Bull. Korean Math. Soc. **54** (2017), No. 5, pp. 1655–1667
https://doi.org/10.4134/BKMS.b160684
pISSN: 1015-8634 / eISSN: 2234-3016

CYLINDERS IN DEL PEZZO SURFACES WITH DU VAL SINGULARITIES

Grigory Belousov

Abstract. We consider del Pezzo surfaces with du Val singularities. We'll prove that a del Pezzo surface X with du Val singularities has a $-K_X$-polar cylinder if and only if there exist tiger such that the support of this tiger does not contain anti-canonical divisor. Also we classify all del Pezzo surfaces X such that X has not any cylinders.

1. Introduction

A *log del Pezzo surface* is a projective algebraic surface X with only quotient singularities and ample anti-canonical divisor $-K_X$. In this paper we assume that X has only du Val singularities and we work over complex number field $\mathbb{C}$. Note that a del Pezzo surface with only du Val singularities is rational.

Definition 1.1. Let X be a proper normal variety. Let D be an effective $\mathbb{Q}$-divisor on X such that $D \equiv -K_X$ and the pair (X, D) is not log canonical. Such divisor D is called *non-log canonical special tiger* (see [4]).

Remark 1.2. In this paper, a non-log canonical special tiger we will call a tiger.

Definition 1.3 (see. [5])**.** Let M be a $\mathbb{Q}$-divisor on a projective normal variety X. An *M-polar cylinder* in X is an open subset $U = X \setminus \operatorname{Supp}(D)$ defined by an effective $\mathbb{Q}$-divisor D in the $\mathbb{Q}$-linear equivalence class of M such that $U \cong Z \times \mathbb{A}^1$ for some affine variety Z.

In this paper, we consider del Pezzo surfaces with du Val singularities over complex number field $\mathbb{C}$. Our interest is a connection between existence of a $-K_X$-polar cylinder in the del Pezzo surface and tigers on this surface.

The existence of a H-polar cylinder in X is important due to the following fact.

Theorem 1.4 (see [6], Corollary 3.2)**.** *Let Y be a normal algebraic variety over $\mathbb{C}$ projective over an affine variety S with $\dim_S Y \geq 1$. Let $H \in Div(Y)$ be an ample divisor on Y, and let $V = \operatorname{Spec} A(Y, H)$ be the associated affine*

Received August 17, 2016; Revised November 14, 2016; Accepted December 26, 2016.
2010 *Mathematics Subject Classification.* 14E07, 14E25, 14J45.
Key words and phrases. cylinder, del Pezzo surface.

quasicone over Y. Then V admits an effective G_a-action if and only if Y contains an H-polar cylinder.

There exists a classification of del Pezzo surfaces X such that X has a $-K_X$-polar cylinder (see [1], [2]). Also, in the papers [1], [2] the authors have proved that if a del Pezzo surface X has not $-K_X$-polar cylinder, then all tigers contain a support at least one element of $|-K_X|$. Now we prove the inverse statement.

The main result of Section 3 is the followings.

Theorem 1.5. *Let X be a del Pezzo surface with du Val singularities. Then X has a $-K_X$-polar cylinder if and only if there exist a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.*

The main result of Section 4 is the followings.

Theorem 1.6. *Let X be a del Pezzo surface with du Val singularities. Then*

- *X has not cylinders if $\rho(X) = 1$ and X has one of the followings collections of singularities*: $4A_2, 2A_1 + 2A_3, 2D_4$;
- *In the rest cases there exist an ample divisor H such that X has a H-polarization.*

The author is grateful to professor I. A. Cheltsov for suggesting me this problem and for his help.

2. Preliminary results

We work over complex number field $\mathbb{C}$. We employ the following notation:

- $(-n)$-curve is a smooth rational curve with self intersection number $-n$.
- K_X: the canonical divisor on X.
- $\rho(X)$: the Picard number of X.

Theorem 2.7 (Riemann–Roch, see, for example, [3], Theorem 1.6, Ch. 5)**.** *Let D be a divisor on the surface X. Then*

$$\chi(D) = \frac{1}{2}D(D - K_X) + \chi(\mathcal{O}_X).$$

Theorem 2.8 (Kawamata–Viehweg Vanishing Theorem, see, for example, [7], Theorem 5-2-3)**.** *Let X be a non-singular projective variety, A an ample $\mathbb{Q}$-divisor such that the fractional part $\lceil A\rceil - A$ has the support with only normal crossings. Then*

$$H^p(X, K_X + \lceil A\rceil) = 0, \quad p > 0.$$

Let X be a del Pezzo surface with du Val singularities. Let d be the degree of X, i.e., $d = K_X^2$.

Theorem 2.9 (see [1], Theorem 1.5)**.** *Let X be a del Pezzo surface of degree d with at most du Val singularities.*

I. *The surface X does not admit a $-K_X$-polar cylinder when*

(1) $d = 1$ *and* X *allows only singular points of types* A_1, A_2, A_3, D_4 *if any*;
(2) $d = 2$ *and* X *allows only singular points of types* A_1 *if any*;
(3) $d = 3$ *and* X *allows no singular point.*

II. *The surface* X *has a* $-K_X$*-polar cylinder if it is not one of the del Pezzo surfaces listed in I.*

3. The proof of Theorem 1.5

In the papers [1] and [2] authors have classified del Pezzo surfaces X such that X has a $-K_X$-polar cylinder. Moreover, they prove that if a del Pezzo surface X has not a $-K_X$-polar cylinder, then every tiger on X contains an element of $|-K_X|$. So, we need prove that if a del Pezzo surface X has a $-K_X$-polar cylinder, then there exist a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.

Lemma 3.10. *Let* X *be a del Pezzo surface with du Val singularities and let* d *be the degree of* X*. Assume that* $d \geq 7$*. Then* X *has a* $-K_X$*-polar cylinder and there exist a tiger such that the support of this tiger does not contain any elements of* $|-K_X|$.

