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GRADED POST-LIE ALGEBRA STRUCTURES, ROTA-BAXTER OPERATORS AND YANG-BAXTER EQUATIONS ON THE W-ALGEBRA $W(2,2)$

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Abstract. In this paper, we characterize the graded post-Lie algebra structures on the W-algebra $W(2,2)$. Furthermore, as applications, the homogeneous Rota-Baxter operators on $W(2,2)$ and solutions of the formal classical Yang-Baxter equation on $W(2,2) \ltimes_{\mathrm{ad}^*} W(2,2)^*$ are studied.

1. Introduction and preliminaries

Throughout the paper, denote by $\mathbb{C}, \mathbb{Z}$ the sets of complex numbers, integers respectively. For a fixed integer k, let $\mathbb{Z}_{>k} = \{t \in \mathbb{Z} \,|\, t > k\}$, $\mathbb{Z}_{<k} = \{t \in \mathbb{Z} \,|\, t < k\}$, $\mathbb{Z}_{\geqslant k} = \{t \in \mathbb{Z} \,|\, t \geqslant k\}$ and $\mathbb{Z}_{\leqslant k} = \{t \in \mathbb{Z} \,|\, t \leqslant k\}$. In this paper, we aim to determine the graded post-Lie algebra structures on W-algebra $W(2,2)$, and classify some Rota-Baxter operators on $W(2,2)$ and solutions of the formal Yang-Baxter equations on $W(2,2) \ltimes_{\mathrm{ad}^*} W(2,2)^*$. Now we recall some related concepts and facts as follows.

1.1. W-algebra $W(2,2)$

The W-algebra $W(2,2)$ is an infinite-dimensional Lie algebra with the $\mathbb{C}$-basis $\{L_m, H_m \mid m \in \mathbb{Z}\}$ and the Lie brackets are given by

$$[L_m, L_n] = (m-n)L_{m+n},$$
$$[L_m, H_n] = (m-n)H_{m+n},$$
$$[H_m, H_n] = 0, \ \forall m, n \in \mathbb{Z}.$$

A class of central extensions of $W(2,2)$ first introduced by [28] in their recent work on the classification of some simple vertex operator algebras, and then

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some scholars studied the theory on structures and representations of $W(2,2)$ or its central extensions, see [7, 12, 15, 19, 26] and so forth.

1.2. Post-Lie algebra

Post-Lie algebras were introduced around 2007 by B. Vallette [25], who found the structure in a purely operadic manner as the Koszul dual of a commutative trialgebra. Since then, post-Lie algebras have aroused the interest of a great many authors, see [1,4–6,9,10,17,18,23]. It should be pointed out that post-Lie algebras appear in many areas of mathematics and physics including the differential geometry [17], Lie groups [6,17], classical Yang-Baxter equation [1], Hopf algebra, classical r-matrices [11] and Rota-Baxter operators [13]. One of the most important problems in the study of post-Lie algebras is to find the post-Lie algebra structures on the (given) Lie algebras. For the finite-dimensional cases, in [18], the authors determined all post-Lie algebra structures on $sl(2,\mathbb{C})$ of special linear Lie algebra of order 2 and in [23] the authors studied the post-Lie algebra structures on the solvable Lie algebra $t(2,\mathbb{C})$ of the Lie algebra of 2×2 upper triangular matrices. For the infinite-dimensional cases, some classes of post-Lie algebra structures on the Witt algebra are considered by [21], and all commutative post-Lie algebra structures on the W-algebra $W(2,2)$ are given in [22]. We now turn to the definition of post-Lie algebra following reference [25].

Definition 1.1. A post-Lie algebra $(V,\circ,[,])$ is a vector space V over a field k equipped with two k-bilinear products $x\circ y$ and $[x,y]$ satisfying that $(V,[,])$ is a Lie algebra and

$$[x,y]\circ z = x\circ(y\circ z) - y\circ(x\circ z) - \langle x,y\rangle\circ z, \tag{1}$$

$$x\circ[y,z] = [x\circ y,z] + [y,x\circ z] \tag{2}$$

for all $x,y,z\in V$, where $\langle x,y\rangle = x\circ y - y\circ x$. We also say that $(V,\circ,[,])$ is a post-Lie algebra structure on the Lie algebra $(V,[,])$. If a post-Lie algebra $(V,\circ,[,])$ satisfies $x\circ y = y\circ x$ for all $x,y\in V$, then it is called a commutative post-Lie algebra.

Suppose that $(L,[,])$ is a Lie algebra. Two post-Lie algebras $(L,[,],\circ_1)$ and $(L,[,],\circ_2)$ on the Lie algebra L are called to be isomorphic if there is an automorphism τ of the Lie algebra $(L,[,])$ satisfies

$$\tau(x\circ_1 y) = \tau(x)\circ_2\tau(y), \forall x,y\in L.$$

By Proposition 2.5 of [17], we have the following result.

Proposition 1.2. *Let $(V,\circ,[,])$ be a post-Lie algebra defined by Definition 1.1. Then the following product*

$$\{x,y\} \triangleq \langle x,y\rangle + [x,y], \tag{3}$$

induces a Lie algebra structure on V, where $\langle x,y\rangle = x\circ y - y\circ x$. Furthermore, if two post-Lie algebras $(V,\circ_1,[,])$ and $(V,\circ_2,[,])$ on the same Lie algebra $(V,[,])$

are isomorphic, then the two induced Lie algebras $(V, \{,\}_1)$ *and* $(V, \{,\}_2)$ *are isomorphic.*

Remark 1.3. The left multiplications of the post-Lie algebra $(V, [,], \circ)$ are denoted by $\mathcal{L}(x)$, i.e., we have $\mathcal{L}(x)(y) = x \circ y$ for all $x, y \in V$. By (2), we see that all operators $\mathcal{L}(x)$ are Lie algebra derivations of the Lie algebra $(V, [,])$.

1.3. Rota-Baxter operator

As a matter of fact, the Rota-Baxter operators were originally defined on associative algebras by G. Baxter to solve an analytic formula in probability [2] and then developed by the Rota school [20]. These operators have showed up in many areas in mathematics and mathematical physics (see [8, 13, 14, 24] and the references therein). Now let us recall the definition of Rota-Baxter operator.

