda$ . Then

$$\pi_2 \circ \pi_3(\mathcal{B}) = \left| \pi_1^* \left( -K_{V_1} \right) - 2E_1 \right| + E_1,$$

but  $|\pi_1^*(-K_{V_1}) - 2E_1|$  does not have base curves in  $E_1$  (see the proof of Lemma 3.10). Then

$$\mathcal{B} \not\sim (\pi_1 \circ \pi_2 \circ \pi_3)^* \left( -K_{V_1} \right) - (\pi_2 \circ \pi_3)^*(E_1) - \pi_3^*(E_2) - E_3,$$

which is a contradiction. □

Let  $B$  and  $D_3$  be general surfaces in  $\mathcal{B}$  and  $\mathcal{D}_3$ , respectively. Then

$$D_3|_B = m_1 \check{L}_1 + m_2 \check{L}_2 + \Delta \equiv \left( (\pi_1 \circ \pi_2 \circ \pi_3)^* \left( -nK_{V_1} \right) - (\nu_1 + \nu_2 + \nu_3)E_3 \right)|_B$$

for some non-negative integers  $m_1$  and  $m_2$ , where  $\Delta$  is an effective divisor such that

$$\check{L}_2 \not\subset \text{Supp}(\Delta) \not\supset \check{L}_2,$$

and  $\check{L}_1$  and  $\check{L}_2$  are proper transforms of the curves  $L_1$  and  $L_2$  on the threefold  $V_3$ , respectively.

**Lemma 6.7.** *The scheme  $V_1|_\Sigma$  is not reduced along  $L_1$  and is not reduced along  $L_2$ .*

*Proof.* Suppose that  $V_1|_\Sigma$  is reduced along  $L_i$ , where  $i \in \{1, 2\}$ . Then  $m_i = \text{mult}_{L_i}(\mathcal{D})$ . But

$$-3m_i \leq m_i \check{L}_i \cdot \check{L}_i \leq \left( m_1 \check{L}_1 + m_2 \check{L}_2 + \Delta \right) \cdot \check{L}_i = n - \nu_1 - \nu_2 - \nu_3 < -3n,$$

because  $\check{L}_i \cdot \check{L}_i \geq -3$ . Then  $\text{mult}_{L_i}(\mathcal{D}) > n$ , which is impossible by Lemma 2.2. □

The quadric hypersurface  $Q_1|_\Sigma \subset \Sigma \cong \mathbb{P}^3$  must be irreducible, because  $V_1$  satisfies the generality conditions E and F. Arguing as in the proof of Lemma 3.14, we see that  $Q_1|_\Sigma$  is smooth. Then

$$V_1|_\Sigma = 2L_1 + 2L_2 + Z,$$

where  $Z$  is a conic such that  $O \notin \text{Supp}(Z)$ , because  $V_1$  satisfies the generality condition I.

**Lemma 6.8.** *The curve  $Z$  is reduced.*

*Proof.* The curve  $Z$  is a divisor of bi-degree  $(1, 1)$  on  $Q_1|_\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$ . Then  $Z$  is reduced.  $\square$

**Lemma 6.9.** *The surface  $B$  is smooth outside of the set  $\check{L}_1 \cup \check{L}_1$ .*

*Proof.* For every point  $Q \in V_1 \setminus \{P, O\}$ , we have  $\text{mult}_Q(V_1|_\Sigma) \leq 3$ , which implies that  $\Sigma$  is not a tangent linear subspace to  $V_1$  at the point  $Q$ . Then

$$\text{Sing}(B) \subset \check{L}_1 \cup \check{L}_2 \cup (E_3 \cap B),$$

because  $Z$  is reduced. But  $B$  is smooth along  $E_3 \cap B$ .  $\square$

Let  $\check{P} \in X_3$  be a point such that  $\pi_1 \circ \pi_2 \circ \pi_3(\check{P}) = P$ . Put  $\check{E}_3 = E_3|_B$ . Then

$$\check{L}_1 \cap \check{E}_3 \not\subset \text{Sing}(B) \not\supset \check{L}_2 \cap \check{E}_3$$

and  $B$  has singularity at  $\check{P}$  of type  $\mathbb{A}_q$ . Then  $q \leq 2$  by Lemma 3.9, because  $\text{mult}_P(V_1|_\Sigma) \leq 4$ .

