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FOURTH HANKEL DETERMINANT FOR THE FAMILY OF FUNCTIONS WITH BOUNDED TURNING

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Abstract. The main aim of this paper is to study the fourth Hankel determinant for the class of functions with bounded turning. We also investigate for 2-fold symmetric and 3-fold symmetric functions.

1. Introduction and definitions

Let $\mathfrak{A}$ denote the family of all functions f that are analytic in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ having the Taylor series expansions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{D}), \tag{1.1}$$

while $\mathcal{S}$ represents a family of functions $f \in \mathfrak{A}$ that are univalent in $\mathbb{D}$. Let $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$ denote the classes of starlike, convex and bounded turning functions respectively and are defined as:

$$\mathcal{S}^* = \left\{ f : f \in \mathfrak{A} \text{ and } \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D} \right\},$$

$$\mathcal{C} = \left\{ f : f \in \mathfrak{A} \text{ and } \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D} \right\},$$

and

$$\mathcal{R} = \{ f : f \in \mathfrak{A} \text{ and } \operatorname{Re}(f'(z)) > 0, \quad z \in \mathbb{D} \}.$$

Let $\mathcal{P}$ denote the family of all analytic functions p of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{1.2}$$

in $\mathbb{D}$ whose real parts are positive in $\mathbb{D}$. It is known that the nth coefficient for the functions belong to the family $\mathcal{S}$, is bounded by n and this bound helps to study its geometric properties. In particular, the growth and distortion

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properties of a normalized univalent function $f \in \mathcal{S}$ are determined by the bound of its second coefficient.

The Hankel determinant $H_{q,n}(f)$ $(q, n \in \mathbb{N} = \{1, 2, \ldots\})$ for a function $f \in \mathcal{S}$ of the form (1.1) was defined by Pommerenke [21, 22], (see also [2, 3]) as

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.3}$$

For fixed integer q and n, the growth of $H_{q,n}(f)$ has been studied for different subfamilies of univalent functions. We include here a few of them. The sharp bounds of $|H_{2,2}(f)|$ for the subfamilies $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$ of the set $\mathcal{S}$ were investigated by Janteng et al. [10, 11]. They proved the bounds

$$|H_{2,2}(f)| \leq \begin{cases} 1 & \text{for} \quad f \in \mathcal{S}^*, \\ \frac{1}{8} & \text{for} \quad f \in \mathcal{C}, \\ \frac{4}{9} & \text{for} \quad f \in \mathcal{R}. \end{cases}$$

For the family of Bazilevič functions, the exact estimate of $|H_{2,2}(f)|$ was obtained by Krishna et al. [13]. For more works on $H_{2,2}(f)$ for subfamilies of $\mathcal{S}$ see the references [5, 9, 12, 14, 17, 19, 20].

Unfortunately, the sharp bound of $|H_{2,2}(f)|$ for the whole class $\mathcal{S}$ is still not known. In [26], Thomas conjectured that if $f \in \mathcal{S}$, then $|H_{2,n}(f)| \leq 1$. As it was shown by Li and Srivastava in [15], this conjecture is not true for $n \geq 4$. Similarly, Răducanu and Zaprawa in [23] proved that it is also false for $n = 2$. In fact, they showed that $\max\{|H_{2,2}(f)| : f \in \mathcal{S}\} \geq 1.175\ldots$.

The estimation of $|H_{3,1}(f)|$ is much more difficult than the case of $|H_{2,2}(f)|$. The first paper on $H_{3,1}(f)$ appears in 2010 by Babalola [4] in which he obtained the upper bound of $H_{3,1}(f)$ for the families of $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$. Later on some other authors [1, 6, 8, 24, 25, 27] published their works concerning $|H_{3,1}(f)|$ for different subfamilies of analytic and univalent functions. Recently in 2016, Zaprawa [28] improved the results of Babalola [4] by proving

$$|H_{3,1}(f)| \leq \begin{cases} 1 & \text{for} \quad f \in \mathcal{S}^*, \\ \frac{49}{540} & \text{for} \quad f \in \mathcal{C}, \\ \frac{41}{60} & \text{for} \quad f \in \mathcal{R}, \end{cases}$$

and claimed that these bounds are still not sharp. Further for the sharpness, he considered the subfamilies of $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$ consisting of functions with m-fold symmetry and obtained the sharp bounds. In this paper, we contribute to the fourth Hankel determinant for the class of functions with positive real part.