Proof. By Theorem 2.9, we see that X has a $-K_X$-polar cylinder. Now, we construct a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. Consider $|-2K_X|$. By Theorem 2.7 and Theorem 2.8, $\dim|-2K_X| = \frac{-2K_X\cdot(-2K_X-K_X)}{2} = 3d$. Let P be an arbitrary smooth point on X. Consider a set Ω of elements $L \in |-2K_X|$ such that $\operatorname{mult}_P L \geq 5$. Then Ω is a linear subsystem of the linear system $|-2K_X|$. Note that $\dim|\Omega| = 3d - 15 \geq 6$ for $d \geq 7$. Hence, Ω is not empty. Let $N \in \Omega$ be a general element of the linear system Ω.

Note that N does not contain a support of anti-canonical divisor. Indeed, assume that there exist an element $M_1 \in |-K_X|$ such that $\operatorname{Supp} M_1 \subseteq \operatorname{Supp} N$. Then $N = M_1 + M_2$, where $M_2 \in |-K_X|$. We see that $\dim|-K_X| = \frac{-K_X\cdot(-K_X-K_X)}{2} = d$. Therefore, $\operatorname{mult}_P M_1 \leq 3$ and $\operatorname{mult}_P M_2 \leq 3$. Hence, we may assume that $\operatorname{mult}_P M_1 = 2$, $\operatorname{mult}_P M_2 = 3$. Let $\tilde{M}_1$ be the linear subsystem of $|-K_X|$ such that $\tilde{M}_1$ consist of elements with multiply two in the point P. Let $\tilde{M}_2$ be the linear subsystem of $|-K_X|$ such that $\tilde{M}_2$ consist of elements with multiply three in the point P. Then

$$\dim|\tilde{M}_1 + \tilde{M}_2| = \dim|\tilde{M}_1| + \dim|\tilde{M}_2| = (d-3) + (d-6) = 2d - 9.$$

Note that $3d - 15 > 2d - 9$ for $d \geq 7$. Hence, a general element N of the linear system Ω does not contain a support of anti-canonical divisor. Then $\frac{1}{2}N$ is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. □

Lemma 3.11. *Let* X *be a del Pezzo surface with du Val singularities and let* d *be the degree of* X*. Assume that* $d = 4, 6$*. Then* X *has a* $-K_X$*-polar cylinder* *and there exists a tiger such that the support of this tiger does not contain any elements of* $|-K_X|$.

Proof. By Theorem 2.9, we see that X has a $-K_X$-polar cylinder. Now, we construct a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. Let $f : \bar{X} \to X$ be the minimal resolution. Let E be a (-1)-curve on $\bar{X}$ and $E' = f(E)$. Put $-3K_{\bar{X}} \backsim 2E + F$. Then $-3K_{\bar{X}} \cdot E = 2E^2 + F \cdot E$. Since $K_{\bar{X}} \cdot E = -1$ and $E^2 = -1$, we see that $F \cdot E = 5$. We have $-3K_{\bar{X}}^2 = 2E \cdot K_{\bar{X}} + F \cdot K_{\bar{X}}$. Since $K_{\bar{X}} \cdot E = -1$ and $K_{\bar{X}}^2 = d$, we see that $F \cdot K_{\bar{X}} = -(3d-2)$. We obtain $-3K_{\bar{X}} \cdot F = 2E \cdot F + F^2$. Since $F \cdot E = 5$ and $F \cdot K_{\bar{X}} = -(3d-2)$ we see that $F^2 = 9d - 16$. Hence, by Theorem 2.7 and Theorem 2.8, $\dim |F| = 6d - 9$. Let P' be a general smooth point on E' and $P' = f(P)$. Consider a set Ω of elements $L \in |F|$ such that $\operatorname{mult}_P L \geq 5$. Note that $\dim |\Omega| = 6d - 9 - 15 = 6d - 24 \geq 0$ for $d \geq 4$, i.e., Ω is non-empty. We see that Ω contains an element N such that $N + E$ does not contain a support of anti-canonical divisor. Indeed, assume that for all $N \in \Omega$ there exist $M_1 \in |-K_{\bar{X}}|$ such that $\operatorname{Supp} M_1 \subseteq \operatorname{Supp}(N + E)$. Then $N + 2E = M_1 + M_2$, where $M_2 \in |-2K_{\bar{X}}|$. We have the followings three cases.

Case 1. $M_1 = 2E + F_1$, M_2 does not contain the curve E. Hence, $F_1 \cdot E = 3$, $F_1 \cdot K_{\bar{X}} = -(d-2)$, $F_1^2 = d - 8 \leq -2$, a contradiction.

Case 2. $M_1 = E + F_1$, $M_2 = E + F_2$. Then $F_1 \cdot E = 2$, $F_1 \cdot K_{\bar{X}} = -(d-1)$, $F_1^2 = d-3$, $F_2 \cdot E = 3$, $F_2 \cdot K_{\bar{X}} = -(2d-1)$, $F_2^2 = 4d-5$. Hence, $\dim |F_1| = d-2$, $\dim |F_2| = 3d - 3$. Note that the multiplicities F_1 and F_2 in the point P are equaled 2 and 3 correspondingly. Let $\tilde{F_1}$ be the linear subsystem of $|F_1|$ such that the multiplicity of elements of $\tilde{F_1}$ is equaled two in the point P, let $\tilde{F_2}$ be the linear subsystem of $|F_2|$ such that the multiplicity of elements of $\tilde{F_2}$ is equaled three in the point P. Then $\dim |\tilde{F_1}| = d - 5$. Hence, $d = 6$. Note that

$$\dim |\tilde{F_1} + \tilde{F_2}| = \dim |\tilde{F_1}| + \dim |\tilde{F_2}| = (d-5) + (3d-9) = 4d - 14 = 10.$$

On the other hand, $\dim |\Omega| = 6d - 24 = 12 > 10$. Therefore, a general element $N \in \Omega$ does not contain $\operatorname{Supp}(-K_{\bar{X}}) \setminus \operatorname{Supp}(E)$.

