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# BIRATIONALLY RIGID COMPLETE INTERSECTIONS OF CODIMENSION TWO

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Abstract. We prove that in the parameter space of $M$-dimensional Fano complete intersections of index one and codimension two the locus of varieties that are not birationally superrigid has codimension at least $\frac{1}{2}(M-9)(M-10)-1$.

## Introduction

**0.1. Statement of the main result.** Birational (super) rigidity is known for almost all families of Fano complete intersections of index one in the projective space, see [21–23]. Typically birational superrigidity was shown for a generic (in particular, non-singular) variety in the family. Now the improved techniques make it possible to obtain more precise results, covering complete intersections with certain simple types of singularities and estimating the codimension of the subset of non-rigid varieties in the parameter space of the family. The first work of this type for a family of Fano varieties was done in [6] for Fano hypersurfaces of index 1. Here we do it for complete intersections of codimension two.

In this paper, the symbol $\mathbb{P}$ stands for the complex projective space $\mathbb{P}^{M+2}$, where $M \geq 13$. Fix two integers $d_2 \geq d_1 \geq 2$, such that $d_1 + d_2 = M+2$ and consider the space

$$\mathcal{P} = \mathcal{P}_{d_1,M+3} \times \mathcal{P}_{d_2,M+3}$$

of pairs of homogeneous polynomials $(f_1, f_2)$ on $\mathbb{P}$ (that is to say, in $M+3$ variables $x_0, \dots, x_{M+2}$) of degrees $d_1$ and $d_2$, respectively. The symbol $V(f_1, f_2)$ denotes the set of common zeros of $f_1$ and $f_2$. The following claim is the main result of this paper.

**Theorem 0.1.** *There exists a Zariski open subset $\mathcal{P}_{\mathrm{reg}} \subset \mathcal{P}$ such that*:

(i) *for every pair $(f_1, f_2) \in \mathcal{P}_{\mathrm{reg}}$ the closed set $V = V(f_1, f_2)$ is irreducible, reduced and of codimension 2 in $\mathbb{P}$ with the singular locus* $\operatorname{Sing} V$ *of codimension at least 10 in $V$, so that $V$ is a factorial projective algebraic variety*; *the*

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*singularities of $V$ are terminal, so that $V$ is a primitive Fano variety of index 1 and dimension $M$*;

(ii) *the estimate*

$$\operatorname{codim}((\mathcal{P}\backslash\mathcal{P}_{\rm reg})\subset\mathcal{P})\geq\frac{1}{2}(M-9)(M-10)-1$$

*holds*;

(iii) *for every pair $(f_1,f_2)\in\mathcal{P}_{\rm reg}$ the Fano variety $V=V(f_1,f_2)$ is birationally superrigid.*

See [21, Chapter 2] for the definitions of birational rigidity and superrigidity as well as for the standard implications of these properties: Theorem 0.1 implies that for every pair $(f_1,f_2)\in\mathcal{P}_{\rm reg}$ the corresponding Fano complete intersection $V=V(f_1,f_2)\subset\mathbb{P}$ admits no structures of a rationally connected fibre space, that is to say, there exists no rational dominant map $\varphi\colon V\dashrightarrow S$ onto a positive dimensional base $S$ such that the fibre of general position is rationally connected. In particular, $V$ is non-rational. Another well known implication is that the groups of birational and biregular self-maps of $V$ are the same: $\operatorname{Bir}V=\operatorname{Aut}V$.

Now we describe the set $\mathcal{P}_{\rm reg}$ by explicit conditions (some of them are global but most of them are local) and outline the proof of Theorem 0.1.

**0.2. Regular complete intersections.** Consider a pair of homogeneous polynomials $(f_1,f_2)\in\mathcal{P}$, both non-zero. Below we list the conditions that these polynomials are supposed to satisfy for a regular pair.

(R0.1) The polynomial $f_1$ is irreducible and the hypersurface $\{f_1=0\}=F_1$ has at most quadratic singularities of rank 5.

*Remark* 0.1. This condition ensures that $F_1$ is a factorial variety so that $\operatorname{Cl}F_1\cong\operatorname{Pic}F_1$ is generated by the class of a hyperplane section and every effective divisor on $F_1$ is cut out by a hypersurface in $\mathbb{P}$.

(R0.2) $f_2|_{F_1}\not\equiv 0$ and moreover the closed set $\{f_2|_{F_1}=0\}$ is irreducible and reduced.

(R0.3) Every point $o\in V=V(f_1,f_2)$ is

- either non-singular,
- or a quadratic singularity,
- or a biquadratic singularity.

For each of the three types the local regularity conditions will be stated separately. Given a point $o\in V$, we fix a system of affine coordinates $z_1,\dots,z_{M+2}$ on an affine subset $o\in\mathbb{A}^{M+2}\subset\mathbb{P}^{M+2}$ with the origin at $o$, and write down the expansions of the polynomials $f_i$:

$$\begin{aligned}f_1&=q_{1,1}+q_{1,2}+\dots+q_{1,d_1},\\ f_2&=q_{2,1}+q_{2,2}+\dots+q_{2,d_1}+\dots+q_{2,d_2},\end{aligned}$$

where $q_{i,j}$ are homogeneous of degree $j$. We list the homogeneous polynomials in the *standard order* as follows:

$$q_{1,1}, q_{2,1}, q_{1,2}, q_{2,2}, \dots, q_{1,d_1}, q_{2,d_1}, \dots, q_{2,d_2},$$

so that polynomials of smaller degrees precede the polynomials of higher degrees and for $j \leq d_1$ the form $q_{1,j}$ precedes $q_{2,j}$.

Every non-singular point $o \in V$ is assumed to satisfy the regularity condition

(R1) the polynomials $q_{i,j}$ in the standard order with the last two of them removed form a regular sequence in $\mathcal{O}_{o,\mathbb{P}}$.

Every quadratic point $o \in V$ is assumed to satisfy a number of regularity conditions. Note that in this case at least one of the linear forms $q_{1,1}, q_{2,1}$ is non-zero and the other one is proportional to it. We denote a non-zero form in the set $\{q_{1,1}, q_{2,1}\}$ by the symbol $q_{*,1}$.

(R2.1) The rank of the quadratic point $o \in V$ is at least 9.

*Remark* 0.2. When we cut $V$ by a general linear subspace $P \subset \mathbb{P}$ of dimension 10, containing the point $o$, we get a complete intersection $V_P \subset P \cong \mathbb{P}^{10}$ of dimension 8 with the point $o$ an isolated singularity resolved by one blow up $V_P^+ \to V_P$, the exceptional divisor of which, $Q_P$, is a non-singular 7-dimensional quadric.

Apart from (R2.1), the quadratic point $o$ is assumed to satisfy the condition

(R2.2) the polynomials

$$q_{*,1}, q_{1,2}, q_{2,2}, \dots, q_{2,d_2}$$

in the standard order with $q_{2,d_2}$ removed, form a regular sequence in $\mathcal{O}_{o,\mathbb{P}}$.

Now let us consider the biquadratic points, that is, the points $o \in V$ for which $q_{1,1} \equiv q_{2,1} \equiv 0$.

(R3.1) For a general linear subspace $P \subset \mathbb{P}$ of dimension 12, containing the point $o$, the intersection $V_P = V \cap P$ is a complete intersection of codimension 2 in $P = \mathbb{P}^{12}$ with the point $o \in V_P$ an isolated singularity resolved by one blow up $V_P^+ \to V_P$ with the exceptional divisor $Q_P$ which is a non-singular complete intersection of two quadrics in $\mathbb{P}^{11}$, $\dim Q_P = 9$.

Apart from (R3.1), the biquadratic point $o$ is assumed to satisfy the condition (R3.2) the polynomials

$$q_{1,2}, q_{2,2}, \dots, q_{2,d_2}$$

form a regular sequence in $\mathcal{O}_{o,\mathbb{P}}$.

The subset $\mathcal{P}_{\rm reg}$ consists of the pairs $(f_1, f_2)$ such that the conditions (R0.1-R0.3) are satisfied and the conditions (R1), (R2.1) and (R2.2), (R3.1) and (R3.2) are satisfied for every non-singular, quadratic and biquadratic point, respectively.

**0.3. The structure of the proof of Theorem 0.1.** By the well known Grothendieck's theorem [2] for every pair $(f_1, f_2) \in \mathcal{P}_{\rm reg}$ the variety $V(f_1, f_2)$ satisfies the conditions of part (i) of Theorem 0.1. Therefore, Theorem 0.1 is implied by the following two claims.

**Theorem 0.2.** *The estimate*

$$\operatorname{codim}((\mathcal{P}\backslash\mathcal{P}_{\rm reg}) \subset \mathcal{P}) \geq \frac{1}{2}(M-9)(M-10) - 1$$

*holds.*

**Theorem 0.3.** *For every pair $(f_1, f_2) \in \mathcal{P}_{\rm reg}$ the variety $V = V(f_1, f_2)$ is birationally superrigid.*

The two claims are independent of each other and for that reason will be shown separately: Theorem 0.2 in Section 3 and Theorem 0.3 in Sections 1 and 2.

In order to prove Theorem 0.3, we fix a mobile linear system $\Sigma \subset |nH|$ on $V$, where $H$ is the class of a hyperplane section. All we need to show is that $\Sigma$ has no maximal singularities. (For all definitions and standard facts and constructions of the method of maximal singularities we refer the reader to [21, Chapters 2 and 3].)

Therefore, we consider the following four options:

- $\Sigma$ has a maximal subvariety,
- $\Sigma$ has an infinitely near maximal singularity, the centre of which on $V$ is not contained in the singular locus $\operatorname{Sing} V$,
- $\Sigma$ has an infinitely near maximal singularity, the centre of which on $V$ is contained in $\operatorname{Sing} V$ but not in the locus of biquadratic points,
- $\Sigma$ has an infinitely near maximal singularity, the centre of which on $V$ is contained in the locus of biquadratic points.

The first two options are excluded in Section 1 (this is fairly straightforward), where we also prove a useful technical claim strengthening the $4n^2$-inequality in the non-singular case. The two remaining options are excluded in Section 2 (which is much harder and requires some additional work).