2. A set of lemmas

In order to find the bound of the fourth Hankel determinant, we need the following sharp estimates for the class $\mathcal{S}^*$ of starlike functions and $\mathcal{P}$ of functions with positive real part.

Lemma 2.1. *If $p \in \mathcal{P}$, then, for $n, k \in \mathbb{N} = \{1, 2 \ldots\}$, the following sharp inequalities hold*

$$|c_{n+k} - \lambda c_n c_k| \leq 2 \quad \text{for } 0 \leq \lambda \leq 1, \tag{2.1}$$

$$|c_n| \leq 2. \tag{2.2}$$

The inequalities (2.1) and (2.2) are proved in [7] and [18] respectively.

Lemma 2.2. *Let $p \in P$ of the form* (1.2)*. Then*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right)$$

for some x with $|x| \leq 1$.

This result is due to Libera and Złotkiewicz [16].

Let $g \in \mathcal{S}^*$ of the form

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \qquad (z \in \mathbb{D}). \tag{2.3}$$

Then for the real number λ, consider the functional

$$\Phi_g(\lambda) = \left|b_2^2\left(b_3 - \lambda b_2^2\right)\right|.$$

Now we prove the upper bound of $\Phi_g(\lambda)$ as follows.

Theorem 2.3. *Let $g \in \mathcal{S}^*$ of the form* (2.3)*. Then*

$$\Phi_g(\lambda) \leq \begin{cases} 4(3 - 4\lambda), & \lambda \leq 5/8, \\ \frac{1}{2(2\lambda-1)}, & \lambda \in [5/8, 3/4], \\ \frac{1}{4(1-\lambda)}, & \lambda \in [3/4, 7/8], \\ 4(4\lambda - 3), & \lambda \geq 7/8. \end{cases}$$

Proof. Let $g \in \mathcal{S}^*$ of the form (2.3). Then

$$\frac{zg'(z)}{g(z)} = p(z),$$

where p is in class $\mathcal{P}$ of functions with positive real part. Then it is easy to see that

$$b_2 = c_1, \quad 2b_3 = c_2 + c_1^2.$$

Hence by applying Lemma 2.2, and the above relations, we get

$$\Phi_g(\lambda) = \frac{1}{4}\left|c_1^2\left[x\left(4 - c_1^2\right) + (3 - 4\lambda)c_1^2\right]\right|$$

for some x such that $|x| \leq 1$. Taking into account of the invariance of Φ_g under rotation, we may assume that c_1 is a non negative real number such that $c_1 = 2r$, $r \in [0, 1]$. Therefore

$$\Phi_g(\lambda) = 4r^2\left|\left(1 - r^2\right)x + (3 - 4\lambda)r^2\right|.$$

1. Now we suppose that $\lambda \leq 3/4$. Then

$$\Phi_g(\lambda) \leq 4r^2\left[2(1 - 2\lambda)r^2 + 1\right].$$

Let $q_1(r) = 4r^2\left[2(1-2\lambda)r^2+1\right]$. Then for $\lambda \le 1/2$ and $r \in [0,1]$, $q_1(r)$ is an increasing function. Hence $q_1(r) \le q_1(1)$. For $\lambda \in (1/2, 3/4]$, we have

$$q_1(r) \le \begin{cases} q_1(1), & \lambda \in (1/2, 5/8], \\ q_1\left(1/\sqrt{4(2\lambda-1)}\right), & \lambda \in [5/8, 3/4]. \end{cases}$$

