East Asian Math. J.
Vol. 34 (2018), No. 5, pp. 615–621
http://dx.doi.org/10.7858/eamj.2018.041

RINGS WITH THE SYMMETRIC PROPERTY FOR IDEMPOTENT-PRODUCTS

Juncheol Han and Hyo-Seob Sim*

Abstract. Let R be a ring with the unity 1, and let e be an idempotent of R. In this paper, we discuss some symmetric property for the set $\{(a_1, a_2, ..., a_n) \in R^n : a_1a_2 \cdots a_n = e\}$. We here investigate some properties of those rings with such a symmetric property for an arbitrary idempotent e; some of our results turn out to generalize some known results observed already when $n = 2$ and $e = 0, 1$ by several authors. We also focus especially on the case when $n = 3$ and $e = 1$. As consequences of our observation, we also give some equivalent conditions to the commutativity for some classes of rings, in terms of the symmetric property.

1. Introduction

Let R be a ring with the unity 1, and let e be an idempotent of R. Then, R is called *e-reversible* if $ab = e$ implies to $ba = e$ for every a, b in R. R is called *e-symmetric* if $abc = e$ implies to $acb = e$ for every a, b, c in R.

For the case when $e = 0$, the term '0-reversible' is usually called simply by 'reversible' (see, for example, Cohn [2]). The term '0-symmetric' is usually called simply by 'symmetric'. Some other notation for the term 0-reversible or 0-symmetric is also used. For examples, Anderson and Camillo [1] uses the notation ZC_2 for 0-reversible, and ZC_3 for 0-symmetric, while Krempa and Niewieczerzal [3] uses C_0 for 0-reversible.

For the case when $e = 1$, the term '1-reversible' is usually called by the well-known term 'directly finite'. It was shown that every reversible ring is directly finite, which amounts to say that every 0-reversible ring is 1-reversible, in fact. The proof can be found in [1]. In Proposition 2.1 in Section 2, we shall generalize the above observation from the unity 1 to arbitrary idempotent e.

As an analogue to the reversible rings, it is natural to ask whether if R is 0-symmetric then R is 1-symmetric; it is shown that it is not true in general in

Received August 14, 2018; Accepted August 31, 2018.
2010 *Mathematics Subject Classification.* 16E50, 16S99.
Key words and phrases. symmetric ring, reversible ring, e-symmetric ring, e-reversible ring, semiperfect ring.
This work was supported by a Research Grant of Pukyong National University (2017 year).
* Corresponding author.

(pISSN 1226-6973, eISSN 2287-2833)

Example 2.6. In Corollary 2.8, we shall give an alternative observation about the question.

It was also shown that a ring R is symmetric if and only if the set

$$\{(a_1, a_2, ..., a_n) \in R^n : a_1 a_2 \cdots a_n = 0\}$$

is invariant under the permutation action. For the proof, see Anderson and Camillo [1].

This characterization of 0-symmetric property can be generalized to an analogous characterization of e-symmetric property for every idempotent e. In Theorem 2.5, we shall establish the characterization for the e-symmetric property.

In Section 2, we also investigate several important properties on the e-symmetric property as well as the e-reversible property for a given idempotent e of a ring. In particular, we shall give a characterization for 1-symmetric property, in terms of the commutativity of the multiplicative group of units in Theorem 2.7, which turns out to be useful for our purpose in Section 3.

In section 3, we shall discuss some applications of results investigated in Section 2. We shall describe some equivalent conditions to the commutativity for some classes of semiperfect rings in terms of the 1-symmetric property, as well as the symmetric property for some mutually orthogonal local idempotent elements.

Throughout this paper all rings are associative with the unity unless otherwise specified. For a ring R, let $J(R)$ denote the Jacobson radical of R, and $U(R)$ the multiplicative group of all units of R. A ring R is called *reduced* if the only nilpotent element of R is zero. A ring R is called *abelian* if every idempotent element of R is central. It is easy to see that every reduced ring is abelian.

The notation and terminology not defined in this paper may be found in some books on related areas, for example, Lam [4].