Definition 1.4. Let L be a complex Lie algebra. A Rota-Baxter operator of weight $\lambda \in \mathbb{C}$ is a linear map $R : L \to L$ satisfying

$$[R(x), R(y)] = R([R(x), y] + [x, R(y)]) + \lambda R([x, y]), \ \forall x, y \in L. \tag{4}$$

Note that if R is a Rota-Baxter operator of weight $\lambda \neq 0$, then $\lambda^{-1}R$ is a Rota-Baxter operator of weight 1. Therefore, one only needs to consider Rota-Baxter operators of weight 0 and 1.

1.4. Yang-Baxter equation

The Yang-Baxter equation first appeared in theoretical physics and statistical mechanics in the works of Yang [27] and Baxter [3] and it has led to several interesting applications in quantum groups and Hopf algebras, knot theory, tensor categories and integrable systems [16]. Let $\mathfrak{g}$ be a Lie algebra. An element $r = \sum_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$ is called a solution of the classical Yang-Baxter equation (CYBE) on $\mathfrak{g}$ if r satisfies

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \quad \text{in} \ \ U(\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}),$$

where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ and

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1, \ r_{13} = \sum_i a_i \otimes 1 \otimes b_i, \ r_{23} = \sum_i 1 \otimes a_i \otimes b_i.$$

For any $r = \sum_i a_i \otimes b_i$, set

$$r^{21} = \sum_i b_i \otimes a_i.$$

It is obvious that r is skew-symmetric if and only if $r = -r^{21}$.

Our results can be briefly summarized as follows: In Section 2, we classify the graded post-Lie algebra structures on the W-algebra $W(2, 2)$, and then we obtain the induced graded Lie algebras. In Section 3, we give the induced Rota-Baxter operators of weight 1 from the post-Lie algebras on $W(2, 2)$. In

Section 4, we give some solutions of the formal classical Yang-Baxter equation on $W(2,2) \ltimes_{\mathrm{ad}^*} W(2,2)^*$.

2. The graded post-Lie algebra structure on the W-algebra $W(2,2)$

Recently the author in [22] proved that any commutative post-Lie algebra structure on the W-algebra $W(2,2)$ is trivial (namely, $x \circ y = 0$ for all $x, y \in W(2,2)$). We now will dedicate on the study of the noncommutative cases. Since the W-algebra $W(2,2)$ is graded, we suppose that the post-Lie algebra structure on the W-algebra $W(2,2)$ to be graded. Namely, we mainly consider the post-Lie algebra structure on W-algebra $W(2,2)$ which satisfies

$$L_m \circ L_n = \phi(m,n)L_{m+n}, \tag{5}$$

$$L_m \circ H_n = \varphi(m,n)H_{m+n}, \tag{6}$$

$$H_m \circ L_n = \theta(m,n)H_{m+n}, \tag{7}$$

$$H_m \circ H_n = 0 \tag{8}$$

for all $m, n \in \mathbb{Z}$, where ϕ, φ, θ are complex-valued functions on $\mathbb{Z} \times \mathbb{Z}$.

Lemma 2.1 (see [12]). *Denote by* $\mathrm{Der}(W(2,2))$ *and by* $\mathrm{Inn}(W(2,2))$ *the space of derivations and the space of inner derivations of* $W(2,2)$ *respectively. Then*

$$\mathrm{Der}(W(2,2)) = \mathrm{Inn}(W(2,2)) \oplus \mathbb{C}D,$$

where D is an outer derivation defined by $D(L_m) = 0$, $D(H_m) = H_m$ for all $m \in \mathbb{Z}$.

Lemma 2.2. *There exists a graded post-Lie algebra structure on $W(2,2)$ satisfying* (5)-(8) *if and only if there are complex-valued functions f, g on $\mathbb{Z}$ and a complex number μ such that*

$$\phi(m,n) = (m-n)f(m), \tag{9}$$

$$\varphi(m,n) = (m-n)f(m) + \delta_{m,0}\mu, \tag{10}$$

$$\theta(m,n) = (m-n)g(m), \tag{11}$$

$$(m-n)(f(m+n) + f(m)f(m+n) + f(n)f(m+n) - f(m)f(n)) = 0, \tag{12}$$

$$(m-n)(g(m+n) + f(m)g(m+n) + g(n)g(m+n) - f(m)g(n)) = 0, \tag{13}$$

$$(m-n)(f(m) + f(n) + 1)\delta_{m+n,0}\mu = 0. \tag{14}$$

Proof. Suppose that there exists a graded post-Lie algebra structure satisfying (5)-(8) on $W(2,2)$. By Remark 1.3, $\mathcal{L}(x)$ is a derivation of $W(2,2)$. It follows by Lemma 2.1 that there are a linear map ψ from $W(2,2)$ into itself and a linear function ρ on $W(2,2)$ such that

$$x \circ y = (\mathrm{ad}\psi(x) + \rho(x)D)(y) = [\psi(x), y] + \rho(x)D(y),$$

where D is given by Lemma 2.1. This, together with (5)-(8), gives that

$$L_m \circ L_n = [\psi(L_m), L_n] = \phi(m,n)L_{m+n}, \tag{15}$$

$$L_m \circ H_n = [\psi(L_m), H_n] + \rho(L_m)H_n = \varphi(m,n)H_{m+n}, \tag{16}$$

$$H_m \circ L_n = [\psi(H_m), L_n] = \theta(m,n)H_{m+n}, \tag{17}$$

$$H_m \circ H_n = [\psi(H_m), H_n] + \rho(H_m)H_n = 0. \tag{18}$$

Let

$$\psi(L_m) = \sum_{i\in\mathbb{Z}} a_i^{(m)} L_i + \sum_{i\in\mathbb{Z}} b_i^{(m)} H_i \text{ and } \psi(H_m) = \sum_{i\in\mathbb{Z}} c_i^{(m)} L_i + \sum_{i\in\mathbb{Z}} d_i^{(m)} H_i,$$

where $a_i^{(m)}$, $b_i^{(m)}$, $c_i^{(m)}$, $d_i^{(m)} \in \mathbb{C}$ for all $i \in \mathbb{Z}$. Then we have by (15)-(18) that

$$\sum_{i\in\mathbb{Z}} (i-n)a_i^{(m)} L_{i+n} + \sum_{i\in\mathbb{Z}} (i-m)b_i^{(m)} H_{i+n} = \phi(m,n)L_{m+n},$$
$$\sum_{i\in\mathbb{Z}} (i-n)a_i^{(m)} H_{i+n} + \rho(L_m)H_n = \varphi(m,n)H_{m+n},$$
$$\sum_{i\in\mathbb{Z}} (i-n)c_i^{(m)} L_{i+n} - \sum_{i\in\mathbb{Z}} (n-i)d_i^{(m)} H_{i+n} = \theta(m,n)H_{m+n},$$
$$\sum_{i\in\mathbb{Z}} (i-n)c_i^{(m)} H_{i+n} + \rho(H_m)H_n = 0.$$