Let  $\check{Z}$  be a proper transform of the curve  $Z$  on the threefold  $X_3$ . Then

$$\begin{aligned} B|_B &\equiv 2\check{L}_1 + 2\check{L}_2 + \check{Z} + \check{E}_3 \equiv \left( (\pi_1 \circ \pi_2 \circ \pi_3)^* (-K_{V_1}) - 3E_3 \right)|_B \\ &\equiv (\pi_1 \circ \pi_2 \circ \pi_3)^* (-K_{V_1})|_B + 3\check{E}_3 \end{aligned}$$

and  $\check{L}_1 \cap \check{Z} \not\subset \text{Sing}(B) \not\supset (\check{L}_1 \setminus \check{P}) \cap \check{Z}$ .

**Lemma 6.10.** *The conic  $Z$  is reducible.*

*Proof.* Suppose that  $Z$  is irreducible. Put

$$\begin{aligned} \mathcal{D}_3|_B &= m_1\check{L}_1 + m_2\check{L}_2 + \text{mult}_Z(\mathcal{D})\check{Z} + \epsilon\check{E}_3 + \mathcal{F} \\ &\equiv (\pi_1 \circ \pi_2 \circ \pi_3)^* (-nK_{V_1})|_B - (\nu_1 + \nu_2 + \nu_3)\check{E}_3, \end{aligned}$$

where  $\epsilon \geq 0$  is an integer, and  $\mathcal{F}$  is a linear system on  $B$  that has no fixed components. Then

$$(2n - m_1)\check{L}_1 + (2n - m_2)\check{L}_2 + (n - \text{mult}_Z(\mathcal{D}))\check{Z} \equiv \mathcal{F} + (\nu_1 + \nu_2 + \nu_3 + \epsilon - 4)\check{E}_3,$$

on the surface  $B$ . Arguing as in the proof of Lemma 3.11, we easily see that intersection form of the curves  $\check{L}_1, \check{L}_2, \check{Z}$  on the surface  $B$  is not negative definite. But

$$\check{L}_1 \cdot \check{Z} = \check{L}_2 \cdot \check{Z} = 1, \quad \check{Z} \cdot \check{Z} = \check{L}_1 \cdot \check{L}_1 = \check{L}_2 \cdot \check{L}_2 = -2, \quad \check{L}_2 \cdot \check{L}_1 = 0$$

in the case when  $P \notin Z$ . Therefore, we have  $P \in Z$ .

The point  $\check{P}$  must be a singular point of the surface  $B$  of type  $A_2$ , because

$$\frac{3}{2} = \check{Z} \cdot \check{Z} + 3 = \check{Z} \cdot (2\check{L}_1 + 2\check{L}_2 + \check{Z}) = 2$$

in the case when  $\check{P}$  is an ordinary double point of the surface  $B$ .

We have  $\check{Z} \cdot \check{L}_2 = 1$  and  $\check{L}_1 \cdot \check{L}_2 = 0$ . But  $\check{Z} \cdot \check{Z} = -4/3$  by the subadjunction formula. Then

$$2\check{Z} \cdot \check{L}_1 + 2/3\check{Z} \cdot (2\check{L}_1 + 2\check{L}_2 + \check{Z}) = 2,$$

which implies that  $\check{Z} \cdot \check{L}_1 = 2/3$  on the surface  $B$ . Then  $\check{L}_1 \cdot \check{L}_1 = -11/6$ , because the equalities

$$2\check{L}_1 \cdot \check{L}_1 + 2/3 = \check{L}_1 \cdot (2\check{L}_1 + 2\check{L}_2 + \check{Z}) = -3$$

hold. Similarly, we easily see that  $\check{L}_2 \cdot \check{L}_2 = -2$ . Therefore, we proved that the intersection form of the curves  $\check{L}_1, \check{L}_2, \check{Z}$  is negative definite, which is a contradiction.  $\square$