Case 3. $M_2 = 2E + F_2$, M_1 does not contain the curve E. Then $F_2 \cdot E = 4$, $F_2 \cdot K_{\bar{X}} = -(2d-2)$, $F_2^2 = 4d - 12$. Hence, $\dim |F_2| = 3d - 7$, $\dim |M_1| = d$. Note that the multiplicities M_1 and F_2 in the point P are equal to 1 and 4 correspondingly. Let $\tilde{M_1}$ be the set of elements of the linear system $|-K_{\bar{X}}|$ that pass through the point P, let $\tilde{F_2}$ be the set of elements of the linear system $|F_2|$ that have multiplicity four in the point P. Note that $\tilde{F_1}$ and $\tilde{M_2}$ are the linear system. Then $\dim |\tilde{F_2}| = 3d - 17$. Hence, $d = 6$. Note that

$$\dim |\tilde{M_1} + \tilde{F_2}| = \dim |\tilde{M_1}| + \dim |\tilde{F_2}| = (d-1) + (3d-17) = 4d - 18 = 6.$$

On the other hand, $\dim |\Omega| = 6d - 24 = 12 > 6$. Therefore, a general element $N \in \Omega$ does not contain any elements of $|-K_{\bar{X}} - E|$.

So, a general element $N \in \Omega$ does not contain any elements of $|-K_{\bar{X}} - E|$. Denote this element by N. Note that $\operatorname{mult}_P(2E + N) \geq 7$. Then $\frac{1}{3}f(N) + \frac{2}{3}E'$

is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. ☐

Lemma 3.12. *Let X be a del Pezzo surface with du Val singularities and let d be the degree of X. Assume that $d = 5$. Then X has a $-K_X$-polar cylinder and there exists a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.*

Proof. By Theorem 2.9, we see that X has a $-K_X$-polar cylinder. Now, we construct a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. Consider $|-4K_X|$. By Theorem 2.7 and Theorem 2.8, we see that $\dim|-4K_X| = 50$. Let P be an arbitrary smooth point on X. Consider a set Ω of elements $L \in |-4K_X|$ such that $\mathrm{mult}_P L \geq 9$. Then Ω is the linear subsystem of the linear system of $|-4K_X|$. Note that $\dim|\Omega| = 50 - 45 = 5$. Hence, Ω is non-empty. Let $N \in \Omega$ be a general element of the linear system Ω. We see that N does not contain a support of anti-canonical divisor. Indeed, assume that there exists an element $M_1 \in |-K_X|$ such that $\mathrm{Supp}\, M_1 \subseteq \mathrm{Supp}\, N$. Then $N = M_1 + M_2$, where $M_2 \in |-3K_X|$. Note that $\dim|-K_X| = 5$, $\dim|-3K_X| = 30$. Put $d_1 = \mathrm{mult}_P M_1$ and $d_2 = \mathrm{mult}_P M_2$. Since

$$\dim|-K_X| - \frac{d_1 \cdot (d_1+1)}{2} = 5 - \frac{d_1 \cdot (d_1+1)}{2} \geq 0$$

and

$$\dim|-3K_X| - \frac{d_2 \cdot (d_2+1)}{2} = 30 - \frac{d_2 \cdot (d_2+1)}{2} \geq 0,$$

we see that $\mathrm{mult}_P M_1 \leq 2$ and $\mathrm{mult}_P M_1 \leq 7$. Hence, $\mathrm{mult}_P M_1 = 2$, $\mathrm{mult}_P M_2 = 7$. Let $\tilde{M}_1$ be the set of elements of the linear system $|-K_X|$ that have multiply 2 in the point P, let $\tilde{M}_2$ be the set of elements of the linear system $|-3K_X|$ that have multiply 7 in the point P. Note that $\tilde{M}_1$ and $\tilde{M}_2$ are the linear system. Then $\dim|\tilde{M}_1| = 5 - 3 = 2$ $\dim|\tilde{M}_2| = 30 - 28 = 2$. Hence,

$$\dim|\tilde{M}_1 + \tilde{M}_2| = 4 < 5 = \dim|\Omega|.$$

So, a general element N of Ω does not contain the support of anti-canonical divisor. Then $\frac{1}{4}N$ is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. ☐

Lemma 3.13. *Let X be a del Pezzo surface with du Val singularities and let d be the degree of X. Assume that $d \geq 3$ and there exists a singular point of type A_1. Then X has a $-K_X$-polar cylinder and there exists a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.*

Proof. Let X be a del Pezzo surface with du Val singularities, and let P be a singular point of type A_1. By Lemmas 3.10, 3.11 and 3.12 we may assume that $d = 3$. Let $f : \bar{X} \to X$ be the minimal resolution of singularities of X, and let $D = \sum_{i=1}^{n} D_i$ be the exceptional divisor of f, where D_i is a (-2)-curve. We may assume that $P = f(D_1)$. By Theorem 2.9, we see that X

has a $-K_X$-polar cylinder. Now, we construct a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. Consider $-4K_{\bar{X}}$. Put $-4K_{\bar{X}} \backsim 3D_1 + F$. Then $F \cdot D_1 = 6$, $F \cdot K_{\bar{X}} = -12$, $F^2 = 30$. Hence, $\dim |F| = 21$. Let Q be a point on D_1. Note that there exists an element $N \in |F|$ such that $\operatorname{mult}_Q N = 6$. Now, we prove that $N + D_1$ does not contain the support of anti-canonical divisor. Indeed, assume that for all $N \in \Omega$ there exists an element $M_1 \in |-K_{\bar{X}}|$ such that $\operatorname{Supp} M_1 \subseteq \operatorname{Supp}(N + D_1)$. Then $N + 3D_1 = M_1 + M_2$, where $M_2 \in |-3K_{\bar{X}}|$. So, we have the following four cases.