2. Idempotent-symmetric property

We first discuss some basic properties on the e-reversible property and e-symmetric property for an arbitrary idempotent e of a ring R.

It was proved in [2] that if a ring R is reversible then R is directly finite. In our terminology, if R is 0-reversible then R is 1-reversible. We here generalize the result with the following proposition.

Proposition 2.1. *Let R be a ring with the unity 1. If R is reversible, then R is e-reversible for every idempotent e in R.*

Proof. Let e be an idempotent in R. For every r in R, $(1-e)er = 0$ and $re(1-e) = 0$. Since R is reversible, it follows that $er(1-e) = 0$ and $(1-e)re = 0$, and so $er = re$. therefore, e is central. Suppose that $ab = e$ for $a, b \in R$. Then $(ba-e)b = b(ab) - be = 0$. It follows that $b(ba-e) = 0$ because R is reversible. So $eba = e$. On the other hand, since $1-e$ is central, $a(1-e)b = (1-e)ab = 0$.

Since R is reversible, $(1-e)ba = 0$. It follows that $ba = eba = e$. So R is e-reversible. □

Proposition 2.2. *Let R be a ring with the unity 1 and let e be an idempotent in R. If R is e-reversible, then e is central.*

Proof. Let r be an element of R, and denote $a = e+er(1-e)$ and $b = e+(1-e)re$. Then $ae = e$ and $eb = e$. Since R is e-reversible, $ea = ae$ and $eb = be$, which yield that $er = ere$ and $re = ere$, and so $er = re$. Therefore, e is central. □

We then have an immediate consequence of the above observations as follows.

Corollary 2.3. *Every reversible ring is abelian.*

It is also easy to see that every reduced ring is reversible, and so abelian.

Note that the converse of Proposition 2.2 may not be true; we observe it with the following example.

Example 2.4. Let K be a field and consider

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in K \right\}.$$

Then R has only two idempotents 0 and 1, which are central. Let E_{ij} denote the 3×3 matrix over K with (i,j)-entry 1, elsewhere 0, and $A = E_{23}+E_{13}$, $B = E_{12} + E_{13} \in R$. Then R is not 0-reversible because $AB = 0$ and $BA = E_{13}$, while R is directly finite, i.e., R is 1-reversible.

We now observe a characteristic property for e-symmetric ring with the following theorem.

Theorem 2.5. *Let R be a ring with the unity, and let e be an idempotent of R. Then R is e-symmetric if and only if for every positive integer n, $a_1 \cdots a_n = e$ in R implies that $a_{\sigma(1)} \cdots a_{\sigma(n)} = e$ for every permutation of degree n.*

Proof. Suppose that R is e-symmetric, it is obvious that R is e-reversible. Let n be an integer such that $n \geq 2$. Suppose that $a_1 a_2 \cdots a_n = e$ in R. Then since R is e-reversible, it follows that $(a_2 \cdots a_n)\, a_1 = e$, and so $a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)} = e$ for $\sigma = (1, 2, \ldots, n)$ in the symmetric group S_n. Moreover, it follows that $a_2 a_1 \,(a_3 \cdots a_n) = e$ since $a_2(a_3 \cdots a_n)a_1 = e$ and R is e-symmetric, and so $a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)} = e$ for $\sigma = (1, 2) \in S_n$. Therefore, $a_{\sigma(1)} a_{\sigma(2)} a_{\sigma(3)} \cdots a_{\sigma(n)} = e$ for all $\sigma \in S_n$, since S_n is generated by $(1, 2)$ and $(1, 2, \ldots, n)$. It is also obvious that the converse is true. In fact, the condition for $n = 3$ implies that R is e-symmetric. □

Note that the same result for $e = 0$ was shown in [1].

It is obvious that every e-symmetric ring R is e-reversible for every idempotent e of R. However, the converse may not be true, as shown in the following example.