We have  $Z = Z_1 + Z_2$ , where  $Z_1$  and  $Z_2$  are lines such that  $Z_1 \cap Z_2 \neq \emptyset$ . We may assume that

$$Z_1 \cap L_2 = Z_2 \cap L_2 = \emptyset$$

and  $Z_1 \cap L_1 \neq \emptyset \neq Z_2 \cap L_1$ . Then it follows from Lemma 3.8 that  $P \notin Z_1$ .

Let  $\check{Z}_1$  and  $\check{Z}_2$  be the proper transform of  $Z_1$  and  $Z_2$  on  $X_3$ , respectively. Then

$$\check{Z}_1 \cdot \check{Z}_1 = \check{Z}_2 \cdot \check{Z}_2 = \check{L}_1 \cdot \check{L}_1 = \check{L}_2 \cdot \check{L}_2 = -2,$$

and  $\check{Z}_1 \cdot \check{Z}_2 = \check{Z}_1 \cdot \check{L}_1 = \check{Z}_2 \cdot \check{L}_2 = 1$  on the surface  $B$ . Therefore, we see that the intersection form of the curves  $\check{L}_1, \check{L}_2, \check{Z}_1, \check{Z}_2$  on the surface  $B$  is negative definite. Arguing as in the proof of Lemma 6.10, we get a contradiction that proves Lemma 4.4.

### 7. Lines in smooth locus

Let us use the assumptions and notation of Lemma 4.7. Suppose that  $k_1 = 1$ . Then  $k_1 k_2 \cdots k_r \neq 1$  by Lemma 4.5. Thus, we may assume that  $k_2 \geq 2$ . Put  $k = k_2$  and  $\alpha = \alpha_2$ .

*Remark 7.1.* The isomorphism  $\mathcal{N}_{L_2/V_1} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$  holds by Remark 2.1.

Let  $\omega_k: \bar{X}_k \rightarrow X_k$  be the blow up of  $L_2^k$ , and let  $G_k$  be the exceptional divisor of  $\omega_k$ .

**Lemma 7.2.** *The isomorphism  $G_k \cong \mathbb{F}_1$  hold.*

*Proof.* Arguing as in the proof of Lemma 4.9, we see that  $G_k \cong \mathbb{F}_1$ .  $\square$

Let  $Z_1$  and  $Z_2$  be curves on  $G_k \cong \mathbb{F}_1$  such that  $Z_1 \cdot Z_1 = -1$ ,  $Z_2 \cdot Z_2 = 0$ ,  $Z_1 \cdot Z_2 = 1$ .

**Lemma 7.3.** *The equivalence  $-G_k|_{G_k} \sim Z_1 + (k + 1)Z_2$  holds.*

*Proof.* There is an integer  $\epsilon$  such that  $-G_k|_{G_k} \sim Z_1 + \epsilon Z_2$ . Then

$$-1 + 2\epsilon = (Z_1 + \epsilon Z_2) \cdot (Z_1 + \epsilon Z_2) = G_k^3 = -c_1(\mathcal{N}_{L_2^k/X_k}) = 2 + K_{X_k} \cdot L_2^k = 2k + 1,$$

which implies that  $\epsilon = k + 1$ .  $\square$

Let  $\bar{\mathcal{D}}_k$  be the proper transform of  $\mathcal{D}$  on  $\bar{X}_k$ . Then

$$\bar{\mathcal{D}}_k|_{G_k} \sim \text{mult}_{L_2}(\mathcal{D})Z_1 + \left( n + (k + 1)\text{mult}_{L_2}(\mathcal{D}) - \sum_{i=1}^k \nu_i \right) Z_2$$

by Lemma 7.3. Put  $\mu = \text{mult}_{L_2}(\mathcal{D})$  and  $\theta = \sum_{i=1}^k \nu_i/k$ . Then  $\mu \leq n$  by Lemma 2.5.