It is not difficult to see by the above equations that (9), (10) and (11) are established with

$$f(m) = a_m^{(m)},\ g(m) = d_m^{(m)},\ \mu = \rho(L_0) = \varphi(0,0).$$

By a simple computation, we see that (1) with $(x, y, z) = (L_m, L_n, L_k)$ holds if and only if the following equation holds:

$$\begin{aligned} &(m-n)(m+n-k)f(m+n) \\ &= (n-k)(m-n-k)f(n)f(m) - (m-k)(n-m-k)f(m)f(n) \\ &\quad - (m-n)(m+n-k)f(m)f(m+n) + (n-m)(n+m-k)f(n)f(m+n). \end{aligned} \tag{19}$$

The above equation can be viewed as a polynomial equation in k, then we see that (19) holds if and only if (12) holds. Similarly, one can see that (1) with $(x, y, z) = (L_m, H_n, L_k)$ or (H_n, L_m, L_k) holds if and only if the following equation holds:

$$\begin{aligned} &(m-n)(m+n-k)g(m+n) \\ &= (n-k)\left((m-n-k)f(m)+\delta_{m,0}\mu\right)g(n) - (m-k)(n-m-k)f(m)g(n) \\ &\quad - ((m-n)f(m) + \delta_{m,0}\mu - (n-m)g(n))(n+m-k)g(m+n). \end{aligned} \tag{20}$$

Viewing (20) as a polynomial equation in k, we see that (20) holds if and only if the coefficients of degrees $0, 1$ and 2, respectively, are the same on both sides of the polynomial equation (20), i.e.,

$$\begin{aligned} &(m-n)(m+n)(g(m+n) + f(m)g(m+n) + g(n)g(m+n) - f(m)g(n)) \\ &= n\delta_{m,0}(ng(n) - (m+n)g(m+n)), \end{aligned}$$

$$(n-m)(g(m+n)+f(m)g(m+n)+g(n)g(m+n)-f(m)g(n))$$
$$=\delta_{m,0}\mu(g(m+n)-g(n))$$

and $0=f(m)g(n)-f(m)g(n)$ hold. Note that $n\delta_{m,0}(ng(n)-(m+n)g(m+n))=0$ and $\delta_{m,0}\mu(g(m+n)-g(n))=0$. This implies that (20) holds if and only if (13) holds. In a similar way as above, we obtain that (1) with $(x,y,z)=(L_m,L_n,H_k)$ holds if and only if (13) and (14) hold. It has been proved that (9)-(14) hold.

Conversely, suppose that there are $\mu\in\mathbb{C}$ and complex-valued functions f,g on $\mathbb{Z}$ satisfying (9)-(14). It is easy to verify that (2) holds by (9)-(11). We have to prove that (1) holds for all $x,y,z\in W(2,2)$. We observe that this is obviously right when at least two elements in x,y,z belong to the set $\{H_k,k\in\mathbb{Z}\}$. Next, the discussion in the above paragraph tells us that (1) with $(x,y,z)=(L_m,L_n,L_k)$ holds by (12); (1) with $(x,y,z)=(L_m,H_n,L_k)$ or (H_n,L_m,L_k) holds by (13); and (1) with $(x,y,z)=(L_m,L_n,H_k)$ holds by (13) and (14). The proof is completed. □

For complex-valued functions f, g on $\mathbb{Z}$, we denote I, J, M and N by

$$I=\{m\in\mathbb{Z}\,|\,f(m)=0\},\quad J=\{m\in\mathbb{Z}\,|\,f(m)=-1\},$$
$$M=\{n\in\mathbb{Z}\,|\,g(n)=0\},\quad N=\{n\in\mathbb{Z}\,|\,g(n)=-1\}.$$

Lemma 2.3. *Suppose that f,g are complex-valued functions on $\mathbb{Z}$. Then* (12) *and* (13) *hold if and only if the following statements hold*:

(i) $I\cup J=M\cup N=\mathbb{Z}\setminus\{0\}$;
(ii) $m,n\in I\Rightarrow m+n\in I$ *and* $m,n\in J\Rightarrow m+n\in J$ *for* $m\neq n$;
(iii) $m\in I,n\in M\Rightarrow m+n\in M$, *and* $m\in J,n\in N\Rightarrow m+n\in N$ *for all* $m\neq n$.

Proof. We first prove the "only if" part. Letting $n=0$ in (12), we have $m(f(m)+f(m)^2)=0$. Thus, for $m\neq 0$, $f(m)=0$ or $f(m)=-1$. Similarly, by letting $m=0$ in (13), it follows that $g(n)=0$ or $g(n)=-1$ for $n\neq 0$. This proves (i). Now we chose a pair of $m,n\in\mathbb{Z}$ with $m\neq n$, then by (12) and (13) we see that

$$f(m+n)+f(m)f(m+n)+f(n)f(m+n)-f(m)f(n)=0, \tag{21}$$
$$g(m+n)+f(m)g(m+n)+g(n)g(m+n)-f(m)g(n)=0. \tag{22}$$

According to (21) and (22), it is easy to verify that (ii) and (iii) hold.