Theorem 0.2 is shown in Section 3, which completes the proof of Theorem 0.1.

**0.4. Historical remarks.** The first complete intersection (which was not a hypersurface in the projective space) that was shown to be birationally rigid was the complete intersection of a quadric and cubic $V_{2\cdot 3} \subset \mathbb{P}^5$, see [10,15] and for a modern exposition [21, Chapter 2]. Higher-dimensional complete intersections were studied in [19, 22, 23]; as a result of that work, birational superrigidity is now proven for all non-singular generic complete intersections of index 1 in the projective space, except for three infinite series $2 \cdot \dots \cdot 2, 2 \cdot \dots \cdot 2 \cdot 3$ and $2 \cdot \dots \cdot 2 \cdot 4$ and finitely many particular families.

Three-dimensional complete intersections of type $2 \cdot 3$ with a double point were studied in [5]. Birational superrigidity of one particular family (complete intersections of type $2{\cdot}4$) of four-folds was proved in [3]. Recently a considerable progress was made in the study of birational geometry of weighted complete intersections and more complicated subvarieties [1,12–14]. Note that Fano double hypersurfaces and cyclic covers [4, 18, 20] are also complete intersections of index two in the weighted projective space. Finally, there is a recent paper [24] claiming birational superrigidity of certain families of complete intersections of index one, but it is based on the ideas of [7], which later turned out to be faulty [8] and even in the corrected version some parts are hard to follow. The classical techniques of the method of maximal singularities remains the only reliable approach to showing birational rigidity.

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## 1. Exclusion of maximal singularities. I. Maximal subvarieties and non-singular points

In this section we exclude maximal subvarieties of the mobile linear system $\Sigma$ (Subsection 1.1) and infinitely near maximal singularities of $\Sigma$, the centre of which is not contained in the singular locus of $V$ (Subsection 1.2). After that we show an improvement of the $4n^2$-inequality (Subsection 1.3), which will be used in Section 2 in the cases where the usual $4n^2$-inequality is insufficient.

**1.1. Exclusion of maximal subvarieties.** We start with the following claim.

**Proposition 1.1.** *The linear system $\Sigma$ has no maximal subvarieties.*

*Proof.* Assume that $B \subset V$ is a maximal subvariety for $\Sigma$. Let us consider first the case $\operatorname{codim}(B \subset V) = 2$. For a general linear subspace $P \subset \mathbb{P}$ of dimension 7 the intersection $V_P = V \cap P$ is a non-singular complete intersection of codimension 2 in $\mathbb{P}^7$, hence for the numerical Chow group of classes of cycles of codimension 2 on $V_P$ we have

$$A^2 V_P = \mathbb{Z} H_P^2,$$

where $H_P$ is the class of a hyperplane section of $V_P$. Now the standard arguments [21, Chapter 2, Section 2] give the inequality

$$\operatorname{mult}_{B \cap P} \Sigma_P \leq n,$$

where $\Sigma_P$ is the restriction of $\Sigma$ onto $V_P$, a mobile subsystem of $|nH_P|$. Therefore, $\operatorname{mult}_B \Sigma \leq n$ and $B$ is not a maximal subvariety — a contradiction.

Now let us consider the case $\operatorname{codim}(B \subset V) \geq 3$, $B \not\subset \operatorname{Sing} V$. In this case we have the inequality

$$\operatorname{mult}_B Z > 4n^2,$$

where $Z=(D_1\circ D_2)$ is the self-intersection of the system $\Sigma$, $D_i\in\Sigma$ are general divisors. As $\deg Z=n^2\deg V=n^2d_1d_2$, we use the inequality

$$\frac{\mathrm{mult}_o}{\deg}Y\leq\frac{4}{d_1d_2},$$

which holds for any smooth point $o\in V$ and any irreducible subvariety $Y\subset V$ of codimension 2 (see Proposition 1.3 below) to obtain a contradiction. Finally, assume that $B\subset\mathrm{Sing}\,V$. In this case $\mathrm{codim}(B\subset V)\geq 10$, so that

$$\mathrm{mult}_B\,\Sigma>\delta n,$$

where $\delta\geq 7$. Therefore, we have the inequality

$$\mathrm{mult}_B\,Z>98n^2,$$

which is impossible as for any singular point $o\in V$ and subvariety $Y$ of codimension 2 the inequality

$$\frac{\mathrm{mult}_o}{\deg}Y\leq\frac{9}{d_1d_2}$$

holds, see Propositions 2.1 and 2.2.

We have excluded all options for $B$. □

**1.2. Exclusion of maximal singularities, the centre of which is not contained in the singular locus.** Our next step is the following.

**Proposition 1.2.** *The centre $B$ of maximal singularity $E$ is contained in the singular locus* $\mathrm{Sing}\,V$.

*Proof.* Assume the converse: $B\not\subset\mathrm{Sing}\,V$. Since $B$ is not a maximal subvariety of $\Sigma$, we see that $\mathrm{codim}(B\subset V)\geq 3$ and the $4n^2$-inequality holds:

$$\mathrm{mult}_B\,Z>4n^2. \tag{1}$$

Now let us show the opposite inequality.

**Proposition 1.3.** *For any non-singular point $o\in V$ and any irreducible subvariety $Y$ of codimension 2 the inequality*

$$\frac{\mathrm{mult}_o}{\deg}Y\leq\frac{4}{d_1d_2}$$

*holds.*

*Proof.* We consider the general case when $d_1+2\leq d_2$; the obvious modifications for the two remaining cases $d_2=d_1+1$ and $d_2=d_1$ are left to the reader.

Our proof is identical to the proof of Theorem 2.1 on birational superrigidity of Fano complete intersections in [21, Chapter 3, Section 2], except for the only point of difference: due to the slightly weaker regularity condition (R1) for smooth points, the procedure of constructing intersections with hypertangent divisors has to terminate one step sooner than in the cited argument. In other words, we use hypertangent divisors

$$D_1,D_2,D'_3,D''_3,\dots,D'_i,D''_i,\dots,D'_{d_1-1},D''_{d_1-1},$$

followed by

$$D_{d_1}, \dots, D_{d_2-3}$$

(as usual, $D_i \in \Lambda_i$ or $D'_i, D''_i \in \Lambda_i$ are generic divisors in the $i$-th hypertangent linear system, $\Lambda_i \subset |iH|$, $\mathrm{mult}_o \Lambda_i \geq i+1$), *but not* $D_{d_2-2}$ as in [21, Chapter 3, Section 2], since the weaker regularity condition does not allow to make that last step.

Assuming that for $Y \ni o$ the claim of Proposition 1.3 does not hold, we apply the technique of hypertangents divisors as outlined above, and obtain an irreducible surface $S \ni o$, satisfying the inequality

$$\begin{aligned}\frac{\mathrm{mult}_o}{\deg} S &\geq \left(\frac{\mathrm{mult}_o}{\deg} Y\right) \cdot \frac{2}{1} \cdot \frac{3}{2} \cdot \left(\frac{4}{3} \cdot \dots \cdot \frac{d_1}{d_1-1}\right)^2 \cdot \frac{d_1+1}{d_1} \cdot \dots \cdot \frac{d_2-2}{d_2-3} \\ &= \left(\frac{\mathrm{mult}_o}{\deg} Y\right) \cdot d_1 \cdot \frac{d_2-2}{3} > \frac{4(d_2-2)}{3d_2} \geq 1\end{aligned}$$

(the last inequality in this sequence holds as $d_2 \geq 8$). Therefore, $\mathrm{mult}_o S > \deg S$, which is impossible. Proposition 1.3 is shown. □

Therefore, the inequality (1) is impossible. Proof of Proposition 1.2 is complete. □

**1.3. An improvement of the $4n^2$-inequality.** Let us consider the following general situation: $X$ is a smooth affine variety, $B \subset X$ a smooth subvariety of codimension at least 3, $\Sigma_X$ a mobile linear system on $X$ such that

$$\mathrm{mult}_B \Sigma_X = \alpha n \leq 2n$$

for some $\alpha \in (1, 2]$ and positive $n \in \mathbb{Q}$, but the pair $\left(X, \frac{1}{n}\Sigma_X\right)$ has a non-canonical singularity with the centre $B$. In other words, for some birational morphism $\varphi\colon \widetilde{X} \to X$ of smooth varieties and a $\varphi$-exceptional divisor $E \subset \widetilde{X}$, such that $\varphi(E) = B$, the Noether-Fano inequality

$$\mathrm{ord}_E \varphi^* \Sigma_X > na(E, X)$$

holds. By the symbol $Z_X = (D_1 \circ D_2)$ we denote the self-intersection of the mobile linear system $\Sigma_X$.

**Theorem 1.1.** *The following inequality holds*:

$$\mathrm{mult}_B Z_X > \frac{\alpha^2}{\alpha - 1} n^2$$

*Remark* 1.1. It is easy to see that the minimum of the real function $\frac{t^2}{t-1}$ on the interval (1,2] is attained at $t = 2$, so that the theorem improves the very well known $4n^2$-inequality [21, Chapter 2, Theorem 2.1]. The proof given below is based on the idea that was first used in [15] and later in several other papers.