Example 2.6. Let $R = D \times D$ for a division ring D which is not commutative. It is easily checked that R is a reduced ring, and so e-reversible for every idempotent $e \in R$ by Proposition 2.1. There are four idempotents $0 = (0, 0), 1 = (1, 1), e_1 = (1, 0), e_2 = (0, 1)$ in R. To show that R is 0-symmetric, assume that $(a_1, a_2)(b_1, b_2)(c_1, c_2) = (0, 0)$ in R. Then $a_i b_i c_i = 0$ in D for each $i = 1, 2$. If $a_i = 0$, then it is obvious that $a_i c_i b_i = 0$. Assume $a_i \neq 0$. Then since each a_i are invertible, $b_i c_i = 0$. It follows that $c_i b_i = 0$ since R is reversible, and hence $a_i c_i b_i = 0$. Consequently, $(a_1, a_2)(c_1, c_2)(b_1, b_2) = (0, 0)$, and so R is 0-symmetric. (Note that it was already shown in [1, Theorem I.3] that every reduced ring is symmetric.) Assume that R is 1-symmetric. Since D is not commutative, there exist nonzero elements a, b in D such that $ab \neq ba$. Consider $(a, 1)(b^{-1}, 1)(ba^{-1}, 1) = (1, 1)$. Since R is 1-symmetric by assumption, $(a, 1)(ba^{-1}, 1)(b^{-1}, 1) = (1, 1)$, and so $aba^{-1}b^{-1} = 1$, which yields a contradiction. Therefore, R is not 1-symmetric. By the similar argument, one can show that R is neither e_1-symmetric nor e_2-symmetric.

We now take our focus on 1-symmetric property. Note that either 0-symmetric or 1-reversible is not sufficient for 1-symmetric, as shown in Example 2.6.

We here describe a necessary and sufficient condition for the 1-symmetric property.

Theorem 2.7. *Let R be a ring with the unity 1. R is 1-symmetric if and only if R is 1-reversible and $U(R)$ is abelian.*

Proof. Suppose that R is 1-symmetric. Then clearly, R is 1-reversible. Let a, b be elements of $U(R)$. Then there exists $u \in U(R)$ such that $u(ab) = (ab)u = 1$. Since R is 1-symmetric, it follows that $u(ba) = (ba)u = 1$. Thus $ab = u^{-1} = ba$, and so $U(R)$ is abelian. Conversely, suppose that R is 1-reversible and $U(R)$ is abelian. Let $abc = 1$ in R. Then $cab = 1 = bca$ since R is 1-reversible. Thus $ab = c^{-1}$ and $bc = a^{-1}$. Therefore $ac = ca$ since a, c are contained in the abelain group $U(R)$, and so $acb = 1$. It follows that R is 1-symmetric. □

Corollary 2.8. *If R is a symmetric ring such that $U(R)$ is abelian, then R is 1-symmetric.*

Proof. Since R is symmetric, R is reversible, and then R is 1-reversible by Proposition 2.1. Hence it follows from Theorem 2.7 R is 1-symmetric. □

Proposition 2.9. *Let R be a ring with the unity 1. If R is 1-symmetric, then eR is e-symmetric for every central idempotent e.*

Proof. Suppose that R is 1-symmetric. Let e be a central idempotent of R and denote $S = eR$. Since e is central, $S = eRe = Re = eR$. Assume that $(ex)(ey)(ez) = e$, where $x, y, z \in R$. Denote $a = ex + (1-e)$, $b = ey + (1-e)$and $c = ez + (1-e)$. Then, since e be central, $abc = (ex)(ey)(ez) + (1-e) = 1$. Since R is 1-symmetric, it follows that $acb = 1$, and so $(ex)(ez)(ey) + (1-e) = acb = 1$. Therefore, $(ex)(ez)(ey) = e$. Consequently S is e-symmetric. □

In virtue of Proposition 2.9, it seems that a homomorphic image of every 1-symmetric ring might be 1-symmetric. Unexpectedly, it is not true in general. Nevertheless, it can be true if the kernel is contained in the Jacobson radical of the ring; we investigate it with the following lemma.