2. For the case $\lambda \ge 3/4$, we have

$$\Phi_g(\lambda) \le 4r^2\left[4(\lambda-1)r^2+1\right].$$

Again, letting $q_2(r) = 4r^2\left[4(\lambda-1)r^2+1\right]$ and using similar arguments, we have

$$q_2(r) \le \begin{cases} q_2\left(1/\sqrt{8(1-\lambda)}\right), & \lambda \in [3/4, 7/8], \\ q_2(1), & \lambda \ge 7/8. \end{cases}$$

Hence, we have the required result. □

3. Bounds of $|H_{4,1}(f)|$ for the set $\mathcal{R}$

First, for any $f \in \mathfrak{A}$ of the form (1.1), we can write $H_{4,1}(f)$ in the form

$$H_{4,1}(f) := a_7 H_3(1) - a_6\Delta_1 + a_5\Delta_2 - a_4\Delta_3, \tag{3.1}$$

where Δ_1, Δ_2 and Δ_3 are determinants of order 3 given by

$$\Delta_1 = (a_3a_6 - a_4a_5) - a_2(a_2a_6 - a_3a_5) + a_4\left(a_2a_4 - a_3^2\right), \tag{3.2}$$

$$\Delta_2 = \left(a_4a_6 - a_5^2\right) - a_2(a_3a_6 - a_4a_5) + a_3\left(a_3a_5 - a_4^2\right), \tag{3.3}$$

$$\Delta_3 = a_2\left(a_4a_6 - a_5^2\right) - a_3(a_3a_6 - a_4a_5) + a_4\left(a_3a_5 - a_4^2\right). \tag{3.4}$$

From (1.3), we observe that $H_{4,1}(f)$ is a polynomial of six successive coefficients a_2, a_3, a_4, a_5, a_6 and a_7 of a function f in a given class. However, in many problems these coefficients are connected to the coefficients of the function p in the set $\mathcal{P}$.

Assume now that $f \in \mathcal{R}$. We have

$$f'(z) = p(z), \tag{3.5}$$

where $p \in \mathcal{P}$ of the form (1.2). From (3.5), we can easily obtain

$$na_n = c_{n-1}. \tag{3.6}$$

Using (3.6) in (3.2), (3.3) and (3.4), it follows that

$$\Delta_1 = \frac{1}{18}c_2c_5 - \frac{1}{20}c_3c_4 - \frac{1}{24}c_1^2c_5 + \frac{1}{30}c_1c_2c_4 + \frac{1}{32}c_1c_3^2 - \frac{1}{36}c_2^2c_3, \tag{3.7}$$

$$\Delta_2 = \frac{1}{24}c_3c_5 - \frac{1}{25}c_4^2 + \frac{1}{40}c_1c_3c_4 - \frac{1}{36}c_1c_2c_5 + \frac{1}{45}c_2^2c_4 - \frac{1}{48}c_2c_3^2, \tag{3.8}$$

$$\Delta_3 = \frac{1}{48}c_1c_3c_5 - \frac{1}{50}c_1c_4^2 + \frac{1}{30}c_2c_3c_4 - \frac{1}{64}c_3^3 - \frac{1}{54}c_2^2c_5. \tag{3.9}$$

Now we can prove our main result.

Theorem 3.1. *If $f \in \mathcal{R}$, then*

$$|H_{4,1}(f)| \leq \frac{73757}{94500} \simeq 0.78050.$$

Proof. Let $f \in \mathcal{R}$. Then we can rewrite (3.7), (3.8) and (3.9) in the following ways