*Proof of Theorem 1.1.* We follow the arguments given in [21, Chapter 2, Section 2], using the notations of the proof of the $4n^2$-inequality given there. Repeating those arguments word for word, we

- resolve the singularity $E$,
- consider the oriented graph $\Gamma$ of the resolution,
- divide the set of vertices of $\Gamma$ into the lower part ($\operatorname{codim} B_{i-1} \geq 3$) and the upper part ($\operatorname{codim} B_{i-1} = 2$),
- employ the technique of counting multiplicities,
- use the optimization procedure for the quadratic function $\sum_{i=1}^{K} p_i \nu_i^2$

and obtain the inequality

$$\operatorname{mult}_B Z > \frac{(2\Sigma_l + \Sigma_u)^2}{\Sigma_l(\Sigma_l + \Sigma_u)} n^2,$$

see Subsection 2.2 in [21, Chapter 2]. Now set $m = \frac{1}{n^2} \operatorname{mult}_B Z$, so that the equality just above can be re-written as

$$(4-m)\Sigma_l^2 + (4-m)\Sigma_l\Sigma_u + \Sigma_u^2 < 0.$$

As the elementary multiplicities $\nu_i = \operatorname{mult}_{B_{i-1}} \Sigma_X^{i-1}$ are non-increasing, we get the inequalities

$$\alpha n = \nu_1 \geq \nu_2 \geq \cdots \geq \nu_i \geq \nu_{i+1} \geq \cdots,$$

so that the Noether-Fano inequality implies the estimate

$$\alpha(\Sigma_l + \Sigma_u) > 2\Sigma_l + \Sigma_u.$$

As $1 < \alpha \leq 2$ by assumption, we conclude that

$$\Sigma_u > \frac{2-\alpha}{\alpha - 1}\Sigma_l.$$

Now the quadratic function $\gamma(t) = t^2 + (4-m)t + (4-m)$ attains the minimum at $t = \frac{1}{2}(m-4) > 0$ and is negative at $t = 0$. Therefore, if $\gamma(t_0) < 0$ for some

$$t_0 > \frac{2-\alpha}{\alpha - 1},$$

then

$$\gamma\left(\frac{2-\alpha}{\alpha-1}\right) = \left(\frac{2-\alpha}{\alpha-1}\right)^2 + (4-m)\left(\frac{2-\alpha}{\alpha-1}\right) + (4-m) < 0,$$

which easily transforms to the required inequality $m > \alpha^2/(\alpha - 1)$. □

The following elementary fact will be useful in Section 2 when maximal singularities, the centre of which is contained in the singular locus of $V$, are excluded.

**Proposition 1.4.** *The function of real argument*

$$\beta(t) = \frac{t^3}{t-1}$$

*is decreasing for $1 < t \leq \frac{3}{2}$ and increasing for $t \geq \frac{3}{2}$, so that it attains its minimum on $(1, \infty)$ at $t = \frac{3}{2}$, which is equal to $\frac{27}{4}$.*

*Proof.* Obvious calculations. □

## 2. Exclusion of maximal singularities. II. Quadratic and biquadratic points.

In this section we exclude infinitely near maximal singularities of the linear system $\Sigma$, the centre of which is contained in the singular locus of $V$. We start with using the technique of hypertangent divisors to obtain estimates for the multiplicities $\mathrm{mult}_o\,\Sigma$ and $\mathrm{mult}_o\,Z$, where $o$ is a general point in the centre of the maximal singularity and $Z$ is the self-intersection of the mobile system $\Sigma$ (Subsection 2.1). After that, we consider separately the cases when the centre is contained in the locus of the quadratic singularities (Subsections 2.2 and 2.3) and biquadratic singularities (Subsections 2.4 and 2.5). We make use of the inversion of adjunction and the connectedness principle, similarly to the arguments of Section 4 in [Book,Chapter 2], with (quite non-trivial) modifications due to the exceptional divisor of the blow up of the point $o$ being either a quadric or a complete intersection of two quadrics.

**2.1. The technique of hypertangent divisors.** Let $o \in \mathrm{Sing}\, V$ be a singularity (either a quadratic or a biquadratic point), $\sigma\colon V^+ \to V$ its blow up with the exceptional divisor $Q \subset V^+$. We consider $\sigma$ as the resriction of the blow up $\sigma_{\mathbb{P}}\colon \mathbb{P}^+ \to \mathbb{P}$ of the same point $o$ on the projective space $\mathbb{P}$ with the exceptional divisor $E_{\mathbb{P}} = \sigma_{\mathbb{P}}^{-1}(o)$, so that $Q$ is either a quadric in a hyperplane in $E_{\mathbb{P}} \cong \mathbb{P}^{M+1}$ or a complete intersection of two quadrics in $E_{\mathbb{P}}$. For a generic divisor $D \in \Sigma$ set

$$D^+ \sim \sigma^* D - \nu Q$$

for some $\nu \in \mathbb{Z}_+$; thus $\mathrm{mult}_o D = 2\nu$ in the quadratic and $4\nu$ in biquadratic case. In the singular case Proposition 1.3 has to be replaced by the following facts. Let $Y \subset V$ be an irreducible subvariety.

**Proposition 2.1.** *Assume that* $\mathrm{mult}_o\, V = 2$.

(i) *If* $\mathrm{codim}(Y \subset V) = 2$, *then the inequality*

$$\frac{\mathrm{mult}_o}{\deg} Y \le \frac{7}{d_1 d_2}$$

*holds.*

(ii) *If* $\mathrm{codim}(Y \subset V) = 3$, *then the inequality*

$$\frac{\mathrm{mult}_o}{\deg} Y \le \frac{72}{7 d_1 d_2}$$

*holds.*

(iii) *The inequality* $\nu \le \sqrt{\frac{7}{2}} n$ *holds.*

Similarly, for the biquadratic case we have:

**Proposition 2.2.** *Assume that* $\mathrm{mult}_o\, V = 4$.

(i) *If* $\operatorname{codim}(Y \subset V) = 2$*, then the inequality*

$$\frac{\operatorname{mult}_o}{\deg} Y \leq \frac{9}{d_1 d_2}$$

*holds.*

(ii) *The inequality* $\nu \leq \frac{3}{2} n$ *holds.*

*Proof of Proposition 2.1.* The claim (iii) follows from (i): for the self-intersection $Z$ of the mobile system $\Sigma$ we have the inequality $\operatorname{mult}_o Z \geq 2\nu^2$. As $\deg Z = n^2 d_1 d_2$, we get the inequality of part (iii), assuming (i).

In order to show the claim (i), we apply the technique of hypertangent divisors in the same way as in the proof of Proposition 1.3, but starting with the second hypertangent divisor and completing the procedure with the hypertangent divisor $D_{d_2-2}$ — one more than in the proof of Proposition 1.3, so that now we use the hypertangent divisors

$$D_2, D'_3, D''_3, \ldots, D'_{d_1-1}, D''_{d_1-1}, D_{d_1}, \ldots, D_{d_2-2}.$$

If the claim (i) is not true, we obtain an irreducible surface $S \ni o$, satisfying the inequality

$$\begin{aligned}\frac{\operatorname{mult}_o}{\deg} S &\geq \left(\frac{\operatorname{mult}_o}{\deg} Y\right) \cdot \frac{3}{2} \cdot \left(\frac{4}{3} \cdot \cdots \cdot \frac{d_1}{d_1 - 1}\right)^2 \cdot \frac{d_1 + 1}{d_1} \cdot \cdots \cdot \frac{d_2 - 1}{d_2 - 2} \\ &= \left(\frac{\operatorname{mult}_o}{\deg} Y\right) \cdot \frac{d_1(d_2 - 1)}{6} > \frac{7(d_2 - 1)}{6 d_2} > 1\end{aligned}$$

which is impossible. The contradiction proves the claim (i).

Finally, to show the claim (ii), we argue in exactly the same way as above, starting with the hypertangent divisors $D'_3, D''_3$ (removing $D_2$), so that if the claim (ii) does not hold, we obtain an irreducible surface $S \ni o$, satisfying the inequality

$$\frac{\operatorname{mult}_o}{\deg} S > \frac{72}{7 d_1 d_2} \cdot \left(\frac{4}{3} \cdot \cdots \cdot \frac{d_1}{d_1 - 1}\right)^2 \cdot \frac{d_1 + 1}{d_1} \cdot \cdots \cdot \frac{d_2 - 1}{d_2 - 2}.$$

The right hand side simplifies to

$$\frac{72(d_2 - 1)}{63 d_2} \geq 1$$

for $d_2 \geq 8$ which gives the desired contradiction and completes the proof of Proposition 2.1. □

*Proof of Proposition 2.2.* Proof of Proposition 2.2 is very similar. First, we note that part (i) implies part (ii) via looking at the multiplicity of the self-intersection $Z$ at the point $o$. In order to show the claim (i), we use the hypertangent divisors

$$D'_3, D''_3, \ldots, D'_{d_1-1}, D''_{d_1-1}, D_{d_1}, \ldots, D_{d_2-1}$$

to obtain the required estimate. □

**2.2. Exclusion of the quadratic case, part I.** In this subsection and in the next one we assume that the centre of the maximal singularity is contained in the singular locus $\operatorname{Sing} V$ but not in the locus of biquadratic points. We will show that this assumption leads to a contradiction. To begin with, fix a general point $o \in V$ in the centre of the maximal singularity.

Let $\Pi \subset P$ be a general 6-plane in a 10-plane in $\mathbb{P}$ through the point $o$. Denote by $V_\Pi$ and $V_P$ the intersections $V \cap \Pi$ and $V \cap P$, respectively. By our assumptions about the singularities of $V$, the varieties $V_\Pi$ and $V_P$ are non-singular outside $o$. Let

$$\begin{array}{ccccccc} & V_\Pi^+ & \subset & & V_P^+ & \subset & V^+ \\ \sigma_\Pi & \downarrow & & \sigma_P & \downarrow & & \downarrow \quad \sigma \\ & V_\Pi & \subset & & V_P & \subset & V \end{array} \tag{2}$$

be the blow ups of the point $o$ on $V_\Pi$, $V_P$ and $V$. The varieties $V_\Pi^+$ and $V_P^+$ are non-singular. Denote the exceptional divisors of $\sigma_\Pi, \sigma_P$ and $\sigma$ by $Q_\Pi$, $Q_P$ and $Q$, respectively. The quadrics $Q_\Pi$ and $Q_P$ are non-singular. The hyperplane sections of $V_\Pi$ and $V_P$ will be written as $H_\Pi$ and $H_P$. Obviously, for a general divisor $D \in \Sigma$ we have

$$D_\Pi^+ \sim nH_\Pi - \nu Q_\Pi, \quad D_P^+ \sim nH_P - \nu Q_P,$$

where $D_\Pi = D|_{V_\Pi}$, $D_P = D|_{V_P}$ (abusing our notations, we write $H_P$ for $\sigma_P^* H_P$ etc.) and the upper index $+$ means the strict transform. By inversion of adjunction the pairs $(V_\Pi, \frac{1}{n} D_\Pi)$ and $(V_P, \frac{1}{n} D_P)$ are not log canonical at the point $o$. As by Proposition 2.1(iii) we have $\nu < 2n$, whereas $a(Q_\Pi, V_\Pi) = 2$, the pair

$$\left( V_\Pi^+, \frac{1}{n} D_\Pi^+ + \frac{(\nu - 2n)}{n} Q_\Pi \right) \tag{3}$$

is not log canonical, and the centre of any of its non-log canonical singularities is contained in the exceptional quadric $Q_\Pi$ (see Lemma 4.1 in [21, Chapter 2]). The union of all centres of non-log canonical singularities of the pair (3) is a connected closed set by the Connectedness Principle [11, 25]. Therefore,

- either it is a point,
- or it is a connected 1-cycle,
- or it contains a surface on the quadric $Q_\Pi$.