**Corollary 7.4.** *The inequality  $(k + 1)\mu + n \geq k\theta$  holds.*

Let  $\bar{\mathcal{H}}$  be the proper transform on  $\bar{X}_k$  of the linear system that is cut out on  $V_1$  by hyperplanes that pass through  $L_2$ . Then  $\bar{\mathcal{H}}|_{G_k} \sim Z_1 + 2Z_2$  by Lemma 7.3. Then  $\bar{\mathcal{H}}$  has no base curves.

**Lemma 7.5.** *The inequality  $\alpha \leq 6n^2 - \mu^2(k + 1) + 2\mu(k\theta - n) - k\theta^2$  holds.*

*Proof.* Let  $\bar{D}_1$  and  $\bar{D}_2$  be general surfaces in  $\bar{\mathcal{D}}_k$ , and let  $\bar{H}$  be general surface in  $\bar{\mathcal{H}}$ . Then

$$\alpha \leq \mu^2 + \bar{D}_1 \cdot \bar{D}_2 \cdot \bar{H} = 6n^2 - \sum_{i=1}^k \nu_i^2 - \mu^2(k + 1) + 2\mu(k\theta - n),$$

because the divisor  $\bar{H}|_{G_k}$  is ample, and the inequality  $\sum_{i=1}^k \nu_i^2 \geq k\theta^2$  holds.  $\square$

Arguing as in the proof of Lemma 4.15, we see that  $\theta > 5n/4$ . Then it follows from by Lemma 7.5 that  $\alpha < 29n^2/8$ .

**Lemma 7.6.** *The equality  $k = 2$  holds.*

*Proof.* Suppose that  $k \geq 3$ . Then  $\theta \leq 5n/3$  by Lemma 7.4. Hence, we have

$$\frac{(N + L)^2}{N} n^2 > \frac{25}{6} L n^2,$$

because  $2N > 3L$  by Lemma 4.11. But it follows from Lemma 7.5 that

$$\alpha \leq 6n^2 - \mu^2(k + 1) + 2\mu(k\theta - n) - k\theta^2 < \frac{205}{64} n^2,$$

because  $k \geq 3$  and  $\theta > 5n/4$ . Then it follows from the inequalities (4.6) that

$$\left(\frac{205(k-1)}{64} + \frac{6L}{k}\right)n^2 \geq (k-1)\alpha + 6n^2\frac{L}{k} > \frac{(N+L)^2}{N}n^2 > \frac{25}{6}Ln^2,$$

which implies that  $k \leq 2$ , because  $L \geq k$ . The obtained contradiction completes the proof.  $\square$

We have  $N > L \geq k = 2$ . Then  $L \leq 3$ , because it follows from the inequalities (4.6) that

$$\left(\frac{29}{8} + 3L\right)n^2 \geq \alpha + 3Ln^2 > \frac{(N+L)^2}{N}n^2 > 4Ln^2.$$

**Lemma 7.7.** *The equality  $L = 2$  holds.*

*Proof.* Suppose that  $L = 3$ . Then it follows from the inequalities (4.6) that

$$\frac{101}{8}n^2 \geq \alpha + 9n^2 > \frac{(N+3)^2}{N}n^2,$$

which implies that  $N = 4$ , because  $N > L = 3$ . Then  $\theta > 7n/4$  by Lemma 4.11, and

$$\alpha \leq 6n^2 - \mu^2(k+1) + 2\mu(k\theta - n) - k\theta^2 < \frac{49}{25}n^2,$$

by Lemma 7.5. Then it follows from the inequalities (4.6) that

$$\frac{274}{25}n^2 \geq \alpha + 9n^2 > \frac{(N+3)^2}{N}n^2 = \frac{49}{4}n^2,$$

which is a contradiction.  $\square$

It follows from the inequalities (4.6) that  $77/8 > (N+2)^2/N$ . Then  $N \leq 4$ .