Case 1. $M_2 = 3D_1 + F_2$, M_1 does not contain the curve D_1. Then $F_2 \cdot D_1 = 6$, $F_2 \cdot K_X = -9$, $F_2^2 = 9$. Hence, $\dim |F_2| = 9$. Therefore, $\operatorname{mult}_Q F_2 \leq 3$. Since M_1 does not meet D_1, we have a contradiction.

Case 2. $M_1 = D_1 + F_1$, $M_2 = 2D_1 + F_2$. Then $F_1 \cdot D_1 = 2$, $F_1 \cdot K_X = -d$, $F_1^2 = 1$. Therefore, $\dim |F_1| = 2$. Hence, $\operatorname{mult}_Q F_2 \leq 1$, a contradiction.

Case 3. $M_1 = 2D_1 + F_1$, $M_2 = D_1 + F_2$. Then $F_1 \cdot D_1 = 4$, $F_1 \cdot K_X = -3$, $F_1^2 = -5$, a contradiction.

Case 4. $M_1 = 3D_1 + F_2$, M_2 does not contain the curve D_1. Then $F_1 \cdot D_1 = 6$, $F_1 \cdot K_X = -3$, $F_1^2 = -15$, a contradiction.

So, $\operatorname{Supp}(N + D_1)$ does not contain the support of anti-canonical divisor. Note that $\operatorname{mult}_Q(3D_1 + N) = 9$. Then $\frac{1}{4}f(N)$ is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. □

Lemma 3.14. *Let X be a del Pezzo surface with du Val singularities and let d be the degree of X. Assume that $d \geq 2$ and there exists a singular point of type A_2 or A_3. Then X has a $-K_X$-polar cylinder and there exists a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.*

Proof. As above, we may assume that $d = 2$ or $d = 3$. By Theorem 2.9, we see that X contains $-K_X$-polar cylinder. Let $f : \bar{X} \to X$ be the minimal resolution of singularities of X, and let $D = \sum_{i=1}^{n} D_i$ be the exceptional divisor of f, where D_i is a (-2)-curve. Consider two cases.

Case 1. There exists a point $P \in X$ such that P of type A_2. We may assume that D_1 and D_2 correspond to P. So, $D_1 \cdot D_2 = 1$. Let Q be the point of intersection of D_1 and D_2. Consider $-2K_{\bar{X}}$. Put $-2K_{\bar{X}} \backsim 2D_1 + 2D_2 + F$. Then $F \cdot D_1 = F \cdot D_2 = 2$, $F \cdot K_{\bar{X}} = -2d$, $F^2 = 4d - 8$. Hence, $\dim |F| = 3d - 4$. Consider the set Ω of elements $L \in |F|$ such that $Q \in L$. Then $\dim \Omega = 3d - 4 - 1 = 3d - 5$. Put $-K_{\bar{X}} \backsim D_1 + D_2 + \tilde{F}$. Then $\tilde{F} \cdot D_1 = \tilde{F} \cdot D_2 = 1$, $\tilde{F} \cdot K_{\bar{X}} = -d$, $\tilde{F}^2 = d - 2$. Hence, $|\tilde{F}| = d - 1$. Consider the set $\tilde{\Omega}$ of elements $L \in |\tilde{F}|$ such that $Q \in L$. Then $\dim \tilde{\Omega} = d - 2$. Note that $\dim \Omega = 3d - 5 > \dim \tilde{\Omega} = d - 2$. So, there exists an element N of Ω such that $f(N)$ does not contain the support of anti-canonical divisor. Note that $\operatorname{mult}_Q(2D_1 + 2D_2 + N) \geq 5$. Then $\frac{1}{2}f(N)$ is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.

Case 2. There exists a point $P \in X$ such that P of type A_3. We may assume that D_1, D_2 and D_3 correspond to P. So, $D_1 \cdot D_2 = D_2 \cdot D_3 = 1$. Let

Q be the point of intersection of D_1 and D_2. Consider $-2K_{\bar{X}}$. Put $-2K_{\bar{X}} \backsim 2D_1 + 2D_2 + D_3 + F$. Then $F \cdot D_1 = 2$, $F \cdot D_2 = 1$, $F \cdot D_3 = 0$ $F \cdot K_{\bar{X}} = -2d$, $F^2 = 4d - 6$. Hence, $\dim |F| = 3d - 3$. So, there exists an element N of $|F|$ such that $Q \in N$. Note that the support $N + 2D_1 + 2D_2 + D_3$ does not contain any elements $|-K_{\bar{X}}|$. So, $f(N)$ does not contain the support of anti-canonical divisor. Note that $\mathrm{mult}_Q(2D_1 + 2D_2 + D_3 + N) \geq 5$. Then $\frac{1}{2}f(N)$ is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. □

Lemma 3.15. *Let X be a del Pezzo surface with du Val singularities and let d be the degree of X. Assume that $d \geq 2$ and there exists a singular point of type D_4. Then X has a $-K_X$-polar cylinder and there exists a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.*

Proof. Let X be a del Pezzo surface with du Val singularities, and let P be a singular point of type D_4. By Theorem 2.9, we see that X has a $-K_X$-polar cylinder. Let $f : \bar{X} \to X$ be the minimal resolution of singularities of X, and let $D = \sum_{i=1}^{n} D_i$ be the exceptional divisor of f, where D_i is a (-2)-curve. We may assume that D_1, D_2, D_3 and D_4 correspond to P. Moreover, D_1 is the central component. Put $-3K_{\bar{X}} \backsim 4D_1 + 3D_2 + 2D_3 + 2D_4 + F$. Then $F \cdot D_1 = 1$, $F \cdot D_2 = 2$, $F \cdot D_3 = F \cdot D_4 = 0$ $F \cdot K_{\bar{X}} = -3d$, $F^2 = 9d - 10 > 0$ for $d \geq 2$. Note that $4D_1 + 3D_2 + 2D_3 + 2D_4 + F$ does not admit representation as $M_1 + M_2$, where $M_1 \in |-K_{\bar{X}}|$ and $M_2 \in |-2K_{\bar{X}}|$. Let N be an element of $|F|$. Note that the support $N + 4D_1 + 3D_2 + 2D_3 + 2D_4$ does not contain any elements $|-K_{\bar{X}}|$. So, $f(N)$ does not contain the support of anti-canonical divisor. Note that $\mathrm{mult}_Q(4D_1 + 3D_2 + 2D_3 + 2D_4 + N) \geq 7$, where Q is the intersection of D_1 and D_2. Then $\frac{1}{3}f(N)$ is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. □