As the union of all centres of non-log canonical singularities of the pair (3) is a section of the union of all centres of non-log canonical singularities of the pair

$$\left( V_P^+, \frac{1}{n} D_P^+ + \frac{(\nu - 2n)}{n} Q_P \right) \tag{4}$$

by $V_\Pi^+ \cap Q_P$ (which is a section of the non-singular quadric $Q_P$ by a general 4-plane in $\langle Q_P \rangle$), we see that the first option is impossible, as the smooth

7-dimensional quadric $Q_P$ can not contain a linear subspace of dimension 4. Therefore, we conclude that the pair (4) is not log canonical at an irreducible subvariety $\Delta \subset Q_P$ of codimension either 1 or 2.

**Proposition 2.3.** *The case* $\operatorname{codim}(\Delta \subset Q_P) = 1$ *is impossible.*

*Proof.* Assume that $\Delta$ is a divisor on $Q_P$. Then by Proposition 4.1 in [21, Chapter 2] we have the following estimate for the multiplicity of the self-intersection $Z_P$ of the system $\Sigma_P = \Sigma|_{V_P}$ at the point $o$:

$$\operatorname{mult}_o Z_P \geq 2\nu^2 + 2 \cdot 4 \left(3 - \frac{\nu}{n}\right) n^2$$

(the factor 2 in the second component of the right hand side appears since we have the inequality $\deg \Delta \geq 2$), and easy calculations give

$$\operatorname{mult}_o Z = \operatorname{mult}_o Z_P \geq 16n^2,$$

which contradicts Proposition 2.1(i). □

Therefore we assume that $\Delta \subset Q_P$ is an irreducible subvariety of codimension 2. That option will be shown to be impossible in the next subsection.

**2.3. Exclusion of the quadratic case, part II.** Our arguments are very similar to those in [21, Chapter 2, Section 4]. Let $D_1, D_2 \in \Sigma$ be general divisors, $Z = (D_1 \circ D_2)$ the self-intersection of the system $\Sigma$. We can write

$$((D_1|_{V_P})^+ \circ (D_2|_{V_P})^+) = Z_P^+ + Z_{P,Q},$$

where $Z_{P,Q}$ is an effective divisor on the quadric $Q_P$. By the standard rules of the intersection theory [9],

$$\operatorname{mult}_o Z = \operatorname{mult}_o Z_P = \deg(Z_P^+ \circ Q_P) = 2\nu^2 + \deg Z_{P,Q}.$$

Let us consider the cases $\deg \Delta = 2$ (when $\Delta$ is a section of $Q_P$ by a linear subspace of codimension 2 in $\langle Q_P \rangle$) and $\deg \Delta \geq 4$ separately. Set $\alpha = \frac{\nu}{n} < 2$. Note that since $\operatorname{mult}_\Delta \Sigma_P^+ > n$ and $\Sigma_P^+|_{Q_P} \sim \nu H_Q$, where $H_Q$ is the hyperplane section of the quadric $Q_P$, we have the inequality $\nu > n$, so that $\alpha > 1$. By Theorem 1.1,

$$\operatorname{mult}_\Delta (Z_P^+ + Z_{P,Q}) > \frac{\alpha^2}{\alpha - 1} n^2.$$

Assume now that $\deg \Delta \geq 4$. By Proposition 2.1(i) we have:

$$4 \operatorname{mult}_\Delta Z_P^+ \leq \deg(Z_P^+ \circ Q_P) \leq 7n^2,$$

so that

$$\operatorname{mult}_\Delta Z_{P,Q} > \left(\frac{\alpha^2}{\alpha - 1} - \frac{7}{4}\right) n^2.$$

However, for $l \in \mathbb{Z}_+$ defined by the equivalence

$$Z_{P,Q} \sim l H_Q$$

we have the estimate $l \geq \operatorname{mult}_{\Delta} Z_{P,Q}$, so that

$$\operatorname{mult}_o Z = 2(\nu^2 + l) > 2\left(\alpha^2 + \frac{\alpha^2}{\alpha - 1}\right) - \frac{7}{4}n^2.$$

The right hand side simplifies as

$$2\left(\frac{\alpha^3}{\alpha - 1} - \frac{7}{4}\right)n^2 \geq 10n^2$$

by Proposition 1.4. Therefore, we obtained the inequality $\operatorname{mult}_o Z > 10n^2$, which contradicts Proposition 2.1(i). The case $\deg \Delta \geq 4$ is now excluded.

From now on, and until the end of this subsection, we assume that $\deg \Delta = 2$, that is, $\Delta$ is cut out on $Q_P$ by a linear subspace in $\langle Q_P \rangle$ of codimension 2. By construction, that means that there is a subvariety $\Delta_V \subset Q$ of codimension 2 and degree 2 (that is, $\Delta_V$ is cut out on the quadric $Q$ by a linear subspace in $\langle Q \rangle$ of codimension 2), such that pair

$$\left(V^+, \frac{1}{n}\Sigma^+ + \frac{\nu - 2n}{n}Q\right)$$

is not log canonical at $\Delta_V$ and

$$\Delta = \Delta_V \cap V_P^+.$$

Let $R$ be a general hyperplane section of $V$, such that $R \ni o$ and the strict transform $R^+$ contains $\Delta_V$. Let $Z_R = (Z \circ R)$ be the self-intersection of the mobile system $\Sigma_R = \Sigma|_R$. Obviously,

$$\operatorname{mult}_o Z_R = \operatorname{mult}_o Z + 2 \operatorname{mult}_{\Delta_V} Z^+.$$

Now set $Z_{P,R} = (Z_P \circ Z_R)$. By generality of both $P$ and $R$ we have the equalities

$$\operatorname{mult}_o Z_{P,R} = \operatorname{mult}_o Z_R, \quad \operatorname{mult}_{\Delta} Z_P^+ = \operatorname{mult}_{\Delta_V} Z^+.$$

Applying Proposition 2.1(iii) and taking into account the equalities above, we get the estimate

$$\operatorname{mult}_o Z_P + 2 \operatorname{mult}_{\Delta} Z_P^+ \leq \frac{72}{7}n^2. \tag{5}$$

On the other hand, $Q_P$ is a non-singular (quadric) hypersurface, so that by [21, Chapter 2, Proposition 2.3] we have the estimate

$$\deg Z_{P,Q} \geq 2 \operatorname{mult}_{\Delta} Z_{P,Q}$$

and for that reason

$$\operatorname{mult}_o Z_P \geq 2\nu^2 + 2 \operatorname{mult}_{\Delta} Z_{P,Q},$$

so that by (5) we get:

$$\frac{72}{7}n^2 \geq 2\nu^2 + 2(\operatorname{mult}_{\Delta} Z_{P,Q} + \operatorname{mult}_{\Delta} Z_P^+)$$

$$> 2\left(\alpha^2 + \frac{\alpha^2}{\alpha - 1}\right) n^2 = 2\frac{\alpha^3}{\alpha - 1} n^2.$$

Now we apply Proposition 1.4 and obtain the inequality $\frac{72}{7} > \frac{27}{2}$, which is false. This contradiction excludes the quadratic case completely.

**2.4. Exclusion of the biquadratic case, part I.** In this section and in the next one we assume that the centre of the maximal singularity is contained in the locus of biquadratic points. Again, we show that this assumption leads to a contradiction. For a start, we fix a general point $o \in V$ in the centre of the maximal singularity.

Now we take a general 7-plane $\Pi$ through the point $o$ and a general 12-plane $P \supset \Pi$. The notations $V_\Pi$, $V_P$ etc. have the same meaning as in quadratic case (Subsection 2.2), the same applies to the diagram (2) and the subsequent introductory arguments. The only difference is that the exceptional divisors $Q_\Pi$ and $Q_P$ of the blow ups of the point $o$ on $V_\Pi$ on $V_P$ are now non-singular complete intersections of two quadrics. Instead of Proposition 2.1, we use Proposition 2.2(ii) to obtain the inequality $\nu \leq \frac{3}{2}n < 2n$ and, once again, to conclude that the pair (3) is non-log canonical. Repeating the arguments of Subsection 2.2, we obtain the following four options for the union of all centres of non-log canonical singularities of the pair (3) in the biquadratic case:

- either it is a point,
- or it is a connected 1-cycle,
- or it is a connected closed set of dimension 2,
- or it contains a divisor on the 4-dimensional complete intersection $Q_\Pi$.

Passing over to the pair (4) in exactly the same way as we did it in the quadratic case, we see that the first option is impossible as a non-singular 9-fold $Q_P$ can not contain a linear subspace of dimension 5. Therefore, the pair (4) is not log canonical at an irreducible subvariety $\Delta \subset Q_P$ of codimension 1,2 or 3. The divisorial case ($\operatorname{codim}(\Delta \subset Q_P) = 1$) is excluded by the arguments of the proof of Proposition 2.3 — in fact, we get a stronger estimate in this case:

$$\operatorname{mult}_o Z_P \geq 4\nu^2 + 4 \cdot 4\left(3 - \frac{\nu}{n}\right) n^2$$

(as $\operatorname{mult}_o V_P = 4$ and $\deg \Delta \geq 4$), so that

$$\operatorname{mult}_o Z = \operatorname{mult}_o Z_P \geq 32n^2,$$

which contradicts Proposition 2.2(i).