**Lemma 7.8.** *The equality  $N = 3$  holds.*

*Proof.* Suppose that  $N = 4$ . Then  $\theta > 3n/2$  by Lemma 4.11, which implies that

$$\alpha \leq 6n^2 - \mu^2(k+1) + 2\mu(k\theta - n) - k\theta^2 < \frac{71}{25}n^2,$$

by Lemma 7.5. Then it follows from the inequalities (4.6) that

$$\frac{221}{25}n^2 \geq \alpha + 6n^2 > \frac{(N+2)^2}{N}n^2 = 9n^2,$$

which is a contradiction.  $\square$

Therefore, the inequality  $\theta > 5n/3$  holds by Lemma 4.11. Then

$$\alpha \leq 6n^2 - \mu^2(k+1) + 2\mu(k\theta - n) - k\theta^2 < \frac{113}{50}n^2,$$

by Lemma 7.5. Then it follows from the inequalities (4.6) that

$$\frac{413}{50}n^2 \geq \alpha + 6n^2 > \frac{(N+2)^2}{N}n^2 = \frac{25}{3}n^2,$$

which is a contradiction. The assertion of Lemma 4.7 is proved.

### 8. Generality conditions

Let us use the assumption and notation of Section 1, and let us prove Theorem 1.13. Put

- $h(y_1, y_2, y_3, y_4) = \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 + \alpha_4 y_4,$
- $q(y_0, y_1, y_2, y_3, y_4) = Ay_0^2 + y_0 \sum_{i=1}^4 \beta_i y_i + \sum_{1 \leq i < j \leq 4} \gamma_{ij} y_i y_j,$
- $t(y_0, y_1, y_2, y_3, y_4) = \sum_{1 \leq i < j < k \leq 4} \delta_{ijk} y_i y_j y_k + y_0^2 \sum_{i=1}^4 \epsilon_i y_i,$

where  $\alpha_i, \beta_i, \gamma_{ij}, A, \delta_{ijk}, \epsilon_i$  are complex numbers. Let  $Q \subset \mathbb{P}^5$  be a quadric in that is given by

$$y_5 \sum \alpha_i y_i = Ay_0^2 + y_0 \sum \beta_i y_i + \sum \gamma_{ij} y_i y_j$$

in  $\text{Proj}(\mathbb{C}[y_0, y_1, y_2, y_3, y_4, y_5]) \cong \mathbb{P}^5$ , let  $T \subset \mathbb{P}^5$  be a cubic hypersurface that is given by

$$y_5 \left( Ay_0^2 - y_0 \sum \beta_i y_i + \sum \gamma_{ij} y_i y_j \right) = \sum \delta_{ijk} y_i y_j y_k + y_0^2 \sum \epsilon_i y_i,$$

and let  $P \in \mathbb{P}^5$  be the point  $\{y_0 = \dots = y_4 = 0\}$ . Put  $V = Q \cap T$ . Suppose that

- the threefold  $V$  satisfy the conditions A, B, C, D,
- the inequality  $A \neq 0$  holds.

*Remark 8.1.* To prove Theorem 1.13, we must show that the threefold  $V$  satisfies the generality conditions E, F, G, H, I in the case when the polynomials  $h, q$  and  $t$  are sufficiently general.

Let  $F \subset \mathbb{P}^5$  be a hyperplane  $\{y_0 = 0\}$ , let  $\iota \in \text{Aut}(\mathbb{P}^5)$  be an involution that is given by

$$y_0 \rightarrow -y_0, y_1 \rightarrow y_1, y_2 \rightarrow y_2, y_3 \rightarrow y_3, y_4 \rightarrow y_4, y_5 \rightarrow y_5,$$

and let  $\zeta \in \mathbb{P}^5$  be the point  $\{y_1 = \dots = y_5 = 0\}$ . Then  $\iota$  fixes  $F$  and  $\zeta$ .