Lemma 2.10. *Let R be a ring with the unity 1, and let $\phi : R \to S$ be a homomorphism from R into S with kernel K. If $U(R)$ is abelian and $K \subseteq J(R)$, then R is 1-reversible if and only if $\phi(R)$ is $\phi(1)$-reversible.*

Proof. Suppose that R is 1-reversible and $\phi(x)\phi(y) = \phi(1)$ for $x, y \in R$. Then $xy - 1 \in K \subseteq J(R)$, and so $xy \in 1 + J(R) \subseteq U(R)$. It follows that $(xy)r = 1 = r(xy)$ for some $r \in U(R)$. Since R is 1-reversible, $y(rx) = 1 = (rx)y$, that is, $y \in U(R)$ and hence $x \in U(R)$. Since $U(R)$ is abelian, $\phi(y)\phi(x) = \phi(1)$. Consequently, $\phi(R)$ is $\phi(1)$-reversible.

Conversely, suppose that $\phi(R)$ is $\phi(1)$-reversible and $xy = 1$ in R. Then $\phi(xy) = \phi(x)\phi(y) = \phi(1)$. Since $\phi(R)$ is $\phi(1)$-reversible, $\phi(yx) = \phi(y)\phi(x) = \phi(1)$. It follows that $xy - 1, yx - 1 \in K \subseteq J(R)$, and so $xy, yx \in 1 + J(R) \subseteq U(R)$. Therefore, $(xy)s = 1 = t(yx)$ for some $s, t \in U(R)$, which yields that $ys = ty = x^{-1}$ and so $x \in U(R)$. It follows from $xy \in U(R)$ that $y \in U(R)$. Since $U(R)$ is abelian, $yx = 1$. Consequently, R is 1-reversible. □

Corollary 2.11. *Let R be a ring with the unity, and let K be an ideal of R such that $K \subseteq J(R)$. If R is 1-symmetric then the factor ring R/K is $\bar{1}$-symmetric, where $\bar{1}$ denotes the unity of R/K.*

Proof. Let $\phi : R \longrightarrow R/K$ be the canonical homomorphism from R onto R/K. Suppose that R is 1-symmetric. Then it is easy to see that $U(\phi(R))$ is contained in $\phi(U(R))$. Since $U(R)$ is abelian, $U(\phi(R))$ is abelian. From Lemma 2.10, $\phi(R)$ is $\phi(1)$-reversible. It now follows from Theorem 2.7 that $\phi(R)$ is $\phi(1)$-symmetric, that is, R/K is $\bar{1}$-symmetric. □

The converse of Corollary 2.11 may not be true, as shown in the next example.

Example 2.12. Let K be a field and consider

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in K \right\}$$

Then $J(R) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in K \right\}$, and R is 1-reversible but $U(R)$ is not abelian. Thus R is not 1-symmetric. However, $R/J(R)$ is clearly $\bar{1}$-symmetric, because $R/J(R)$ is isomorphic to $K \times K$.

3. 1-symmetric semiperfect rings

In this section, we consider a certain class of rings, in which every 1-symmetric ring is commutative. As consequences of our observation in the previous section, we give some equivalent conditions to the commutativity for some classes of semiperfect rings in terms of the 1-symmetric property, as well as the symmetric property for some mutually orthogonal local idempotent elements.

We first remark immediate consequences of Theorem 2.7.

Lemma 3.1. *Let R be a semiperfect ring. Then, R is 1-symmetric if and only if $U(R)$ is abelian.*

Proof. As found in Lam [4, Proposition (20.8)], every semilocal ring is 1-reversible. The result now follows from Theorem 2.7, since every semiperfect ring is a semilocal ring. □

Lemma 3.2. *Let R be a finite direct sum of local rings. Then the following conditions are equivalent:*

(1) *R is 1-symmetric.*
(2) *$U(R)$ is abelian.*
(3) *R is commutative.*

Proof. Let $R = R_1 \oplus \cdots \oplus R_n$ be the direct sum of local rings $R_1, ..., R_n$. By Theorem 2.7, (1) implies (2). To show that (2) implies (3), suppose that $U(R)$ is abelian. Since $U(R) = U(R_1) \oplus \cdots \oplus U(R_n)$, each $U(R_i)$ is abelian. Since each R_i is a local ring, for each $x \in R_i$ either $x \in U(R_i)$ or $1 - x \in U(R_i)$. It follows that every R_i is commutative, and so $R = R_1 \oplus \cdots \oplus R_n$ is commutative. It is now obvious that (3) implies (1). □

It was shown in [4, Theorem (23.6)] that a ring R is semiperfect if and only if the unity 1 can be decomposed into some mutually orthogonal local idempotents $e_1, e_2, \ldots, e_n$ as $1 = e_1 + e_2 + \cdots + e_n$.