Next, we prove the "if" part. In fact, if $m=n$, then (12) and (13) are obvious. Now we suppose that $m\neq n$. In this case, if $m=0$ then $n\neq 0$, then we also can obtain (12) and (13) since $f(n),g(n)\in\{0,-1\}$. Finally, we assume that $m\neq n$ with $m,n\neq 0$. By (i), we know $f(m),f(n),g(m),g(n)\in\{0,-1\}$. It is easy to verify that (12) and (13) hold one by one according to values of f,g. □

Lemma 2.4. *Suppose that f, g are complex-valued functions on $\mathbb{Z}$. Then* (12) *and* (13) *hold if and only if f and g meet one of the situations listed in Table 2.*

Proof. The proof of the "if" direction can be directly verified. We now prove the "only if" direction. In view of f satisfies (12), by Theorem 2.4 of [21] we know that f is determined by Table 1. Next, we discuss the cases of $g(1), g(-1), g(2)$

TABLE 1. Values of f satisfying (12), where $a \in \mathbb{C}$.

Cases	$f(n)$
$\mathcal{P}_1$	$f(\mathbb{Z}) = 0$
$\mathcal{P}_2$	$f(\mathbb{Z}) = -1$
$\mathcal{P}_3^a$	$f(\mathbb{Z}_{>0}) = -1$, $f(\mathbb{Z}_{<0}) = 0$ and $f(0) = a$
$\mathcal{P}_4^a$	$f(\mathbb{Z}_{>0}) = 0$, $f(\mathbb{Z}_{<0}) = -1$ and $f(0) = a$
$\mathcal{P}_5$	$f(\mathbb{Z}_{\geqslant 2}) = -1$ and $f(\mathbb{Z}_{\leqslant 1}) = 0$
$\mathcal{P}_6$	$f(\mathbb{Z}_{\geqslant 2}) = 0$ and $f(\mathbb{Z}_{\leqslant 1}) = -1$
$\mathcal{P}_7$	$f(\mathbb{Z}_{\geqslant -1}) = 0$ and $f(\mathbb{Z}_{\leqslant -2}) = -1$
$\mathcal{P}_8$	$f(\mathbb{Z}_{\geqslant -1}) = -1$ and $f(\mathbb{Z}_{\leqslant -2}) = 0$

and $g(-2)$. Lemma 2.3(i) tells us that $g(1)$, $g(-1)$, $g(2)$, $g(-2) \in \{-1, 0\}$, and so that there are $2^4 = 16$ cases for $g(x)$ where $x = \pm 1, \pm 2$. Using Lemma 2.3(ii) and (iii), it follows by a simple discussion that 30 cases listed in Tabular 2 are established. ☐

Lemma 2.5. *Let $(\mathcal{P}(\phi_i, \varphi_i, \theta_i), \circ_i)$, $i = 1, 2$ be two algebras with the same linear space as $W(2,2)$ and equipped with $\mathbb{C}$-bilinear products $x \circ_i y$ such that*

$$L_m \circ_i L_n = \phi_i(m,n) L_{m+n}, \qquad L_m \circ_i H_n = \varphi_i(m,n) H_{m+n},$$
$$H_m \circ_i L_n = \theta_i(m,n) H_{m+n}, \qquad H_m \circ_i H_n = 0$$

for all $m, n \in \mathbb{Z}$, where ϕ_i, φ_i, θ_i, $i = 1, 2$ are complex-valued functions on $\mathbb{Z} \times \mathbb{Z}$. Furthermore, let $\tau : \mathcal{P}(\phi_1, \varphi_1, \theta_1) \to \mathcal{P}(\phi_2, \varphi_2, \theta_2)$ be a linear map determined by $\tau(L_m) = -L_{-m}$, $\tau(H_m) = -H_{-m}$ for all $m \in \mathbb{Z}$. In addition, suppose that $(\mathcal{P}(\phi_1, \varphi_1, \theta_1), [,], \circ_1)$ is a post-Lie algebra. Then $(\mathcal{P}(\phi_2, \varphi_2, \theta_2), [,], \circ_2)$ is a post-Lie algebra and τ is a isomorphism from $\mathcal{P}(\phi_1, \varphi_1, \theta_1)$ to $\mathcal{P}(\phi_2, \varphi_2, \theta_2)$ if and only if

$$\begin{cases} \phi_2(m,n) = -\phi_1(-m,-n), \\ \varphi_2(m,n) = -\varphi_1(-m,-n), \\ \theta_2(m,n) = -\theta_1(-m,-n). \end{cases} \tag{23}$$

Proof. Clearly, τ is a Lie automorphism of the W-algebra $W(2,2)$. Suppose that $(\mathcal{P}(\phi_2, \varphi_2, \theta_2), [,], \circ_2)$ is a post-Lie algebra and τ is a post-Lie isomorphism from $\mathcal{P}(\phi_1, \varphi_1, \theta_1)$ to $\mathcal{P}(\phi_2, \varphi_2, \theta_2)$. Then from

$$\tau(L_m \circ_1 L_n) = -\phi_1(m,n) L_{-(m+n)},$$

$$\tau(L_m \circ_1 H_n) = -\varphi_1(m,n)H_{-(m+n)},$$
$$\tau(H_m \circ_1 L_n) = -\theta_1(m,n)H_{-(m+n)}$$

and

$$\tau(L_m) \circ_2 \tau(L_n) = \phi_2(-m,-n)L_{-(m+n)},$$

TABLE 2. Values of f and g satisfying (12) and (13), where $a, b \in \mathbb{C}$.