*Remark 8.2.* It follows from  $A \neq 0$  that  $\zeta \notin V$ .

Let  $L$  be a line that pass through  $P$  and  $\zeta$ . Then  $L \not\subset V$ , and  $L$  is given by the equations

$$y_1 = y_2 = y_3 = y_4 = 0.$$

**Lemma 8.3.** *Let  $L_1$  be a line in  $V$  such that  $P \notin L_1$ . Then  $L \cap L_1 = \emptyset$ .*

*Proof.* Suppose that  $L \cap L_1 \neq \emptyset$ . Let  $\Pi \subset \mathbb{P}^5$  be a two-dimensional linear subspace that contains both lines  $L_1$  and  $L$ . We may assume that  $L_1 \cap F = \{y_0 = y_2 = \dots = y_5 = 0\}$ . Then  $\Pi$  is given by  $y_2 = y_3 = y_4 = 0$ , which implies that  $\Pi \not\subset Q$ , because  $A \neq 0$ . Then the conic  $Q|_\Pi$  is given by

$$Ay_0^2 + \beta_1 y_0 y_1 + \gamma_{11} y_1^2 - \alpha_1 y_1 y_5 = y_2 = y_3 = y_4 = 0,$$

but  $L_1 \subset \text{Supp}(Q|_\Pi)$ . Therefore, we have  $\alpha_1 = 0$ , because the homogeneous polynomial

$$Ay_0^2 + \beta_1 y_0 y_1 + \gamma_{11} y_1^2 - \alpha_1 y_1 y_5$$

must be a product of two linear forms. Then  $P \in L_1$ , which is a contradiction.  $\square$

Now we suppose that the polynomials  $h, q$  and  $t$  are sufficiently general.

**Lemma 8.4.** *Let  $L_1$  be a line in  $V$  such that  $P \notin L_1$ . Then  $L_1 \not\subset F$ .*

*Proof.* Let  $\phi: \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$  be a projection from the point  $P$ . Put  $X = \phi(V)$ . Then  $X$  is given by

$$h(y_1, \dots, y_4)t(y_0, \dots, y_4) = q(y_0, \dots, y_4)q(-y_0, \dots, y_4)$$

in  $\text{Proj}(\mathbb{C}[y_0, \dots, y_4]) \cong \mathbb{P}^4$ .

Put  $\bar{L}_1 = \phi(L_1)$  and  $\bar{F} = \phi(F)$ . Suppose that  $\bar{L}_1 \subset \bar{F}$ . Then  $X|_{\bar{F}}$  is given by

$$(8.5) \quad \left(\sum \alpha_i y_i\right) \left(\sum \delta_{ijk} y_i y_j y_k\right) = \left(\sum \gamma_{ij} y_i y_j\right)^2$$

in  $\text{Proj}(\mathbb{C}[y_1, y_2, y_3, y_4]) \cong \mathbb{P}^3$ .

We may assume  $\bar{L}_1$  is given by  $y_0 = y_3 = y_4 = 0$ . Then it follows from  $\bar{L}_1 \subset X$  that

$$(8.6) \quad \begin{cases} \alpha_1 \delta_{111} + \gamma_{11}^2 = 0, \\ \alpha_1 \delta_{112} + \alpha_2 \delta_{111} + 2\gamma_{11} \gamma_{12} = 0, \\ \alpha_1 \delta_{122} + \alpha_2 \delta_{112} + 2\gamma_{11} \gamma_{22} + \gamma_{12}^2 = 0, \\ \alpha_1 \delta_{222} + \alpha_2 \delta_{122} + 2\gamma_{12} \gamma_{22} = 0, \\ \alpha_2 \delta_{222} + \gamma_{22}^2 = 0. \end{cases}$$