The case $\operatorname{codim}(\Delta \subset Q_P) = 2$ is excluded by the arguments of Subsection 2.3 as $\deg \Delta \geq 4$ and the resulting estimate $\operatorname{mult}_o Z > 10n^2$ contradicts Proposition 2.2(i).

It remains to exclude the last option, when $\operatorname{codim}(\Delta \subset Q_P) = 3$, for which there is no analog in the quadratic case.

**2.5. Exclusion of the biquadratic case, part II.** From now on, and until the end of this section, $\Delta \subset Q_P$ is an irreducible subvariety of codimension 3. Slightly abusing our notations, which should not generate any misunderstanding, we show first the following claim.

**Proposition 2.4.** *Let $Q = G_1 \cap G_2 \subset \mathbb{P}^N$, $N \geq 11$, be a non-singular complete intersection of two quadrics $G_1$ and $G_2$, $W \subset Q$ an irreducible subvariety of codimension 2 and $\Delta \subset Q$ an irreducible subvariety of codimension 3. Let $l \in \mathbb{Z}_+$ be defined by the relation*

$$W \sim lH_Q^2,$$

*where $H_Q$ is the class of a hyperplane section of $Q$. Then the inequality*

$$\mathrm{mult}_\Delta W \leq l$$

*holds.*

*Proof.* Assume the converse. For a point $p \in Q$ we denote by the symbol $|H_Q - 2p|$ the pencil of tangent hyperplane sections at that point.

**Lemma 2.1.** *Let $Y \subset Q$ be an irreducible subvariety of codimension 2, containing the subvariety $\Delta$. For a general point $p \in \Delta$ and any divisor $T \in |H_Q - 2p|$ we have $Y \not\subset T$.*

*Proof of Lemma 2.1.* Assume the converse. Then for general points $p, q \in \Delta$ and some hyperplane sections $T_p \in |H_Q - 2p|$ and $T_q \in |H_Q - 2q|$ we have $Y \subset T_p \cap T_q$, so that $Y = T_p \cap T_q$ is a section of $Q$ by a linear subspace of codimension 2. Since $\mathrm{Sing}(T_p \cap T_q)$ is at most 1-dimensional (see, for instance, [17]) and $\mathrm{codim}(\Delta \subset Q) = 3$, we obtain a contradiction, varying the points $p, q$. □

We conclude that for a general point $p \in \Delta$ and an *arbitrary* hyperplane section $T_p \in |H_Q - 2p|$ the cycle $W_p = (W \circ T_p)$ is well defined. It is an effective cycle of codimension 3 on $Q$ and 2 on $T_p$ (the latter variety is a complete intersection of two quadrics in $\mathbb{P}^{N-1}$ with at most 0-dimensional singularities). Let $H_p \in \mathrm{Pic}\, T_p$ be the class of a hyperplane section. Then we can write $W_p \sim lH_p^2$. Set

$$\Delta_p = \Delta \cap T_p.$$

Obviously, for a general point $p$ the closed set $\Delta_p$ is of codimension 3 on $T_p$. For any point $q \in \Delta_p$ the inequality

$$\mathrm{mult}_q W_p > l$$

holds. Besides, by construction $\mathrm{mult}_p W_p > 2l$.

Now let us consider a point $q \in \Delta_p$ of general position. Repeating the proof of Lemma 2.1 word for word (and taking into account that the complete intersection of two quadrics $T_p$ has zero-dimensional singularities), we see that for *any* divisor $T_q \in |H_Q - 2q|$ none of the components of the effective cycle $W_p$ is contained in $T_q$, so that

$$W_{pq} = (W_p \circ T_q)$$

is well defined effective cycle of codimension 2 on $T_p \cap T_q$, of codimension 3 on $T_p$ and 4 on $Q$. Since $T_q$ is an arbitrary hyperplane section in the pencil $|H_Q - 2q|$, we can choose it to be the one containing the point $p$. Now $W_{pq}$ is an effective cycle of codimension 6 on $\mathbb{P}^N$ of degree $\deg W_{pq} = 4$, satisfying the inequalities

$$\mathrm{mult}_p W_{pq} > 2l \quad \text{and} \quad \mathrm{mult}_q W_{pq} > 2l.$$

Taking a general projection onto $\mathbb{P}^{N-6}$, we conclude that the line $[p, q] \subset \mathbb{P}^N$, joining the points $p$ and $q$, is contained in the support of the cycle $W_{pq}$. Therefore, for *any* point $q \in \Delta_p$ we have $[p, q] \subset W$ and so for *any* point $q \in \Delta$ we have $[p, q] \subset W$. Since $\Delta$ is not a linear subspace in $\mathbb{P}^N$ ($Q$ cannot contain linear subspaces of dimension $N - 5$) and $\dim W = N - 4$, we conclude that $\Delta$ is a hypersurface in a linear subspace of dimension $N - 4$ and $W$ is that linear subspace, which is again impossible. The proof of Proposition 2.4 is now complete. □

Now coming back to the biquadratic case and using the notations of that case, we write for general divisors $D_1, D_2 \in \Sigma$:

$$((D_1|_{V_P})^+ \circ (D_2|_{V_P})^+) = Z_P^+ + Z_{P,Q},$$

where again $Z_{P,Q}$ is an effective divisor on the exceptional divisor of the blow up $\sigma_P$ of the point $o$, which is a non-singular complete intersection of two quadrics. Again,

$$\mathrm{mult}_o Z = \mathrm{mult}_o Z_P = \deg(Z_P^+ \circ Q_P) = 4\nu^2 + \deg Z_{P,Q}. \tag{6}$$

We set $\alpha = \frac{\nu}{n} \leq \frac{3}{2}$. By Theorem 1.1,

$$\mathrm{mult}_\Delta(Z_P^+ \circ Q_P) + \mathrm{mult}_\Delta Z_{P,Q} > \frac{\alpha^2}{\alpha - 1} n^2.$$

By Proposition 2.4,

$$\mathrm{mult}_\Delta(Z_P^+ \circ Q_P) \leq \frac{1}{4} \deg(Z_P^+ \circ Q_P) = \mathrm{mult}_o Z_P.$$

As $\deg Q_P = 4$, we also have the estimate

$$\mathrm{mult}_\Delta Z_{P,Q} \leq \frac{1}{4} \deg Z_{P,Q},$$

so that

$$\mathrm{mult}_o Z_P + \deg Z_{P,Q} > 4\frac{\alpha^2}{\alpha - 1} n^2.$$

Using (6), we get finally:

$$2\,\mathrm{mult}_o Z > 4\left(\alpha^2 + \frac{\alpha^2}{\alpha - 1}\right) n^2 = 4\frac{\alpha^3}{\alpha - 1} n^2.$$

Applying Proposition 1.4, we conclude that

$$\mathrm{mult}_o\, Z > \frac{27}{2}n^2,$$

which contradicts Proposition 2.2(i).

Proof of Theorem 0.3 is now complete.

## 3. Regularity conditions

In this section we will prove Theorem 0.2 in several steps. We first notice that

$$\mathrm{codim}((\mathcal{P}\backslash\mathcal{P}_{\mathrm{reg}}) \subset \mathcal{P}) = \min_{\{*\in S\}}\{\mathrm{codim}((\mathcal{P}\backslash\mathcal{P}_*) \subset \mathcal{P})\},$$

where $S = \{(\mathrm{R0.1}),(\mathrm{R0.2}),\dots,(\mathrm{R3.2})\}$ and

$$\mathcal{P}_* = \{(f_1, f_2) \in \mathcal{P} \mid \text{the pair satifies the regularity condition } *\}.$$

We first deal with the global conditions (R0.1-R0.3) (Subsection 3.1). Then move onto estimating the codimension of the bad set for the condition (R1) (that is, the set of pairs $(f_1, f_2)$ that do not satisfy that condition) and show that the same estimates work for the conditions (R2.2) and (R3.2) (Subsections 3.2 and 3.3). Lastly, we deal with the conditions (R2.1) and (R3.1) to get our total estimate (Subsection 3.4).

**3.1. Global conditions.** We first start by splitting the condition (R0.1) up into two conditions. The first is the irreducibility condition for the hypersurface $\{f_1 = 0\}$; the set of pairs $(f_1, f_2)$ with $f_1$ irreducible is denoted by $\mathcal{P}_{\mathrm{irred}}$. The second condition is that the hypersurface $\{f_1 = 0\}$ has at most quadratic singularities of rank at least 5; the corresponding subset of $\mathcal{P}$ is denoted by $\mathcal{P}_{\mathrm{qsing}\geq 5}$.

**Proposition 3.1.** *The codimension of* $\mathcal{P} \setminus \mathcal{P}_{\mathrm{irred}}$ *in* $\mathcal{P}$ *is at least* $\frac{M(M+3)}{2}$.