Cases	$f(n)$ from Table 1	$g(n)$
$\mathcal{W}_1^{\mathcal{P}_1}$	$\mathcal{P}_1$	$g(\mathbb{Z}) = 0$
$\mathcal{W}_2^{\mathcal{P}_1}$	$\mathcal{P}_1$	$g(\mathbb{Z}) = -1$
$\mathcal{W}_1^{\mathcal{P}_2}$	$\mathcal{P}_2$	$g(\mathbb{Z}) = 0$
$\mathcal{W}_2^{\mathcal{P}_2}$	$\mathcal{P}_2$	$g(\mathbb{Z}) = -1$
$\mathcal{W}_1^{\mathcal{P}_3^a}$	$\mathcal{P}_3^a$	$g(\mathbb{Z}) = 0,$
$\mathcal{W}_2^{\mathcal{P}_3^a}$	$\mathcal{P}_3^a$	$g(\mathbb{Z}) = -1$
$\mathcal{W}_3^{\mathcal{P}_3^{a,b}}$	$\mathcal{P}_3^a$	$g(\mathbb{Z}_{>0}) = -1, g(\mathbb{Z}_{<0}) = 0, g(0) = b$
$\mathcal{W}_4^{\mathcal{P}_3^a}$	$\mathcal{P}_3^a$	$g(\mathbb{Z}_{\geqslant 2}) = -1,\ g(\mathbb{Z}_{\leqslant 1}) = 0$
$\mathcal{W}_5^{\mathcal{P}_3^a}$	$\mathcal{P}_3^a$	$g(\mathbb{Z}_{\geqslant -1}) = -1,\ g(\mathbb{Z}_{\leqslant -2}) = 0$
$\mathcal{W}_1^{\mathcal{P}_4^a}$	$\mathcal{P}_4^a$	$g(\mathbb{Z}) = 0$
$\mathcal{W}_2^{\mathcal{P}_4^a}$	$\mathcal{P}_4^a$	$g(\mathbb{Z}) = -1$
$\mathcal{W}_3^{\mathcal{P}_4^{a,b}}$	$\mathcal{P}_4^a$	$g(\mathbb{Z}_{>0}) = 0, g(\mathbb{Z}_{<0}) = -1, g(0) = b$
$\mathcal{W}_4^{\mathcal{P}_4^a}$	$\mathcal{P}_4^a$	$g(\mathbb{Z}_{\geqslant -1}) = 0,\ g(\mathbb{Z}_{\leqslant -2}) = -1$
$\mathcal{W}_5^{\mathcal{P}_4^a}$	$\mathcal{P}_4^a$	$g(\mathbb{Z}_{\geqslant 2}) = 0,\ g(\mathbb{Z}_{\leqslant 1}) = -1$
$\mathcal{W}_1^{\mathcal{P}_5}$	$\mathcal{P}_5$	$g(\mathbb{Z}) = 0$
$\mathcal{W}_2^{\mathcal{P}_5}$	$\mathcal{P}_5$	$g(\mathbb{Z}) = -1$
$\mathcal{W}_3^{\mathcal{P}_5}$	$\mathcal{P}_5$	$g(\mathbb{Z}_{\geqslant 2}) = -1, g(\mathbb{Z}_{\leqslant 1}) = 0$
$\mathcal{W}_4^{\mathcal{P}_5}$	$\mathcal{P}_5$	$g(\mathbb{Z}_{>0}) = -1, g(\mathbb{Z}_{\leqslant 0}) = 0$
$\mathcal{W}_1^{\mathcal{P}_6}$	$\mathcal{P}_6$	$g(\mathbb{Z}) = 0$
$\mathcal{W}_2^{\mathcal{P}_6}$	$\mathcal{P}_6$	$g(\mathbb{Z}) = -1$
$\mathcal{W}_3^{\mathcal{P}_6}$	$\mathcal{P}_6$	$g(\mathbb{Z}_{\geqslant 2}) = 0, g(\mathbb{Z}_{\leqslant 1}) = -1$
$\mathcal{W}_4^{\mathcal{P}_6}$	$\mathcal{P}_6$	$g(\mathbb{Z}_{>0}) = 0, g(\mathbb{Z}_{\leqslant 0}) = -1$
$\mathcal{W}_1^{\mathcal{P}_7}$	$\mathcal{P}_7$	$g(\mathbb{Z}) = 0$
$\mathcal{W}_2^{\mathcal{P}_7}$	$\mathcal{P}_7$	$g(\mathbb{Z}) = -1$
$\mathcal{W}_3^{\mathcal{P}_7}$	$\mathcal{P}_7$	$g(\mathbb{Z}_{\geqslant -1}) = 0, g(\mathbb{Z}_{\leqslant -2}) = -1$
$\mathcal{W}_4^{\mathcal{P}_7}$	$\mathcal{P}_7$	$g(\mathbb{Z}_{\geqslant 0}) = 0,\ g(\mathbb{Z}_{<0}) = -1$
$\mathcal{W}_1^{\mathcal{P}_8}$	$\mathcal{P}_8$	$g(\mathbb{Z}) = 0,$
$\mathcal{W}_2^{\mathcal{P}_8}$	$\mathcal{P}_8$	$g(\mathbb{Z}) = -1,$
$\mathcal{W}_3^{\mathcal{P}_8}$	$\mathcal{P}_8$	$g(\mathbb{Z}_{\geqslant -1}) = -1,\ g(\mathbb{Z}_{\leqslant -2}) = 0,$
$\mathcal{W}_4^{\mathcal{P}_8}$	$\mathcal{P}_8$	$g(\mathbb{Z}_{\geqslant 0}) = -1,\ g(\mathbb{Z}_{<0}) = 0.$

$$\tau(L_m) \circ_2 \tau(H_n) = \varphi_2(-m, -n)H_{-(m+n)},$$
$$\tau(H_m) \circ_2 \tau(L_n) = \theta_2(-m, -n)H_{-(m+n)}$$

we see that (23) holds. Conversely, suppose that (23) holds. Then, by using Lemma 2.2 and $(\mathcal{P}(\phi_1, \varphi_1, \theta_1), [,], \circ_1)$ is a post-Lie algebra, we know that there are complex-valued functions f_1, g_1 on $\mathbb{Z}$ and a complex number μ_1 such that

$$\phi_1(m, n) = (m - n)f_1(m), \tag{24}$$
$$\varphi_1(m, n) = (m - n)f_1(m) + \delta_{m,0}\mu_1, \tag{25}$$
$$\theta_1(m, n) = (m - n)g_1(m), \tag{26}$$

$$(m-n)(f_1(m+n)+f_1(m)f_1(m+n)+f_1(n)f_1(m+n)-f_1(m)f_1(n)) = 0, \tag{27}$$
$$(n-m)(g_1(m+n)+f_1(m)g_1(m+n)+g_1(n)g_1(m+n)-f_1(m)g_1(n)) = 0, \tag{28}$$
$$(m-n)(f_1(m)+f_1(n)+1)\delta_{m+n,0}\mu_1 = 0 \tag{29}$$

for all $m, n \in \mathbb{Z}$. It follows by (24), (25), (26) and (23) that

$$\phi_2(m, n) = -\phi_1(-m, -n) = -(n - m)f_1(-m) = (m - n)f_2(m), \tag{30}$$
$$\begin{aligned}\varphi_2(m, n) &= -\varphi_1(-m, -n) = -(n - m)f_1(-m) - \delta_{m,0}\mu_1 \\ &= (m - n)f_2(m) + \delta_{m,0}\mu_2,\end{aligned} \tag{31}$$
$$\theta_2(m, n) = -\theta_1(-m, -n) = -(n - m)g_1(-m) = (m - n)g_2(m), \tag{32}$$

where f_2, g_2 are complex-valued functions on $\mathbb{Z}$ and μ_2 is a complex number determined by $f_2(m) = f_1(-m)$, $g_2(m) = g_1(-m)$ and $\mu_2 = -\mu_1$.