Lemma 3.16. *Let X be a del Pezzo surface with du Val singularities and let d be the degree of X. Assume that there exists a singular point of type A_k, where $k = 4, 5, 6, 7, 8$. Then X has a $-K_X$-polar cylinder and there exists a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.*

Proof. Let X be a del Pezzo surface with du Val singularities, and let P be a singular point of type A_k. By Theorem 2.9, we see that X has a $-K_X$-polar cylinder. Let $f : \bar{X} \to X$ be the minimal resolution of singularities of X, and let $D = \sum_{i=1}^{n} D_i$ be the exceptional divisor of f, where D_i is a (-2)-curve. We may assume that $D_1, D_2, \ldots, D_k$ correspond to P. Moreover, $D_i \cdot D_{i+1} = 1$ for $i = 1, 2, \ldots, k-1$. Consider the following cases.

Case 1. $k = 4$. Put $-2K_{\bar{X}} \backsim D_1 + 2D_2 + 2D_3 + D_4 + F$. Let Q be the intersection of D_2 and D_3. We obtain $F \cdot D_1 = F \cdot D_4 = 0$, $F \cdot D_2 = F \cdot D_3 = 1$, $F \cdot K_{\bar{X}} = -2d$, $F^2 = 4d - 4$. Then $\dim |F| = 3d - 2$. So, there exists an element $N \in |F|$ such that N passes through Q. Note that $D_1 + 2D_2 + 2D_3 + D_4 + N$ does not admit representation as $M_1 + M_2$, where $M_1, M_2 \in |-K_{\bar{X}}|$. Hence, $f(N)$ does not contain the support of anti-canonical divisor. Note that $\mathrm{mult}_Q(D_1 + 2D_2 + 2D_3 + D_4 + N) \geq 5$. Then $\frac{1}{2}f(N)$ is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.

Case 2. $k=5$. Put $-3K_{\bar{X}} \backsim D_1+2D_2+3D_3+3D_4+2D_5+F$. Let Q be the intersection of D_3 and D_4. We obtain $F\cdot D_1 = F\cdot D_2 = 0$, $F\cdot D_3 = F\cdot D_4 = F\cdot D_5 = 1$, $F\cdot K_{\bar{X}} = -3d$, $F^2 = 9d-8$. Then $\dim|F| = 6d-4$. So, there exists an element $N \in |F|$ such that N passes through Q. Note that $D_1+2D_2+3D_3+3D_4+2D_5+N$ does not admit representation as M_1+M_2, where $M_1 \in |-K_{\bar{X}}|$ and $M_2 \in |-2K_{\bar{X}}|$. Hence, $f(N)$ does not contain the support of anti-canonical divisor. Note that $\mathrm{mult}_Q(D_1+2D_2+3D_3+3D_4+2D_5+N) \geq 7$. Then $\frac{1}{3}f(N)$ is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.

Case 3. $k=6$. Put $-3K_{\bar{X}} \backsim D_1+2D_2+3D_3+3D_4+2D_5+D_6+F$. Let Q be the intersection of D_3 and D_4. We obtain

$$F\cdot D_1 = F\cdot D_2 = F\cdot D_5 = F\cdot D_6 = 0,$$

$$F\cdot D_3 = F\cdot D_4 = 1, F\cdot K_{\bar{X}} = -3d, F^2 = 9d-6.$$

Then $\dim|F| = 6d-3$. So, there exists an element $N \in |F|$ such that N passes through Q. Note that $D_1+2D_2+3D_3+3D_4+2D_5+N$ does not admit representation as M_1+M_2, where $M_1 \in |-K_{\bar{X}}|$ and $M_2 \in |-2K_{\bar{X}}|$. Hence, $f(N)$ does not contain the support of anti-canonical divisor. Note that $\mathrm{mult}_Q(D_1+2D_2+3D_3+3D_4+2D_5+D_6+N) \geq 7$. Then $\frac{1}{3}f(N)$ is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.

Case 4. $k=7$. Put

$$-4K_{\bar{X}} \backsim D_1+2D_2+3D_3+4D_4+4D_5+3D_6+2D_7+F.$$

Let Q be the intersection of D_4 and D_5. We obtain

$$F\cdot D_1 = F\cdot D_2 = F\cdot D_3 = F\cdot D_6 = 0,$$

$$F\cdot D_4 = F\cdot D_5 = F\cdot D_7 = 1, F\cdot K_{\bar{X}} = -4d, F^2 = 16d-10.$$

Then $\dim|F| = 10d-5$. So, there exists an element $N \in |F|$ such that N passes through Q. Note that $D_1+2D_2+3D_3+4D_4+4D_5+3D_6+2D_7+N$ does not admit representation as M_1+M_2, where $M_1 \in |-K_{\bar{X}}|$ and $M_2 \in |-3K_{\bar{X}}|$. Hence, $f(N)$ does not contain the support of anti-canonical divisor. Note that

$$\mathrm{mult}_Q(D_1+2D_2+3D_3+4D_4+4D_5+3D_6+2D_7+N) \geq 9.$$

Then $\frac{1}{4}f(N)$ is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.