We then characterize the commutativity for a semiperfect ring in terms of symmetric property for those mutually orthogonal local idempotents.

Proposition 3.3. *Let R be a semiperfect ring, and $1 = e_1 + e_2 + \cdots + e_n$ a decomposition of 1 into orthogonal local idempotents $e_1, e_2, \ldots, e_n$. Then, R is commutative if and only if R is e_i-symmetric for all $i = 1, 2, \ldots, n$.*

Proof. Suppose that R is e_i-symmetric for all $i = 1, \ldots, n$. Since R is e_i-reversible, all e_i are central for all i by Proposition 2.2, and so $R = e_1R \oplus \cdots \oplus e_nR$ is a finite direct sum of the local rings $e_iR = e_iRe_i$. Since R is e_i-symmetric, so e_iR is also e_i-symmetric. Therefore, $U(e_iR)$ is abelian for all $i = 1, \ldots, n$ by Theorem 2.7, since e_i is the unity of e_iR. It follows that $U(R) = U(e_IR) \oplus \cdots \oplus U(e_nR)$ is also abelian. Consequently, that R is commutative from Lemma 3.2. The converse is obvious. □

We also characterize the commutativity for a semiperfect ring in terms of 1-symmetric property.

Corollary 3.4. *Let R be a semiperfect ring. Then, R is commutative if and only if R is abelian and R is 1-symmetric.*

Proof. Suppose that R is abelian and R is 1-symmetric. Since R is a semiperfect ring, $1 = e_1 + \cdots + e_n$ for some mutually orthogonal local idempotents $e_1, \ldots, e_n$ of R. Since R is abelian, each e_i is central, and so each $e_iR = e_iRe_i$ is a local ring. Therefore, $R = e_1R \oplus \cdots \oplus e_nR$ is a finite direct sum of the local rings. Since $U(R)$ is abelian, it now follows from Corollary 3.2 that R is commutative. □

Corollary 3.5. *Let R be a semiperfect abelian ring. Then the following conditions are equivalent:*

(1) *R is 1-symmetric.*
(2) *$U(R)$ is abelian.*
(3) *R is commutative.*

Proof. The result is an immediate consequence of Lemma 3.1 and Corollary 3.4. □

Note that the result of Corollary 3.5 may not be true in general. The following example provides a counter-example for it.

Example 3.6. Let $R = F[x]$ be the polynomial ring with an indeterminate x over a field F and δ be the derivation on R defined by formal differentiation with respect to x. Let $S = R[y; \delta]$ be the differential polynomial ring where x and y are commuting independent variables. Since there is no zero-divisor in S, S is 0-symmetric. We observe that the only idempotents of S are $0, 1$ and $U(S) = U(F)$ is abelian, which implies that S is 1-symmetric. Clearly, S is not commutative.

Question. Is every 1-symmetric semiperfect ring is commutative?

References

[1] D. D. Anderson and V. Camillo, *Semigroups and rings whose zero products commute*, Comm. Algebra **27** (1999), 2847–2852.
[2] P.M. Cohn, *Reversible rings*, Bull. London Math. Soc. **31** (1999), 641–648.
[3] J. Krempa and D. Niewieczerzal, *Rings in which annihilators are ideals and their application to semigroup rings*, Bull. Acad. Polon. Sci., Math. Astronom. Phys. **25** (1977), 851-856.
[4] T. Y. Lam, *A First Course in Noncommutative Rings*, Springer-Verlag, New York, 1991.

Juncheol Han
Department of Mathematics Education,
Pusan National University, Busan 46277, Korea
E-mail address: jchan@pusan.ac.kr

Hyo-Seob Sim
Department of Applied Mathematics,
Pukyong National University, Busan 48513, Korea
E-mail address: hsim@pknu.ac.kr