Furthermore, by (27), (28) and (29) with $f_2(m) = f_1(-m)$, $\mu_2 = -\mu_1$ we obtain

$$(m-n)(f_2(m+n)+f_2(m)f_2(m+n)+f_2(n)f_2(m+n)-f_2(m)f_2(n)) = 0, \tag{33}$$
$$(n-m)(g_2(m+n)+f_2(m)g_2(m+n)+f_2(n)g_2(m+n)-f_2(m)g_2(n)) = 0, \tag{34}$$
$$(m-n)(f_2(m)+f_2(n)+1)\delta_{m+n,0}\mu_2 = 0. \tag{35}$$

In view of (30)-(35), it follows by Lemma 2.2 that $\mathcal{P}(\phi_2, \varphi_2, \theta_2)$ is a post-Lie algebra. The remainder is to prove that τ is a isomorphism between post-Lie algebras. But one has

$$\tau(L_m \circ_1 L_n) = -\phi_1(m, n)L_{-(m+n)} = \phi_2(-m, -n)L_{-(m+n)} = \tau(L_m) \circ_2 \tau(L_n),$$
$$\tau(L_m \circ_1 H_n) = -\varphi_1(m, n)H_{-(m+n)} = \varphi_2(-m, -n)H_{-(m+n)} = \tau(L_m) \circ_2 \tau(H_n),$$
$$\tau(H_m \circ_1 L_n) = -\theta_1(m, n)H_{-(m+n)} = \theta_2(-m, -n)H_{-(m+n)} = \tau(H_m) \circ_2 \tau(L_n),$$

and $\tau(H_m \circ_1 H_n) = 0 = \tau(H_m) \circ_2 \tau(H_n)$, which completes the proof. □

We now can prove the main theorem of this paper as follows.

Theorem 2.6. *A graded post-Lie algebra structure on $W(2, 2)$ satisfying* (5)-(8) *must be one of the following types* (*in every case* $H_m \circ H_n = 0$) *for all* $m, n \in \mathbb{Z}$,

$(\mathcal{W}_1^{\mathcal{P}_1})$: $L_m \circ_1^{\mathcal{P}_1} L_n = 0$, $L_m \circ_1^{\mathcal{P}_1} H_n = 0$, $H_m \circ_1^{\mathcal{P}_1} L_n = 0$;

$(\mathcal{W}_2^{\mathcal{P}_1}):\ L_m \circ_2^{\mathcal{P}_1} L_n = 0,\ L_m \circ_2^{\mathcal{P}_1} H_n = 0,\ H_m \circ_2^{\mathcal{P}_1} L_n = (n-m)H_{m+n};$

$(\mathcal{W}_1^{\mathcal{P}_2}):\ L_m \circ_1^{\mathcal{P}_2} L_n = (n-m)L_{m+n},\ L_m \circ_1^{\mathcal{P}_2} H_n = (n-m)H_{m+n},\ H_m \circ_1^{\mathcal{P}_2} L_n = 0;$

$(\mathcal{W}_2^{\mathcal{P}_2}):\ L_m \circ_2^{\mathcal{P}_2} L_n = (n-m)L_{m+n},\ L_m \circ_2^{\mathcal{P}_2} H_n = (n-m)H_{m+n},$
$H_m \circ_2^{\mathcal{P}_2} L_n = (n-m)H_{m+n};$

$(\mathcal{W}_{i,\mu}^{\mathcal{P}_3}):\ i = 1, 2, \ldots, 5,$

$$L_m \circ_{i,\mu}^{\mathcal{P}_3^a} L_n = \begin{cases} (n-m)L_{m+n}, & m > 0, \\ -naL_n, & m = 0, \\ 0, & m < 0; \end{cases}$$

$$L_m \circ_{i,\mu}^{\mathcal{P}_3^a} H_n = \begin{cases} (n-m)H_{m+n}, & m > 0, \\ (-na+\mu)H_n, & m = 0, \\ 0, & m < 0; \end{cases}$$

$$\begin{aligned} H_m \circ_{i,\mu}^{\mathcal{P}_3^{a,b}} L_n = {} & \delta_{i,2}(n-m)H_{m+n} \\ & + \delta_{i,3} \begin{cases} (n-m)H_{m+n}, & m > 0, \\ -nbH_n, & m = 0, \\ 0, & m < 0; \end{cases} \\ & + \delta_{i,4} \begin{cases} (n-m)H_{m+n}, & m \geqslant 2, \\ 0, & m \leqslant 1; \end{cases} \\ & + \delta_{i,5} \begin{cases} (n-m)H_{m+n}, & m \geqslant -1, \\ 0, & m \leqslant -2; \end{cases} \end{aligned}$$

$(\mathcal{W}_{i,\mu}^{\mathcal{P}_4^a}):\ i = 1, 2, \ldots, 5,$

$$L_m \circ_{i,\mu}^{\mathcal{P}_4^a} L_n = \begin{cases} (n-m)L_{m+n}, & m < 0, \\ -naL_n, & m = 0, \\ 0, & m > 0; \end{cases}$$

$$L_m \circ_{i,\mu}^{\mathcal{P}_4^a} H_n = \begin{cases} (n-m)H_{m+n}, & m < 0, \\ (-na+\mu)H_n, & m = 0, \\ 0, & m > 0; \end{cases}$$

$$\begin{aligned} H_m \circ_{i,\mu}^{\mathcal{P}_4^{a,b}} L_n = {} & \delta_{i,2}(n-m)H_{n+m} \\ & + \delta_{i,3} \begin{cases} (n-m)H_{m+n}, & m < 0, \\ -nbH_n, & m = 0, \\ 0, & m > 0; \end{cases} \\ & + \delta_{i,4} \begin{cases} (n-m)H_{m+n}, & m \leqslant -2, \\ 0, & m \geqslant -1; \end{cases} \\ & + \delta_{i,5} \begin{cases} (n-m)H_{m+n}, & m \leqslant 1, \\ 0, & m \geqslant 2; \end{cases} \end{aligned}$$