Case 5. $k=8$. Put

$$-4K_{\bar{X}} \backsim D_1+2D_2+3D_3+4D_4+4D_5+3D_6+2D_7+D_8+F.$$

Let Q be the intersection of D_4 and D_5. We obtain

$$F\cdot D_1 = F\cdot D_2 = F\cdot D_3 = F\cdot D_6 = F\cdot D_7 = F\cdot D_8 = 0,$$

$$F\cdot D_4 = F\cdot D_5 = 1, F\cdot K_{\bar{X}} = -4d, F^2 = 16d-8.$$

Then $\dim|F| = 10d-4$. So, there exists an element $N \in |F|$ such that N passes through Q. Note that $D_1+2D_2+3D_3+4D_4+4D_5+3D_6+2D_7+D_8+N$ does not admit representation as M_1+M_2, where $M_1 \in |-K_{\bar{X}}|$ and $M_2 \in |-3K_{\bar{X}}|$. Hence, $f(N)$ does not contain the support of anti-canonical divisor. Note that

$$\mathrm{mult}_Q(D_1 + 2D_2 + 3D_3 + 4D_4 + 4D_5 + 3D_6 + 2D_7 + D_8 + N) \geq 9.$$

Then $\frac{1}{4}f(N)$ is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. □

Lemma 3.17. *Let X be a del Pezzo surface with du Val singularities and let d be the degree of X. Assume that there exists a singular point of type D_k, where $k = 5, 6, 7, 8$. Then X has a $-K_X$-polar cylinder and there exists a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.*

Proof. Let X be a del Pezzo surface with du Val singularities, and let P be a singular point of type D_k. By Theorem 2.9, we see that X has a $-K_X$-polar cylinder. Let $f : \bar{X} \to X$ be the minimal resolution of singularities of X, and let $D = \sum_{i=1}^{n} D_i$ be the exceptional divisor of f, where D_i is a (-2)-curve. We may assume that $D_1, D_2, \ldots, D_k$ correspond to P. Moreover, D_3 is the central component, D_1, D_2 meet only D_3, and $D_i \cdot D_{i+1} = 1$ for $i = 3, 4, \ldots, k-1$. Consider the following cases.

Case 1. $k = 5$. Put $-2K_{\bar{X}} \backsim 2D_1 + 2D_2 + 3D_3 + 2D_4 + D_5 + F$. Then $F \cdot D_1 = F \cdot D_2 = 1$, $F \cdot D_3 = F \cdot D_4 = F \cdot D_5 = 0$, $F \cdot K_{\bar{X}} = -2d$, $F^2 = 4d-4$. Then $\dim|F| = 3d-2$. So, there exists an element $N \in |F|$. Note that $2D_1+2D_2+3D_3+2D_4+D_5+N$ does not admit representation as M_1+M_2, where $M_1, M_2 \in |-K_{\bar{X}}|$. Hence, $f(N)$ does not contain the support of anti-canonical divisor. Note that $\mathrm{mult}_Q(2D_1 + 2D_2 + 3D_3 + 2D_4 + D_5 + N) \geq 5$, where Q is the intersection of D_3 and D_4. Then $\frac{1}{2}f(N)$ is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.

Case 2. $k = 6$. Put $-2K_{\bar{X}} \backsim 2D_1 + 2D_2 + 4D_3 + 3D_4 + 2D_5 + D_6 + F$. Then $F \cdot D_3 = 1$, $F \cdot D_i = 0$ for $i \neq 3$, $F \cdot K_{\bar{X}} = -2d$, $F^2 = 4d-4$. Then $\dim|F| = 3d-2$. So, there exists an element $N \in |F|$. Note that $2D_1+2D_2+4D_3+3D_4+2D_5+D_6+N$ does not admit representation as M_1+M_2, where $M_1, M_2 \in |-K_{\bar{X}}|$. Hence, $f(N)$ does not contain the support of anti-canonical divisor. Note that $\mathrm{mult}_Q(2D_1+2D_2+4D_3+3D_4+2D_5+D_6+N) \geq 7$, where Q is the intersection of D_3 and D_4. Then $\frac{1}{2}f(N)$ is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.

Case 3. $k = 7$. Put

$$-3K_{\bar{X}} \backsim 3D_1 + 3D_2 + 6D_3 + 5D_4 + 4D_5 + 3D_6 + 2D_7 + F.$$

Then $F \cdot D_3 = F \cdot D_7 = 1$, $F \cdot D_i = 0$ for $i \neq 3, 7$, $F \cdot K_{\bar{X}} = -3d$, $F^2 = 9d-8$. Then $\dim|F| = 6d-4$. So, there exists an element $N \in |F|$. Note that $3D_1+3D_2+6D_3+5D_4+4D_5+3D_6+2D_7+N$ does not admit representation as M_1+M_2, where $M_1 \in |-K_{\bar{X}}|$ and $M_2 \in |-2K_{\bar{X}}|$. Hence, $f(N)$ does not contain the support of anti-canonical divisor. Note that $\mathrm{mult}_Q(3D_1 + 3D_2 +$

$6D_3 + 5D_4 + 4D_5 + 3D_6 + 2D_7 + N) \geq 11$, where Q is the intersection of D_3 and D_4. Then $\frac{1}{3}f(N)$ is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.