*Proof.* This is independent of the choice of $f_2$, hence it reduces to looking at $f \in \mathcal{P}_{d_1,M+3}$ such that $f = g_1 \cdot g_2$ with $\deg g_1 = a$ and $\deg g_2 = d_1 - a$, $a = 1, 2, \dots, d_1 - 1$. Then we define

$$\mathcal{F}_i = \mathcal{P}_{i,M+3} \times \mathcal{P}_{d_1-i,M+3}.$$

Obviously, we have

$$\dim \mathcal{P} \setminus \mathcal{P}_{\mathrm{irred}} \leq \max\{\dim \mathcal{F}_i \mid i = 1, 2, \dots, d_1 - 1\}.$$

We calculate:

$$\dim\, \mathcal{F}_i = \binom{i + M + 2}{M + 2} + \binom{d_1 - i + M + 2}{M + 2}.$$

By assumption $d_1 \leq \frac{M}{2} + 1$. We see that this gives the maximum dimension occurring at $i = 1$, or $i = d_1 - 1$ as $\mathcal{F}_i = \mathcal{F}_{d_1-i}$. Then

$$\dim\, \mathcal{F}_1 = (M + 3) + \binom{d_1 + M + 1}{M + 2},$$

which immediately estimates the codimension of $\mathcal{P}\setminus\mathcal{P}_{\mathrm{irred}}$ in $\mathcal{P}$ from below by

$$\binom{d_1+M+2}{M+2}-\left((M+3)+\binom{d_1+M+1}{M+2}\right)=\binom{d_1+M+1}{M+1}-(M+3).$$

The minimal value occurs at $d_1=2$ to get the estimate claimed by our proposition. $\square$

**Proposition 3.2.** *The codimension of $\mathcal{P}\setminus\mathcal{P}_{\mathrm{qsing}\geq 5}$ in $\mathcal{P}$ is at least $\binom{M-1}{2}+1$.*

*Proof.* This is essentially a calculation about the rank of quadratic forms which has been done in many places, see [6]. $\square$

As $\mathcal{P}_{(R0.1)}=\mathcal{P}_{\mathrm{irred}}\cap\mathcal{P}_{\mathrm{qsing}\geq 5}$, we get that the codimension of $\mathcal{P}\setminus\mathcal{P}_{(R0.1)}$ in $\mathcal{P}$ is at least $\binom{M-1}{2}+1$.

Now we consider $\mathcal{P}_{(R0.2)}\subset\mathcal{P}$ consisting of pairs $(f_1,f_2)$ satisfying the regularity condition (R0.2). We have two cases to consider: the first is if the hypersurfaces contains a common component; the second is if the intersection is non-reduced or reducible. The second case is the only one which needs considering as the first one gives a much higher codimension of the bad set. Fixing $f_1$ we consider the set $\mathcal{H}\subset\mathcal{P}_{d_2,M+3}$ such that $F_1\cap F_2$ is reducible or non-reduced.

**Proposition 3.3.** *The codimension of $\mathcal{H}$ in $\mathcal{P}_{d_2,M+3}$ is at least $\binom{M+2}{2}-2$.*

*Proof.* Taking into account Remark 0.1, we see that if $f_2\in\mathcal{H}$, then:

$$f_2|_{F_1}\in\mathcal{P}_{i,M+3}|_{F_1}\times\mathcal{P}_{d_2-i,M+3}|_{F_1}$$

for some $i=1,2,\dots,d_2-1$. Arguing like in the proof of Proposition 3.1, we get: the codimension of $\mathcal{H}$ in $\mathcal{P}_{d_2,M+3}$ is greater or equal than

$$\binom{d_2+M+2}{d_2}-\left((M+3)+\binom{d_2+M+1}{d_2-1}+\binom{d_2-d_1+M+2}{d_2-d_1}\right)$$
$$=\frac{1}{(M+2)!}\left(\frac{(M+2)(d_2+M+1)!}{d_2!}-\frac{(d_2-d_1+M+2)!}{(d_2-d_1)!}\right)-(M+3).$$

Using the substitution $s=d_2-d_1$, we see that for a fixed $s$ the minimum of the above expression occurs for $d_2=s+2$ and is equal to

$$\frac{1}{(M+2)!}\left(\frac{(M+2)(s+M+3)!}{d_2!}-\frac{(s+M+2)!}{s!}\right)-(M+3).$$

An easy check shows that this is an increasing function of $s$, so that the minimum occurs at $s=0$ to give us the required estimate. $\square$

**3.2. Regularity conditions for smooth points.** Recall that a smooth point satisfies the regularity condition (R1) if the homogeneous components $q_{i,j}$ in the standard order with the last two terms (that is, the two terms of highest degree) removed, form a regular sequence. If $d_1<d_2$, then we need

$$W=\{q_{1,1}=q_{1,2}=\cdots=q_{1,d_1}=q_{2,1}=q_{2,2}=\cdots=q_{2,d_2-2}=0\}$$

to be a finite set of surfaces in $\mathbb{A}^{M+2}$. If $d_1 = d_2$, then we need

$$W = \{q_{1,1} = q_{1,2} = \cdots = q_{1,d_1-1} = q_{2,1} = q_{2,2} = \cdots = q_{2,d_2-1} = 0\}$$

to be a finite set of surfaces in $\mathbb{A}^{M+2}$.

The linear forms $q_{1,1}$ and $q_{2,1}$ define the tangent space $T_pV$ at the point $p$, so in the case $d_2 > d_1$

$$W = \{q_{1,2}|_{T_pV} = \cdots = q_{1,d_1}|_{T_pV} = q_{2,2}|_{T_pV} = \cdots = q_{2,d_2-2}|_{T_pV} = 0\} \subset \mathbb{A}^M$$

and similarly for the case $d_1 = d_2$. Finally as all the terms above are homogeneous we can consider the projective variety defined by the same equations in the projectivized tangent space. Denote this by $\widetilde{W} \subset \mathbb{P}^{M-1}$. We have now redefined the regularity condition under consideration to be $\operatorname{codim}(\widetilde{W} \subset \mathbb{P}^{M-1}) = M-2$, that is, $\widetilde{W}$ is a finite set of curves.

**Proposition 3.4.** *The codimension of $\mathcal{P} \setminus \mathcal{P}_{(R1)}$ in $\mathcal{P}$ is at least*

$$\lambda(M) = \frac{(M-5)(M-6)}{2} - (M+1).$$

*Proof.* We follow the methods given in [16, 19] to estimate the codimension of the space of varieties which violate the regularity conditions. The scheme of these methods will be briefly outlined here, firstly we introduce the necessary definitions.

We say a sequence of polynomials $p_1, p_2, \ldots, p_l$ is *k-regular*, with $k \leq l$ if the subsequence $p_1, p_2, \ldots, p_k$ is regular.

We re-label our polynomials in their standard ordering by $h_1 = q_{1,2}$, $h_2 = q_{2,2}$, etc. Also define $\deg h_i = m_i$ to get our sequence $h_1, \ldots, h_{M-2}$, with $m_i \leq m_{i+1}$ in the space

$$\mathcal{L} = \prod_{i=1}^{M-2} \mathcal{P}_{m_i,M}.$$

We further look at the partial products defined by:

$$\mathcal{L}_k = \prod_{i=1}^{k} \mathcal{P}_{m_i,M}.$$

We also define

$$Y_k(p) = \{(h_*) \in \mathcal{L}_k \mid (h_*) \text{ is a nonregular sequence at the point } p\},$$

emphasising the choice of fixing the point $p$ as our origin of affine coordinates. We will now consider $k = 1, 2, \ldots, M-2$ and denote

$$Y(p) = \bigcup_{k=1}^{M-2} Y_k(p),$$

the set of sequences which are not regular at some stage. Clearly, it is sufficient to check that the codimension of $Y_k$ in $\mathcal{L}_k$ is at least $\lambda(M)+M$. Now we outline the two methods of estimating the codimension of the bad set, with the most important cases considered explicitly.

**Method 1.** We will use this method to get estimates for all cases but the one when the regularity fails at the last stage, this method is given in [16].

**Case 1.** For a start, let us consider the trivial case $k = 1$. Here

$$Y_1(p) = \{h_1 \equiv 0 \in \mathcal{P}_{2,M}\},$$

so that

$$\operatorname{codim}(Y_1(x) \subset \mathcal{L}_1) = \dim \mathcal{P}_{2,M} = \binom{M+1}{2}.$$

**Case 2.** Now assume that $k = 2$. This is the first non-trivial case and all the following cases follow this method. We have that

$$Y_2(p) = \{(h_1, h_2) \in \mathcal{P}_{2,M} \times \mathcal{P}_{2,M} \mid \operatorname{codim}\{h_1 = h_2 = 0\} < 2\}.$$

Now we have $Q = \{h_1 = 0\} = \bigcup Q_i \subset \mathbb{P}^{M-1}$, the decomposition into its irreducible components and we assume that $h_1 \not\equiv 0$. Pick a general point $r \in \mathbb{P}^{M-1}$ not on $Q_i$ and consider the projection from this point to get the map $\pi\colon \mathbb{P}^{M-1} \dashrightarrow \mathbb{P}^{M-2}$, so that restricting this projection onto each $Q_i$ we get a finite map $\pi_{Q_i}$, see the figure 1 below. Now take some $g \in H^0(\mathbb{P}^{M-2}, \mathcal{O}_{\mathbb{P}^{M-2}}(2))$

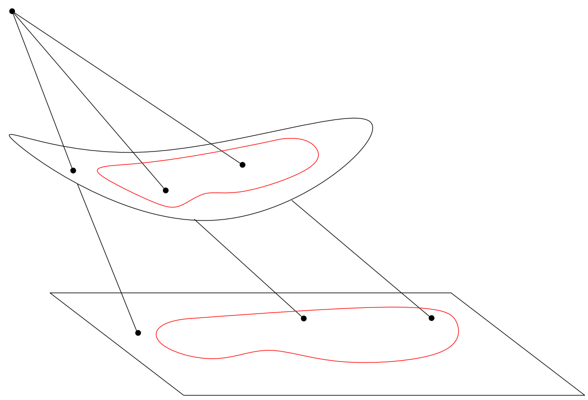

FIGURE 1

and look at $\pi^*_{Q_i}(g)$: as the map is finite, we get that $\pi^*_{Q_i}$ is injective. Therefore, for the closed subset

$$W_2 = \pi^* H^0(\mathbb{P}^{M-2}, \mathcal{O}_{\mathbb{P}^{M-2}}(2)) \subset \mathcal{P}_{2,M-1}$$

we have $W_2 \cap Y_2(x) = \{0\}$. Now we know $\dim W_2 = \binom{M}{2}$ so that $\operatorname{codim} Y_2(x) \geq \binom{M}{2}$. Therefore in the case $k = 2$ we obtain the estimate

$$\operatorname{codim}(Y_2(p) \subset \mathcal{L}_2) \geq \binom{M}{2}.$$

**The remaining cases.** We follow this method for the other values of $k = 3, \ldots, M-3$; we deal with the case $k = M-2$ separately (and by means of a different technique) later. Using this method we obtain for $k \geq 2$ ($k = 1$ is a special case) the inequality

$$\operatorname{codim}(Y_k(p) \subset \mathcal{L}_k) \geq \binom{\alpha_k}{\beta_k},$$

where the values of $\alpha_k$ and $\beta_k$ are listed in the following table ($k$ is changing from 1 to $k = M-3$:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha_k:$ | $M+1,$ | $M,$ | $M,$ | $M-1,$ | $M-1,$ | $\cdots$ | $d_2,$ | $d_2,$ | $d_2,$ | $\cdots$ | $d_2;$ |
| $\beta_k:$ | $2,$ | $2,$ | $3,$ | $3,$ | $4,$ | $\cdots$ | $d_1,$ | $d_1+1,$ | $d_1+2,$ | $\cdots$ | $d_2-3.$ |

If $d_1 = 2$, then the smallest estimate is given by $\binom{M}{2}$, so we assume $d_1 \geq 3$ and the smallest estimate is given by $\binom{d_2}{3}$. Now as $d_2 \geq \frac{M}{2} + 1$ we get

$$\binom{d_2}{3} \geq \frac{M(M+2)(M-2)}{48},$$

which is better than what we need.