Let  $\mathcal{X}$  be the set of all quartic hypersurfaces in  $\mathbb{P}^3$  that are given by the equations (8.5). Put

$$\mathcal{I} = \left\{ (\Gamma, \bar{X}) \mid \Gamma \subset \bar{X} \right\} \subset Gr(2, 4) \times \mathcal{X},$$

and let  $\omega: \mathcal{I} \rightarrow \mathcal{X}$  be the natural projections. Then it follows from the equations (8.6) that

$$\dim(\mathcal{I}) = \dim(\mathcal{X}) - 5 + \dim(Gr(2, 4)) = \dim(\mathcal{X}) - 1 < \dim(\mathcal{X}),$$



which implies that  $\omega$  is not surjective. But the polynomials  $h, q$  and  $t$  are chosen to be sufficiently general by assumption, which implies that  $\bar{L}_1 \subset \bar{F}$ . Therefore, we see that  $L_1 \not\subset F$ .  $\square$

Let  $L_1$  be a line on  $V$  such that  $P \notin L_1$ , and let  $[x_0 : \dots : x_5]$  be coordinates on  $\mathbb{P}^5$  such that

- the hyperplane  $F \subset \mathbb{P}^5$  is given by  $x_0 = 0$ ,
- the line  $L_1 \subset V$  is given by  $x_2 = x_3 = x_4 = x_5 = 0$ ,
- the point  $P$  is given by  $x_0 = x_1 = x_2 = x_3 = x_4 = 0$ ,

and  $\zeta = (1 : -a_1 : -a_2 : -a_3 : -a_4 : -a_5)$ . Then

$$(8.7) \quad \begin{cases} y_0 = x_0, \\ y_1 = x_1 + a_1x_0, \\ y_2 = x_2 + a_2x_0, \\ y_3 = x_3 + a_3x_0, \\ y_4 = x_4 + a_4x_0, \\ y_5 = x_5 + bx_1 + (a_5 + ba_1)x_0, \end{cases}$$

where  $b$  and  $a_i$  are complex numbers. Then the reflection  $\iota$  acts as

$$x_0 \rightarrow -x_0, \quad x_1 \rightarrow x_1 + 2a_1x_0, \quad \dots, \quad x_5 \rightarrow x_5 + 2a_5x_0.$$

Let  $\Pi \subset \mathbb{P}^5$  be a two-dimensional linear subspace such that  $L_1 \subset \Pi$ .

**Lemma 8.8.** *The scheme  $V|_\Pi$  is reduced along  $L_1$ .*

*Proof.* It follows from Lemmas 8.3 and 8.4 that  $L \cap L_1 = \emptyset$  and  $L_1 \not\subset F$ . Let us consider an open subset of a variety of  $(1, 2)$ -flags

$$\mathcal{T} = \left\{ (\Gamma, \Sigma) \mid \Gamma \subset \Sigma, \Gamma \not\subset F, \Gamma \cap L = \emptyset \right\},$$

and a closed subset  $\mathcal{S} = \{(\Gamma, \Sigma) \mid \Sigma \subset \langle \Gamma, L \rangle\} \subset \mathcal{T}$ , where  $\langle \Gamma, L \rangle$  is a three-dimensional linear subspace in  $\mathbb{P}^5$  that contains the lines  $\Gamma$  and  $L$ . Then  $\dim(\mathcal{T}) = 11$  and  $\dim(\mathcal{S}) = 6$ .

Choose the coordinates  $[x_0 : \dots : x_5]$  such that  $\Sigma$  is given by  $x_3 = x_4 = x_5 = 0$ .

Suppose that  $V|_\Sigma$  is not reduced along  $\Gamma$ . Then the scheme

$$x_3 = x_4 = x_5 = Ay_0^2 + y_0 \sum \beta_i y_i + \sum \gamma_{ij} y_i y_j - y_5 \sum \alpha_i y_i = 0$$

is not reduced along the line  $L_1$ , and the scheme

$$\begin{aligned} &x_3 = x_4 = x_5 \\ &= y_5 \left( Ay_0^2 - y_0 \sum \beta_i y_i + \sum \gamma_{ij} y_i y_j \right) - \sum \delta_{ijk} y_i y_j y_k - y_0^2 \sum \epsilon_i y_i = 0 \end{aligned}$$

is not reduced along  $L_1$ , where  $y_0, y_1, \dots, y_5$  are given by the equations 8.7.