Case 4. $k = 8$. Put

$$-3K_{\bar{X}} \backsim 3D_1 + 3D_2 + 6D_3 + 5D_4 + 4D_5 + 3D_6 + 2D_7 + D_8 + F.$$

Then $F \cdot D_3 = 1$, $F \cdot D_i = 0$ for $i \neq 3$, $F \cdot K_{\bar{X}} = -3d$, $F^2 = 9d - 6$. Then $\dim |F| = 6d - 3$. So, there exists an element $N \in |F|$. Note that $3D_1+3D_2+6D_3+5D_4+4D_5+3D_6+2D_7+D_8+N$ does not admit representation as $M_1 + M_2$, where $M_1 \in |-K_{\bar{X}}|$ and $M_2 \in |-2K_{\bar{X}}|$. Hence, $f(N)$ does not contain the support of anti-canonical divisor. Note that $\mathrm{mult}_Q(3D_1 + 3D_2 + 6D_3 + 5D_4 + 4D_5 + 3D_6 + 2D_7 + D_8 + N) \geq 11$, where Q is the intersection of D_3 and D_4. Then $\frac{1}{3}f(N)$ is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. □

Lemma 3.18. *Let X be a del Pezzo surface with du Val singularities and let d be the degree of X. Assume that there exists a singular point of type E_k, where $k = 6, 7, 8$. Then X has a $-K_X$-polar cylinder and there exists a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.*

Proof. Let X be a del Pezzo surface with du Val singularities, and let P be a singular point of type D_k. By Theorem 2.9, we see that X has a $-K_X$-polar cylinder. Let $f : \bar{X} \to X$ be the minimal resolution of singularities of X, and let $D = \sum_{i=1}^{n} D_i$ be the exceptional divisor of f, where D_i is a (-2)-curve. We may assume that $D_1, D_2, \ldots, D_k$ correspond to P. Moreover, D_4 is the central component, D_1 meets only D_4, D_3 meets D_2 and D_4, D_2 meets only D_3, and $D_i \cdot D_{i+1} = 1$ for $i = 3, 4, \ldots, k-1$. Consider the following cases.

Case 1. $k = 6$. Put $-2K_{\bar{X}} \backsim 2D_1 + D_2 + 2D_3 + 3D_4 + 2D_5 + D_6 + F$. Then $F \cdot D_1 = 1$, $F \cdot D_i = 0$ for $i \geq 2$, $F \cdot K_{\bar{X}} = -2d$, $F^2 = 4d - 2$. Then $\dim |F| = 3d - 1$. So, there exists an element $N \in |F|$. Note that $2D_1+D_2+2D_3+3D_4+2D_5+D_6+N$ does not admit representation as M_1+M_2, where $M_1, M_2 \in |-K_{\bar{X}}|$. Hence, $f(N)$ does not contain the support of anti-canonical divisor. Note that $\mathrm{mult}_Q(2D_1+D_2+2D_3+3D_4+2D_5+D_6+N) \geq 5$, where Q is the intersection of D_4 and D_5. Then $\frac{1}{2}f(N)$ is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.

Case 2. $k = 7$. Put

$$-2K_{\bar{X}} \backsim 2D_1 + 2D_2 + 3D_3 + 4D_4 + 3D_5 + 2D_6 + D_7 + F.$$

Then $F \cdot D_2 = 1$, $F \cdot D_i = 0$ for $i \neq 2$, $F \cdot K_{\bar{X}} = -2d$, $F^2 = 4d - 2$. Then $\dim |F| = 3d - 1$. So, there exists an element $N \in |F|$. Note that $2D_1 + 2D_2 + 3D_3 + 4D_4 + 3D_5 + 2D_6 + D_7 + N$ does not admit representation as M_1+M_2, where $M_1, M_2 \in |-K_{\bar{X}}|$. Hence, $f(N)$ does not contain the support of anti-canonical divisor. Note that $\mathrm{mult}_Q(2D_1 + 2D_2 + 3D_3 + 4D_4 + 3D_5 + 2D_6 + D_7 + N) \geq 7$, where Q is the intersection of D_4 and D_5. Then $\frac{1}{2}f(N)$

is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$.

Case 3. $k = 8$. Put

$$-2K_{\bar{X}} \backsim 3D_1 + 2D_2 + 4D_3 + 6D_4 + 5D_5 + 4D_6 + 3D_7 + 2D_8 + F.$$

Then $F \cdot D_8 = 1$, $F \cdot D_i = 0$ for $i \neq 8$, $F \cdot K_{\bar{X}} = -2d$, $F^2 = 4d - 2$. Then $\dim|F| = 3d-1$. So, there exists an element $N \in |F|$. Note that $3D_1 + 2D_2 + 4D_3 + 6D_4 + 5D_5 + 4D_6 + 3D_7 + 2D_8 + N$ does not admit representation as $M_1 + M_2$, where $M_1, M_2 \in |-K_{\bar{X}}|$. Hence, $f(N)$ does not contain the support of anti-canonical divisor. Note that $\mathrm{mult}_Q(2D_1 + 2D_2 + 3D_3 + 4D_4 + 3D_5 + 2D_6 + D_7 + N) \geq 7$, where Q is the intersection of D_4 and D_5. Then $\frac{1}{2}f(N)$ is a tiger such that the support of this tiger does not contain any elements of $|-K_X|$. □

So, Theorem 1.5 follows from Lemmas 3.10, 3.11, 3.12, 3.13, 3.14, 3.15, 3.16, 3.17, and 3.18.

4. The proof of theorem 1.6

Assume that $\rho(X) = 1$. Then X has a H-polar cylinder if and only if X has a $-K_X$-polar cylinder, where H is an arbitrary ample divisor. On the other hand, there exists a classification of del Pezzo surfaces X such that X has a $-K_X$-polar cylinder (see [1]). By a classification of a del Pezzo surface X has not cylinders if X has one of the following collections of singularities: $4A_2, 2A_1 + 2A_3, 2D_4$. So, we may assume that $\rho(X) > 1$.

Let $f : \bar{X} \to X$ be the minimal resolution of singularities of X, and let $D = \sum_{i=1}^{n} D_i$ be the exceptional divisor of f, where D_i is a (-2)-curve.