$$\begin{aligned}
\Delta_1 &= \frac{c_5\left(c_2 - c_1^2\right)}{24} + \frac{c_3\left(c_4 - c_2^2\right)}{36} - \frac{c_3\left(c_4 - c_1c_3\right)}{32} - \frac{67c_4\left(c_3 - c_1c_2\right)}{1440} \\
&\quad + \frac{19c_2\left(c_5 - c_1c_4\right)}{1440} + \frac{c_2c_5}{1440}, \\
\Delta_2 &= \frac{c_5\left(c_3 - c_1c_2\right)}{36} - \frac{c_4\left(c_4 - c_2^2\right)}{45} + \frac{c_3\left(c_5 - c_2c_3\right)}{48} - \frac{4c_4\left(c_4 - c_1c_3\right)}{225} \\
&\quad - \frac{13c_3\left(c_5 - c_1c_4\right)}{1800} + \frac{c_3c_5}{3600}, \\
\Delta_3 &= \frac{c_5\left(c_4 - c_2^2\right)}{54} - \frac{c_5\left(c_4 - c_1c_3\right)}{48} + \frac{c_3\left(c_6 - c_3^2\right)}{64} - \frac{c_3\left(c_6 - c_2c_4\right)}{64} \\
&\quad + \frac{c_4\left(c_5 - c_1c_4\right)}{50} - \frac{17c_4\left(c_5 - c_2c_3\right)}{960} + \frac{c_4c_5}{43200}.
\end{aligned}$$

Using the triangle inequality along with the inequalities (2.1) and (2.2), we obtain

$$\begin{aligned}
|\Delta_1| &\leq \frac{1}{6} + \frac{1}{9} + \frac{1}{8} + \frac{67}{360} + \frac{19}{360} + \frac{1}{360} = \frac{29}{45}, \\
|\Delta_2| &\leq \frac{1}{9} + \frac{4}{45} + \frac{1}{12} + \frac{16}{225} + \frac{26}{900} + \frac{1}{900} = \frac{173}{450},
\end{aligned}$$

and

$$|\Delta_3| \leq \frac{2}{27} + \frac{1}{12} + \frac{1}{16} + \frac{1}{16} + \frac{2}{25} + \frac{17}{240} + \frac{1}{10800} = \frac{13}{30}.$$

Now putting the values $|H_{3,1}(f)| \leq \frac{41}{60}$, $|\Delta_1| \leq \frac{29}{45}$, $|\Delta_2| \leq \frac{173}{450}$, $|\Delta_3| \leq \frac{13}{30}$ along with the inequality $|a_n| \leq \frac{2}{n}$ for $n \geq 2$ in (3.1), we obtain

$$\begin{aligned}
|H_{4,1}(f)| &\leq |a_7|\,|H_3(1)| + |a_6|\,|\Delta_1| + |a_5|\,|\Delta_2| + |a_4|\,|\Delta_3| \\
&\leq \frac{2}{7}\frac{41}{60} + \frac{1}{3}\frac{29}{45} + \frac{2}{5}\frac{173}{450} + \frac{1}{2}\frac{13}{30} \\
&= \frac{73757}{94500} \simeq 0.7805
\end{aligned}$$

and this completes the proof. □

4. Bounds of $|H_{4,1}(f)|$ for the sets $\mathcal{R}^{(2)}$ and $\mathcal{R}^{(3)}$

Let $m \in \mathbb{N} = \{1, 2, \ldots\}$. A domain Λ is said to be m-fold symmetric if a rotation of Λ about the origin through an angle $2\pi/m$ carries Λ on itself. A function f is said to be m-fold symmetric in $\mathbb{D}$, if

$$f\left(e^{2\pi i/m}z\right) = e^{2\pi i/m} f(z), \ (z \in \mathbb{D}).$$

By $\mathcal{S}^{(m)}$, we mean the set of m-fold univalent functions having the following Taylor series form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \quad (z \in \mathbb{D}). \tag{4.1}$$

The sub-family $\mathcal{R}^{(m)}$ of $\mathcal{S}^{(m)}$ is the set of m-fold symmetric bounded turning functions. More intuitively, an analytic function f of the form (4.1) belongs to the family $\mathcal{R}^{(m)}$ if and only if

$$f'(z) = p(z) \text{ with } p \in \mathcal{P}^{(m)},$$

where the set $\mathcal{P}^{(m)}$ is defined by

$$\mathcal{P}^{(m)} = \left\{ p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} c_{mk} z^{mk}, \quad (z \in \mathbb{D}) \right\}. \tag{4.2}$$