**Method 2.** It remains to consider the case $k = M-2$. The previous projection method outlined above in this case does not produce the estimate we need and so we use a different method that was developed in [19]. We fix $Y^* = Y_{M-2}(p)$. Note that for any $(h_*) \in Y^*$ the sequence $h_1, \ldots, h_{M-3}$ is regular.

If a sequence $(h_*)$ belongs to $Y^*$ this means there exists an irreducible component $B \subseteq Z(h_1, \ldots, h_{M-3})$ which is a surface with $h_{M-2}|_B \equiv 0$, where $Z(h_1, \ldots, h_{M-3}) \subset \mathbb{P}^{M-1}$ is the set of common zeros of these polynomials restricted to the projectivized tangent space.

We look at the linear span $\langle B \rangle$ of $B$ and consider all possible values of:

$$b = \operatorname{codim}(\langle B \rangle \subset \mathbb{P}^{M-1}).$$

Now we split $Y^*$ up into the union

$$Y^* = \bigcup_{b=0}^{M-2} Y^*(b),$$

where $Y^*(b)$ is the set of $(M-3)$-uples $(h_*) \in Y^*$ such that for some irreducible curve $B \subseteq Z(h_1, \ldots, h_{M-3})$ such that $\operatorname{codim}\langle B \rangle = b$, the polynomial $h_{M-2}$ vanishes on $B$.

To begin with, let us consider the case $b = 0$. This means that $\langle B \rangle = \mathbb{P}^{M-1}$. Notice that non-zero linear forms in $z_1, \ldots z_M$, the coordinates on $\mathbb{P}^{M-1}$, do not vanish on $B$. As $h_{M-2}$ has degree $d_2 - 2$ or $d_2 - 1$, we consider the worst case with the smaller degree, that is, the space:

$$W = \left\{ \prod_{i=1}^{d_2-2} (a_{i,1}z_1 + \cdots + a_{1,M}z_M) \right\} \subset \mathcal{P}_{d_2-1,M-1}.$$

$W$ is a closed set with $\dim W = (M-1)(d_2-2)+1$; as $d_2 \geq \frac{M}{2}+1$ we have $\dim W \geq \frac{(M-1)(M-2)}{2}+1$. As $Y^*(0) \cap W = \{0\}$, we have

$$\operatorname{codim} Y^*(0) \geq \frac{(M-2)(M-1)}{2} + 1.$$

Now let us deal with the case $1 \leq b < M-3$. We use the technique of good sequences and associated subvarieties, developed and described in detail in [19].

Let us fix some linear subspace $P \subset \mathbb{P}^{M-1}$ of codimension $b$. Let $Y^*(P)$ be the set of all $(M-2)$-uples $(h_*) \in Y^*(b)$ such that the closed subset $Z(h_1, \ldots, h_{M-3})$ contains an irreducible component $B$ such that $\langle B \rangle = P$ and $h_{M-2}|_B \equiv 0$.

Although our intuition may suggest that we could choose a subset

$$\{h_{i_1}, \ldots, h_{i_{M-3-b}}\}$$

of $(M-3-b)$ distinct polynomials in the set $\{h_1, \ldots, h_{M-3}\}$, such that $B$ is an irreducible component of the zero set

$$\{h_{i_1}|_P = \cdots = h_{i_{M-3-b}}|_P = 0\},$$

this is in general not true (see a simple example in [19]). Instead, we have to choose a *good sequence* $h_{i_1}, \ldots, h_{i_{M-3-b}}$ that admits a sequence of irreducible subvarieties $R_0, R_1, \ldots, R_{M-3-b}$ in $P$ such that:

- $R_0 = P$ and $\operatorname{codim}(R_j \subset P) = j$,
- $h_{i_a}|_{R_{a-1}} \not\equiv 0$ and $R_a$ is an irreducible component of the closed set $h_{i_a}|_{R_{a-1}} = 0$,
- $R_{M-3-b} = B$.

In this case we say that $B$ is an associated subvariety of the good sequence $h_{i_1}, \ldots, h_{i_{M-3-b}}$.

We know [19] that good sequences form an open set in the space of tuples of polynomials and that the number of associated subvarieties is bounded from above by a constant, depending on their degrees. Therefore, we may assume that some $(M-3-b)$ polynomials from the set $(h_1|_P, \ldots, h_{M-3}|_P)$ form a good sequence and $B$ is one of its associated subvarieties. The worst estimate corresponds to the case when the polynomials

$$h_{b+1}|_P, \ldots, h_{M-3}|_P$$

of the *highest* possible degrees form a good sequence and $B$ is one of its associated subvarieties, and we will assume that this is the case.

So we fix the polynomials $h_{b+1}, \dots, h_{M-3}$ and estimate the number of independent conditions imposed on the polynomials $h_1, \dots, h_b, h_{M-2}$ by the requirement that they vanish on $B$, arguing as in the case $b = 0$. Subtracting the dimension of the Grassmannian of linear subspaces of codimension $b$ in $\mathbb{P}^{M-1}$, we get the estimate

$$\operatorname{codim}(Y^*(b) \subset \mathcal{L}) \geq (M-1-b) \cdot \left( \sum_{j=1}^{b} \deg h_j + \deg h_{M-2} - b \right) + 1.$$

Denote the right hand side of this inequality by $\theta_b$.

**Proposition 3.5.** *The following inequality*

$$\theta_b \geq \frac{(M-2)(M-1)}{2} + 1 \tag{7}$$

*holds for all* $b = 1, 2, \dots, M-4$.

*Proof.* It is easy to check that

$$\gamma_b = \theta_{b+1} - \theta_b = (M-2-b)(\deg h_{b+1} - 1) - \left( \sum_{j=1}^{b} \deg h_j - b + \deg h_{M-2} \right),$$

and since for $b \geq 2(d_1 - 1)$ we have $\deg h_{b+1} = \deg h_b + 1$, for these values of $b$ the equality

$$\gamma_b = \gamma_{b-1} + (M-2-b) - 2(\deg h_b - 1)$$

holds. From this equality we can see that the sequence $\theta_b$, where $b = 2(d_1 - 1), 2d_1 - 1, \dots, M-4$, has one of the following three types of behaviour:

- either it is non-decreasing,
- or it is first increasing for $b = 2d_1 - 2, \dots, a$, and then decreasing,
- or it is decreasing.

Below it is checked that $\theta_{M-4}$ satisfies the inequality (7). Therefore, in order to show (7) for $b = 2(d_1 - 1), \dots, M-4$, we only need to show this inequality for $b = 2(d_1 - 1)$, which is a part of the computation that we start now.

Assume that $b = 2l$, where $l = 1, \dots, d_1 - 1$. Here

$$\theta_b = (M-1-b) \cdot \left( 2 \sum_{j=1}^{l} (j+1) + \deg h_{M-2} - b \right) + 1 = \omega_1(l),$$

where

$$\omega_1(t) = (M-1-2t)(t^2 + t + d_2 - 2) + 1.$$

It is easy to check that $\omega_1'(t) \geq 0$ for $1 \leq t \leq t_1$ for some $t_1 > 1$, and $\omega_1'(t) < 0$ for $t > t_1$, so that the function of real argument $\omega_1(t)$ is first increasing (on the interval $[1, t_1]$) and then decreasing (on $[t_1, \infty)$). It follows that

$$\min\{\theta_{2l} \mid l = 1, \dots, d_1 - 1\} = \min\{\theta_2, \theta_{2(d_1-1)}\}.$$

Now $\theta_2 = \omega_1(1) = (M-3)d_2 + 1 \geq \frac{1}{2}(M+2)(M-3) + 1$, which satisfies (7).

Let us consider the second option: for $t = d_1 - 1$ we get

$$\omega_1(d_1 - 1) = (M - 2d_1 + 1)(d_1^2 - 2d_1 + M) + 1.$$

As $2d_1 - 2 \leq M - 4$, we get the bound $d_1 \leq \frac{M}{2} - 1$. Looking at the derivative of the function

$$\omega_2(t) = (M - 2t + 1)(t^2 - 2t + M) + 1,$$

we conclude that its minimum on the interval $[2, \frac{M}{2} - 1]$ is attained at one of the endpoints, so is equal to the minimum of the two numbers:

$$M(M-3) + 1 \quad \text{and} \quad \frac{3}{4}(M^2 - 4M + 12) + 1.$$

Clearly, both satisfy the inequality (7).