Lemma 4.19. *Assume that there exists a $\mathbb{P}^1$-fibration $g\colon \bar{X} \to \mathbb{P}^1$ such that at most one irreducible component of the exceptional divisor D not contained in any fiber of g. Moreover, this component is an 1-section. Then there exists an ample divisor H such that X has a H-polar cylinder.*

Proof. Let F be a unique exception curve not contained in any fiber of g (if there exist no such component, then F is an arbitrary 1-section). Put

$$-K_{\bar{X}} \sim_{\mathbb{Q}} 2F + \sum a_i E_i.$$

Note that all E_i are contained in fibers of g. Consider an ample divisor $H = -K_{\bar{X}} + mC$, where C is a fiber of g, m is a sufficiently large number. Then there exists a divisor $\hat{H} \sim_{\mathbb{Q}} H$ such that

$$\hat{H} = 2F + \sum b_i \hat{E}_i,$$

where $b_i > 0$ and the set of $\hat{E}_i$ contains all irreducible curves in singular fibers of g. Then

$$\bar{X} \setminus \mathrm{Supp}(\hat{H}) \cong \mathbb{A}^1 \times (\mathbb{P}^1 \setminus \{p_1, \ldots, p_k\}),$$

where $p_1, \dots, p_k$ correspond to singular fibers of g. So, $\bar{X}$ has a H-polarization. Hence, X has a $f(\hat{H})$-polarization. □

Run MMP for X. We obtain

$$X = X_1 \to X_2 \to \cdots \to X_n.$$

Assume that $X_n = \mathbb{P}^1$. Consider the composition of the minimal resolution and MMP. We have a $\mathbb{P}^1$-fibration $g\colon \bar{X} \to \mathbb{P}^1$. Note that all exception curves of f are contained in fibers of g. Hence, by Lemma 4.19, we see that there exists an ample divisor H such that X has a H-polar cylinder.

So, we may assume that X_n is a del Pezzo surface with $\rho(X_n) = 1$ and du Val singularities.

Lemma 4.20. *Assume that X_n has a $-K_{X_n}$-polar cylinder. Then there exists an ample divisor H such that X has a H-polar cylinder.*

Proof. Put $h\colon X \to X_n$. Assume that h contracts extremal rays in points $p_1, p_2, \dots, p_m$. Let M be an anti-canonical divisor such that $X_n \setminus \mathrm{Supp}(M) \cong Z \times \mathbb{A}^1$. Let $\phi\colon X_n \setminus \mathrm{Supp}(M) \to Z$ be the projection on first factor. Let $C_1, C_2, \dots, C_k$ be the fibers of ϕ such that $C_1, C_2, \dots, C_k$ contain $p_1, p_2, \dots, p_m$, and let $\bar{C}_1, \bar{C}_2, \dots, \bar{C}_k$ be the closure of $C_1, C_2, \dots, C_k$ on X_n. Since $\rho(X_n) = 1$, we see that $C_i \sim_{\mathbb{Q}} -a_i K_{X_n}$. Consider the divisor

$$L = M + m_1\bar{C}_1 + m_2\bar{C}_2 + \cdots + m_k\bar{C}_k,$$

where $m_1, m_2, \dots, m_k$ are sufficiently large numbers. Note that the divisor $L \sim_{\mathbb{Q}} -\alpha K_{X_n}$. Let $\hat{L}$ be the proper transform of the divisor L. Consider $H = \hat{L} + \sum \epsilon_i E_i$, where E_i are irreducible components of the exceptional divisor of h and ϵ_i are positive numbers. Note that for sufficiently large m_i and for sufficiently small ϵ_i, the divisor H is ample. Moreover, $X \setminus \mathrm{Supp}(H) \cong (Z \setminus \{q_1, \dots, q_k\}) \times \mathbb{A}^1$, where $q_1, \dots, q_k$ are k points on Z. So, X has a H-polar cylinder. □

Let X be a del Pezzo surface with du Val singularities. Assume that $\rho(X) = 1$. Then X has a H-polar cylinder if and only if X has a $-K_X$-polar cylinder, where H is an arbitrary ample divisor. On the other hand, there exists a classification of del Pezzo surfaces X such that X has a $-K_X$-polar cylinder (see [1]). By a classification of a del Pezzo surface X has not cylinders if X has one of the following collections of singularities: $4A_2, 2A_1 + 2A_3, 2D_4$. So, we may assume that $\rho(X) > 1$ and X has not cylinders. Run MMP for X. We obtain

$$X = X_1 \to X_2 \to \cdots \to X_n.$$

By Lemma 4.19 we may assume that X_n is a del Pezzo surface with $\rho(X_n) = 1$ and du Val singularities. By Lemma 4.20 we see that X_n is a del Pezzo surface with one of the following collect of singularities: $4A_2, 2A_1 + 2A_3, 2D_4$. On the other hand, the surface X has a smaller degree than X_n. But degree of X_n is equal to one. So, $X = X_n$. On the other hand, $\rho(X_n) = 1$, a contradiction.

This completes the proof of Theorem 1.6.

References

[1] I. Cheltsov, J. Park, and J. Won, *Cylinders in singular del Pezzo surfaces*, Compos. Math. **152** (2016), no. 6, 1198–1224.

[2] ———, *Affine cones over smooth cubic surfaces*, J. Eur. Math. Soc. **18** (2016), no. 7, 1537–1564.

[3] R. Hartshorne, *Algebraic Geometry*, Springer Science+Business Media, Inc., 1977.

[4] S. Keel and J. McKernan, *Rational curves on quasi-projective surfaces*, Mem. Amer. Math. Soc. **140** (1999), no. 669, 153 pp.

[5] T. Kishimoto, Yu. Prokhorov, and M. Zaidenberg, *Group actions on affine cones*, Affine algebraic geometry, 123–163, CRM Proc. Lecture Notes, 54, Amer. Math. Soc., Providence, RI, 2011.

[6] ———, *G_a-actions on affine cones*, Transform. Groups **18** (2013), no. 4, 1137–1153.

[7] K. Matsuki, *Introduction to the Mori Program*, Springer-Verlag New York, Inc., 2002.

Grigory Belousov
Plekhanov Russian University of Economics
Moscow, Russia
E-mail address: belousov_grigory@mail.ru