Theorem 4.1. *If $f \in f \in \mathcal{R}^{(3)}$, then*

$$|H_{4,1}(f)| \leq \frac{1}{49}.$$

Proof. Now, let $f \in \mathcal{R}^{(3)}$. Then there exists a function $\widetilde{g}(z) = z + d_4 z^4 + d_7 z^7 + \cdots \in \mathcal{S}^{*(3)}$ such that $\frac{z\widetilde{g}'(z)}{\widetilde{g}(z)} = f'(z)$. Since $f \in \mathcal{R}^{(3)}$, using the series form (4.1) for $m = 3$, we get

$$1 + 3d_4 z^3 + (6d_7 - 3d_4^2) z^6 + \cdots = 1 + 4a_4 z^3 + 7a_7 z^6 + \cdots.$$

Comparing the coefficients of z^3 and z^6 on both sides, we obtain

$$3d_4 = 4a_4, \quad 6d_7 - 3d_4^2 = 7a_7. \tag{4.3}$$

Since $\widetilde{g} \in \mathcal{S}^{*(3)}$, there exists a function g in $\mathcal{S}^*$ of the form (2.3) such that $\widetilde{g}(z) = \sqrt[3]{g(z^3)}$. Therefore

$$z + d_4 z^4 + d_7 z^7 + \cdots = z + \frac{1}{3} b_2 z^4 + \left(\frac{1}{3} b_3 - \frac{1}{9} b_2^2 \right) z^7 + \cdots.$$

Comparing the coefficients of z^4 and z^7, we get

$$d_4 = \frac{1}{3} b_2, \quad d_7 = \frac{1}{3} b_3 - \frac{1}{9} b_2^2. \tag{4.4}$$

Now from (4.3) and (4.4), it follows that

$$a_4 = \frac{b_2}{4}, \quad a_7 = \frac{1}{7}\left(2b_3 - b_2^2\right). \tag{4.5}$$

We observe that $a_2 = a_3 = a_5 = a_6 = 0$ for the function $f \in \mathcal{R}^{(3)}$. Also it is clear that $H_{4,1}(f) = a_4^2\left(a_4^2 - a_7\right)$. This implies that

$$|H_{4,1}(f)| = \frac{1}{56}\left| b_2^2 \left(b_3 - \frac{23}{32} b_2^2 \right) \right|.$$

Using Theorem 2.3 for $\lambda = \frac{23}{32} \in [5/8, 3/4]$, we have the required result. □

Theorem 4.2. *If $f \in f \in \mathcal{R}^{(2)}$, then*

$$|H_{4,1}(f)| \leq \frac{368}{2625}.$$

Proof. It is clear that for $f \in \mathcal{R}^{(2)}$ we have $a_2 = a_4 = a_6 = 0$. Consequently

$$H_{4,1}(f) := a_3a_5a_7 - a_3^3a_7 + a_3^2a_5^2 - a_5^3.$$

Since $f \in \mathcal{R}^{(2)}$, there exists a function $p \in \mathcal{P}^{(2)}$ such that $f'(z) = p(z)$. For $f \in \mathcal{R}^{(2)}$, using the series form (4.1) and (4.2) when $m = 2$, we can write

$$3a_3 = c_2,\ 5a_5 = c_4,\ 7a_7 = c_6.$$

Therefore

$$\begin{aligned} H_{4,1}(f) &= \frac{1}{105}c_2c_4c_6 - \frac{1}{189}c_2^3c_6 + \frac{1}{225}c_2^2c_4^2 - \frac{1}{125}c_4^3 \\ &= \frac{1}{105}\left(c_2c_6 - \frac{21}{25}c_4^2\right)\left(c_4 - \frac{5}{9}c_2^2\right). \end{aligned}$$

Using Lemma 2.1 and the triangle inequality, we get

$$|H_{4,1}(f)| \leq \frac{368}{2625}.$$

Hence the proof is complete. □

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