In order to complete the proof of our proposition, it remains to consider the case $B = 2l + 1$, where $l = 0, \ldots, d_1 - 2$. Here $\theta_b = \omega_3(l)$, where

$$\omega_3(t) = (M - 2 - 2t)(t^2 + 2t + d_2 - 1) + 1.$$

For $d_1 \geq 3$ it is easy to check that the function $\omega_3(t)$ behaves similarly to $\omega_1(t)$, first increasing and then decreasing, so that it is sufficient ti show that $\omega_3(0)$ and $\omega_3(d_1 - 2)$ satisfy the estimate (7). Indeed,

$$\omega_3(0) = (M - 2)(d_2 - 1) + 1$$

satisfies (7) as $d_2 \geq \frac{M}{2} + 1$ and for $t = d_1 - 2$ we get $\omega_3(d_1 - 2) = \omega_4(d_1)$, where

$$\omega_4(t) = (M - 2t + 2)(t^2 - 3t + M - 1)$$

and easy computations show that (7) is satisfied here as well.

Finally, in the case $d_1 = 2$ we get the number

$$\omega_3(0) = (M - 2)(M - 1) + 1.$$

Now the only case to consider is $b = M - 4$. Here we get

$$\operatorname{codim}(Y^*(b) \subset \mathcal{L}) \geq \frac{3}{4}(M^2 - 4M + 6) + 1.$$

Proof of Proposition 3.5 is complete. □

In order to complete the proof of Proposition 3.4, we have to consider the only remaining case $b = M - 3$. Here $\langle B \rangle = \mathbb{P}^2$, which clearly implies $B \subset \mathbb{P}^{M-1}$ itself is a plane. We do an easy dimension count, for a polynomial $h$ to satisfy $h|_B \equiv 0$ with $\deg h = e$ we get a closed algebraic set of polynomials of codimension $\binom{e+2}{2}$ in $\mathcal{P}_{e,M}$. Therefore

$$\operatorname{codim}(Y^*(M-3) \subset \mathcal{L}) \geq \sum_{i=1}^{M-2} \binom{m_i + 2}{2} - 3(M - 3).$$

The sum takes the minimum value when $d_1 = d_2$ and then we have the estimate

$$\operatorname{codim}(Y^*(M-3) \subset \mathcal{L}) \geq \frac{M(M+4)(M+2)}{24} - 3M + 1.$$

Combining the results of both methods and simple calculation gives the estimate

$$\operatorname{codim}(Y(p) \subset \mathcal{L}) \geq \frac{(M-5)(M-6)}{2} + 1.$$

Now Proposition 3.4 follows from a standard dimension count argument. □

*Remark* 3.3. This is clearly not the tightest bound possible; however, in Proposition 3.8 we have a weaker estimate.

**3.3. Regularity conditions for singular points.** Recall that a point is a quadratic singularity if $q_{1,1}$ and $q_{2,1}$ are proportional and at least one of the terms is non-zero. We say a point is a biquadratic singularity is $q_{1,1} = q_{2,1} = 0$. The regularity conditions (R2.2) and (R3.2) for both of these cases are similar to the smooth case (R1). The arguments used for smooth points (R1) follow in a similar way for the two cases (R2.2) and (R3.2). For quadratic points we work in $\mathbb{P}^M$ and for biquadratic points we work in $\mathbb{P}^{M+1}$, instead of $\mathbb{P}^{M-1}$ and calculations are almost identical. We obtain larger estimates for the codimension of non-regular sequences given below.

**Proposition 3.6.** *The codimension of $\mathcal{P} \setminus \mathcal{P}_*$ in $\mathcal{P}$ is at least*

$$\lambda(M) = \frac{(M-5)(M-6)}{2} - (M+1)$$

*for* $* =$ (R2.2) *and* (R3.2).

*Proof.* We will outline the proof for the quadratic case (R2.2) and the biquadratic case is treated in the same way. Instead of restricting to the tangent space we restrict to the Zariski tangent space $\{q_{i,1} = 0\}$ (for $q_{i,1}$ that is non-zero: the other linear form is proportional to it) and work in $\mathbb{P}^M$. We now have one extra polynomial to get our standard ordering to be given by $h_1, \ldots, h_{M-1}$ and our polynomials now belong to $\mathcal{P}_{m_i,M+1}$. For the method 1, case 1 we get the estimate:

$$\operatorname{codim}(Y_1(x) \subset \mathcal{L}_1) = \dim \mathcal{P}_{2,M+1} = \binom{M+2}{2}.$$

The remaining cases follow in the same way with the table given now

$$\begin{array}{rrrrrcrrrcr}
\alpha_k: & M+2, & M+1, & M+1, & M, & \cdots & d_2+1, & d_2+1, & d_2+1, & \cdots & d_2+1; \\
\beta_k: & 2, & 2, & 3, & 3, & \cdots & d_1, & d_1+1, & d_1+2, & \cdots & d_2-2.
\end{array}$$

Note that we get an extra term as we have an extra polynomial $h_{M-1}$. Again if $d_1 = 2$, then the minimum is given by $\binom{M+1}{2}$ and if $d_1 \geq 3$, then the minimum is given by $\binom{d_2+1}{3}$. Now when using the method 2 for the last case $k = M-1$, we first get codim $Y^*(0) \geq \frac{1}{2}M^2 + 1$, so that in the notations of the proof of Proposition 3.5 we have possible values $b = 1, \ldots, M-2$. For $b < M-2$ we consider good sequences and get that:

$$\operatorname{codim}(Y^*(b) \subset \mathcal{L}) \geq (M-b) \cdot \left( \sum_{j=1}^{b} \deg h_j + \deg h_{M-1} - b \right) + 1.$$

It follows easily that

$$\operatorname{codim}(Y^*(b) \subset \mathcal{L}) \geq (M-1-b) \cdot \left( \sum_{j=1}^{b} \deg h_j + \deg h_{M-2} - b \right) + 1$$

for $b = 1, \dots, M-3$. For $b = M-2$ we now get

$$\operatorname{codim}(Y^*(M-2) \subset \mathcal{L}) \geq \sum_{i=1}^{M-1} \binom{m_i+2}{2} - 3(M-2),$$

and again see the estimate in the case (R1) works here also. □

We are left with the remaining two cases to consider now, that is, (R2.1) and (R3.1).

**Proposition 3.7.** *The codimension of the set of complete intersections with quadratic singularities of rank at most 8, that is, the set $\mathcal{P} \setminus \mathcal{P}_{(R2.1)}$ in $\mathcal{P}$ is at least $\binom{M-5}{2} + 1$.*

*Proof.* Without loss of generality assume $q_{1,1} \neq 0$ and $q_{2,1} = \lambda q_{1,1}$ with $\lambda \in \mathbb{C}$. The rank of the quadratic point is then given by the rank of the quadratic form $(q_{2,2} - \lambda q_{1,2})|_{\{q_{1,1}=0\}}$. The result is due now to well know results on the codimension of quadrics of rank at most $k$ (here $k = 8$), see, for instance, [6], where a similar computation has been done for Fano hypersurfaces. □

**Proposition 3.8.** *The codimension of the set violating the condition* (R3.1)*, that is the set $\mathcal{P} \setminus \mathcal{P}_{(R3.1)}$ in $\mathcal{P}$ is at least $\binom{M-9}{2} - 1$.*

*Proof.* Here we work with the space

$$\mathcal{Q} = \mathcal{P}_{2,M+2} \times \mathcal{P}_{2,M+2}$$

of pairs of quadratic forms on $\mathbb{P}^{M+1}$ (the latter projective space interpreted as the exceptional divisor of the blow up of a point $o \in \mathbb{P}^{M+2}$). Let $(g_1, g_2) \in \mathcal{Q}$ be a pair of forms. The codimension of the closed set of quadratic forms of rank less than 5 is $\frac{(M-4)(M-3)}{2}$, so removing a closed set of that codimension we may assume that $\operatorname{rk} g_1 \geq 5$. This means that the quadric $G_1 = \{g_1 = 0\}$ is factorial, $\operatorname{Pic} G_1 = \operatorname{Cl} G_1 = \mathbb{Z} H_{G_1}$, where $H_{G_1}$ is the class of a hyperplane section. Now for $g_2|_{G_1}$ to be non-reduced or reducible it has to split up into hyperplane sections which gives dimension $2M + 4$. This has codimension $\frac{(M+2)(M-1)}{2}$ in $\mathcal{P}_{2,M-2}$. Therefore, removing a closed set of codimension $\frac{(M-4)(M-3)}{2}$, we obtain a set $\mathcal{Q}^* \subset \mathcal{Q}$ of pairs $(g_1, g_2)$ such that the closed set $\{g_1 = g_2 = 0\}$ is an irreducible and reduced complete intersection of codimension 2.

Let us consider the singular set of such a complete intersection, which we denote by $\mathrm{Sing}(g_1, g_2)$. Note that $\mathrm{Sing}(g_1, g_2)$ is the set of the points $p \in \{g_1 = g_2 = 0\}$ where the Jacobian matrix of $g_1$ and $g_2$ has linearly dependent rows, that is, there exists some $[\lambda_1 : \lambda_2] \in \mathbb{P}^1$ with $p \in \mathrm{Sing}\{\lambda_1 g_1 + \lambda_2 g_2\}$ (where the symbol $\mathrm{Sing}(g)$ denotes the singular locus of the hypersurface $\{g = 0\}$). Therefore,

$$\mathrm{Sing}(g_1, g_2) \subset \bigcup_{[\lambda_1:\lambda_2]\in\mathbb{P}^1} \mathrm{Sing}\{\lambda_1 g_1 + \lambda_2 g_2\},$$

so that if

$$\operatorname{codim}(\mathrm{Sing}(g_1, g_2) \subset \{g_1 = g_2 = 0\}) \leq k, \tag{8}$$

then the line joining $g_1$ and $g_2$ in $\mathcal{P}_{2,M+2}$ meets the closed set of quadratic forms of rank at most $(k + 2)$. We conclude that the set of pairs $(g_1, g_2) \in \mathcal{Q}^*$ satisfying the inequality (8), has codimension at least

$$\frac{(M-k+1)(M-k)}{2} - 1$$

in $\mathcal{Q}$. Putting $k = 10$ (and comparing the result with the codimension of the complement $\mathcal{Q} \setminus \mathcal{Q}^*$ obtained at the previous step), we complete the proof. □

Now the last thing to do is to compare the codimensions of the bad sets for all regularity conditions and to find the minimum.

Proof of Theorem 0.2 is now complete.

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