$(\mathcal{W}_j^{\mathcal{P}_5}):\ j=1,\dots,4,$

$$L_m \circ_j^{\mathcal{P}_5} L_n = \begin{cases} (n-m)L_{m+n}, & m \geqslant 2,\\ 0, & m \leqslant 1; \end{cases}$$

$$L_m \circ_j^{\mathcal{P}_5} H_n = \begin{cases} (n-m)H_{m+n}, & m \geqslant 2,\\ 0, & m \leqslant 1; \end{cases}$$

$$\begin{aligned} H_m \circ_j^{\mathcal{P}_5} L_n = {} & \delta_{j,2}(n-m)H_{m+n}\\ & + \delta_{j,3}\begin{cases} (n-m)H_{m+n}, & m \geqslant 2,\\ 0, & m \leqslant 1; \end{cases}\\ & + \delta_{j,4}\begin{cases} (n-m)H_{m+n}, & m > 0,\\ 0, & m \leqslant 0; \end{cases} \end{aligned}$$

$(\mathcal{W}_j^{\mathcal{P}_6}):\ j=1,\dots,4,$

$$L_m \circ_j^{\mathcal{P}_6} L_n = \begin{cases} (n-m)L_{m+n}, & m \leqslant 1,\\ 0, & m \geqslant 2; \end{cases}$$

$$L_m \circ_j^{\mathcal{P}_6} H_n = \begin{cases} (n-m)H_{m+n}, & m \leqslant 1,\\ 0, & m \geqslant 2; \end{cases}$$

$$\begin{aligned} H_m \circ_j^{\mathcal{P}_6} L_n = {} & \delta_{j,2}(n-m)H_{m+n}\\ & + \delta_{j,3}\begin{cases} (n-m)H_{m+n}, & m \leqslant 1,\\ 0, & m \geqslant 2; \end{cases}\\ & + \delta_{j,4}\begin{cases} (n-m)H_{m+n}, & m \leqslant 0,\\ 0, & m > 0; \end{cases} \end{aligned}$$

$(\mathcal{W}_j^{\mathcal{P}_7}):\ j=1,\dots,4,$

$$L_m \circ_j^{\mathcal{P}_7} L_n = \begin{cases} (n-m)L_{m+n}, & m \leqslant -2,\\ 0, & m \geqslant -1; \end{cases}$$

$$L_m \circ_j^{\mathcal{P}_7} H_n = \begin{cases} (n-m)H_{m+n}, & m \leqslant -2,\\ 0, & m \geqslant -1; \end{cases}$$

$$\begin{aligned} H_m \circ_j^{\mathcal{P}_7} L_n = {} & \delta_{j,2}(n-m)H_{m+n}\\ & + \delta_{j,3}\begin{cases} (n-m)H_{m+n}, & m \leqslant -2,\\ 0, & m \geqslant -1; \end{cases}\\ & + \delta_{j,4}\begin{cases} (n-m)H_{m+n}, & m < 0,\\ 0, & m \geqslant 0; \end{cases} \end{aligned}$$

$(\mathcal{W}_j^{\mathcal{P}_8}):\ j=1,\dots,4,$

$$L_m \circ_j^{\mathcal{P}_8} L_n = \begin{cases} (n-m)L_{m+n}, & m \geqslant -1,\\ 0, & m \leqslant -2; \end{cases}$$

$$L_m \circ_j^{\mathcal{P}_8} H_n = \begin{cases} (n-m)H_{m+n}, & m \geqslant -1, \\ 0, & m \leqslant -2; \end{cases}$$

$$H_m \circ_j^{\mathcal{P}_8} L_n = \delta_{j,2}(n-m)H_{m+n} + \delta_{j,3}\begin{cases} (n-m)H_{m+n}, & m \geqslant -1, \\ 0, & m \leqslant -2; \end{cases} + \delta_{j,4}\begin{cases} (n-m)H_{m+n}, & m \geqslant 0, \\ 0, & m < 0; \end{cases}$$

where $a, b, \mu \in \mathbb{C}$. Conversely, the above types are all the graded post-Lie algebra structure satisfying (5)-(8) *on* $W(2,2)$*. Furthermore, the post-Lie algebras $\mathcal{W}_i^{\mathcal{P}_3^a}$, $\mathcal{W}_j^{\mathcal{P}_5}$, $\mathcal{W}_j^{\mathcal{P}_6}$ and $\mathcal{W}_{i,\mu}^{\mathcal{P}_4^a}$ are isomorphic to the post-Lie algebras $\mathcal{W}_i^{\mathcal{P}_4^a}$, $\mathcal{W}_j^{\mathcal{P}_7}$, $\mathcal{W}_j^{\mathcal{P}_8}$ and $\mathcal{W}_{i,\mu}^{\mathcal{P}_3^a}$, $i \in \{1,2,3,4,5\}$ and $j \in \{1,2,3,4\}$, respectively, and other post-Lie algebras are not mutually isomorphic.*

Proof. Suppose that $(W, [,], \circ)$ is a post-Lie algebra structure satisfying (5)-(8) on $W(2,2)$. By Lemma 2.2, there are complex-valued functions f, g on $\mathbb{Z}$ and $\mu \in \mathbb{C}$ such that (9)-(14) hold. Below two cases of μ are discussed.

Case (I) $\mu = 0$. In this case, f and g satisfy (12) and (13) but (14) is disappeared due to $\mu = 0$. By Lemma 2.4, the 30 cases of f, g listed in Table 2 are established. Thus, by (9)-(11) with $\mu = 0$, we know that the graded post-Lie algebra structure on $W(2,2)$ algebra must be one of the above 30 types. They are exactly the 30 forms described in the theorem but the cases of $\mathcal{W}_{i,\mu}^{\mathcal{P}_k}$, $k = 3, 4, i = 1, 2, \ldots, 5$, should with condition $\mu = 0$.