Suppose that  $(L_1, \Pi) \notin \mathcal{S}$ . Then  $a_3 \neq 0$  or  $a_4 \neq 0$ , because  $\Pi \not\subset \langle L_1, L \rangle$ , which implies that the non-reducedness of the scheme  $V|_\Pi$  along the line  $L_1$  imposes 12 independent linear conditions on the coefficients  $\alpha_i, \beta_i, \gamma_{ij}, A, \delta_{ijk}, \epsilon_i$ .

Let  $\mathcal{R}$  be a family of threefolds that are constructed like the threefold  $V$ . Put

$$\mathcal{I} = \left\{ \left( (\Gamma, \Sigma), Y \right) \mid \Gamma \subset Y, P \notin \Gamma, Y \Big|_{\Sigma} \text{ is not reduced along } \Gamma \right\} \subset \mathcal{T} \setminus \mathcal{S} \times \mathcal{R},$$

and let  $\alpha: \mathcal{I} \rightarrow \mathcal{R}$  be the natural projection. Then

$$\dim(\mathcal{I}) = \dim(\mathcal{T} \setminus \mathcal{S}) + \dim(\mathcal{R}) - 12 = \dim(\mathcal{R}) - 1,$$

which implies that  $\alpha$  is not surjective. Thus, the scheme  $V|_{\Pi}$  is reduced along  $L_1$  if  $\Pi \notin \langle L_1, L_2 \rangle$ .

We see that  $(L_1, \Pi) \in \mathcal{S}$ . Then  $a_3 = a_4 = 0$ , but  $a_2 \neq 0$ , because  $L_1 \cap L = \emptyset$ , which implies that the non-reducedness of the scheme  $V|_{\Pi}$  along  $L_1$  imposes at least 9 independent linear conditions on  $\alpha_i, \beta_i, \gamma_{ij}, A, \delta_{ijk}, \epsilon_i$ . But  $\dim(\mathcal{S}) = 6$ , which is a contradiction.  $\square$

Arguing as in the proof of Lemma 8.8, we see that  $V$  satisfies the conditions E, F, G, H, I.

### References

- [1] M. Artin, *Some numerical criteria of contractability of curves on algebraic surfaces*, Amer. J. Math. **84** (1962), 485–496.
- [2] A. Corti, *Singularities of linear systems and 3-fold birational geometry*, Explicit birational geometry of 3-folds, 259–312, London Math. Soc. Lecture Note Ser., 281, Cambridge Univ. Press, Cambridge, 2000.
- [3] A. Corti and M. Mella, *Birational geometry of terminal quartic 3-folds I*, Amer. J. Math. **126** (2004), no. 4, 739–761.
- [4] V. Iskovskikh and A. Pukhlikov, *Birational automorphisms of multidimensional algebraic manifolds*, J. Math. Sci. **82** (1996), no. 4, 3528–3613.
- [5] M. Kawakita, *Divisorial contractions in dimension three which contracts divisors to smooth points*, Invent. Math. **145** (2001), no. 1, 105–119.
- [6] A. Pukhlikov, *Essentials of the method of maximal singularities*, Explicit birational geometry of 3-folds, 73–100, London Math. Soc. Lecture Note Ser., 281, Cambridge Univ. Press, Cambridge, 2000.
- [7] ———, *Birationally rigid varieties*, Mathematical Surveys and Monographs **190**, AMS, Providence, RI, 2013.
- [8] K. Shramov,  *$\mathbb{Q}$ -factorial quartic threefolds*, Sb. Math. **198** (2007), no. 7-8, 1165–1174.

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