Case (II) $\mu \neq 0$. Because f and g satisfy (12) and (13), it follows by Lemma 2.4 that the 30 cases of f, g listed in Table 2 can happen. In view of (14), we obtain

$$f(m) + f(-m) = -1 \text{ for all } m \neq 0.$$

This, together with a simple checking, yields the only 10 cases as $\mathcal{W}_{i,\mu}^{\mathcal{P}_k}$, $k = 3, 4, i = 1, 2, \ldots, 5$, with $\mu \neq 0$ are right. Thus, by (9)-(11) with $\mu \neq 0$, we get the corresponding post-Lie algebra structures.

Clearly, they are all graded post-Lie algebra structures on the $W(2,2)$ algebra. Finally, by Lemma 2.5 we know that the post-Lie algebras $\mathcal{W}_{i,\mu}^{\mathcal{P}_3^a}$, $\mathcal{W}_j^{\mathcal{P}_5}$ and $\mathcal{W}_j^{\mathcal{P}_6}$ are isomorphic to the post-Lie algebras $\mathcal{W}_{i,\mu}^{\mathcal{P}_4^a}$, $\mathcal{W}_j^{\mathcal{P}_7}$ and $\mathcal{W}_j^{\mathcal{P}_8}$ respectively, and the other post-Lie algebras are not mutually isomorphic. □

Remark 2.7. Theorem 2.6 tells us that, up to isomorphism, there are 17 types of graded post-Lie algebra structures satisfying (5)-(8) on the $W(2,2)$ algebra, that is $\mathcal{W}_k^{\mathcal{P}_1}$, $\mathcal{W}_k^{\mathcal{P}_2}$, $\mathcal{W}_{i,\mu}^{\mathcal{P}_3^a}$, $\mathcal{W}_j^{\mathcal{P}_5}$ and $\mathcal{W}_j^{\mathcal{P}_6}$ where $k \in \{1,2\}$, $i \in \{1,2,3,4,5\}$ and $j \in \{1,2,3,4\}$.

From Theorem 2.6 and Proposition 1.2 we can give some Lie algebras as follows.

Proposition 2.8. *Up to isomorphism, the post-Lie algebras in Theorem 2.6 give rise to the following* 11 *Lie algebras on the space with* $\mathbb{C}$*-basis* $\{L_i, H_i \mid i \in \mathbb{Z}\}$, *and with the bracket* $\{,\}$ *defined by Proposition 1.2* (*in every case* $\{H_m, H_n\} = 0$):

$$(\mathcal{LW}_1^{\mathcal{P}_1}): \ \{L_m, L_n\}_1^{\mathcal{P}_1} = (m-n)L_{m+n} \text{ for all } m,n \in \mathbb{Z};$$
$$\{L_m, H_n\}_1^{\mathcal{P}_1} = (m-n)H_{m+n} \text{ for all } m,n \in \mathbb{Z};$$
$$(\mathcal{LW}_2^{\mathcal{P}_1}): \ \{L_m, L_n\}_2^{\mathcal{P}_1} = (m-n)L_{m+n} \text{ for all } m,n \in \mathbb{Z};$$
$$\{L_m, H_n\}_2^{\mathcal{P}_1} = 0 \text{ for all } m,n \in \mathbb{Z};$$
$$(\mathcal{LW}_{1,\mu}^{\mathcal{P}_3^a}): \ \{L_m, L_n\}_{1,\mu}^{\mathcal{P}_3^a} = \begin{cases} (n-m)L_{m+n}, & m,n > 0, \\ (m-n)L_{m+n}, & m,n < 0, \\ -naL_n, & m = 0, n > 0, \\ -n(a+1)L_n, & m = 0, n < 0, \\ 0, & \text{otherwise}; \end{cases}$$
$$\{L_m, H_n\}_{1,\mu}^{\mathcal{P}_3^a} = \begin{cases} (m-n)H_{m+n}, & m < 0, \\ (-n(a+1)+\mu)H_n, & m = 0, \\ 0, & m > 0; \end{cases}$$
$$(\mathcal{LW}_{2,\mu}^{\mathcal{P}_3^a}): \ \{L_m, L_n\}_{2,\mu}^{\mathcal{P}_3^a} = \{L_m, L_n\}_{1,\mu}^{\mathcal{P}_3^a},$$
$$\{L_m, H_n\}_{2,\mu}^{\mathcal{P}_3^a} = \begin{cases} (n-m)H_{m+n}, & m > 0, \\ (-na+\mu)H_n, & m = 0, \\ 0, & m < 0; \end{cases}$$
$$(\mathcal{LW}_{3,\mu}^{\mathcal{P}_3^{a,b}}): \ \{L_m, L_n\}_{3,\mu}^{\mathcal{P}_3^{a,b}} = \{L_m, L_n\}_{1,\mu}^{\mathcal{P}_3^a},$$
$$\{L_m, H_n\}_{3,\mu}^{\mathcal{P}_3^{a,b}} = \begin{cases} (n-m)H_{m+n}, & m,n > 0, \\ (m-n)H_{m+n}, & m,n < 0, \\ (-na+\mu)H_n, & m = 0, n > 0, \\ (-n(a+1)+\mu)H_n, & m = 0, n < 0, \\ mbH_m, & m > 0, n = 0, \\ m(b+1)H_m, & m < 0, n = 0, \\ 0, & \text{otherwise}; \end{cases}$$
$$(\mathcal{LW}_{4,\mu}^{\mathcal{P}_3^a}): \ \{L_m, L_n\}_{4,\mu}^{\mathcal{P}_3^a} = \{L_m, L_n\}_{1,\mu}^{\mathcal{P}_3^a},$$
$$\{L_m, H_n\}_{4,\mu}^{\mathcal{P}_3^a} = \begin{cases} (n-m)H_{m+n}, & m > 0, n \geqslant 2, \\ (m-n)H_{m+n}, & m < 0, n \leqslant 1, \\ (-na+\mu)H_n, & m = 0, n \geqslant 2, \\ (-n(a+1)+\mu)H_n, & m = 0, n \leqslant 1, \\ 0, & \text{otherwise}; \end{cases}$$
$$(\mathcal{LW}_{5,\mu}^{\mathcal{P}_3^a}): \ \{L_m, L_n\}_{5,\mu}^{\mathcal{P}_3^a} = \{L_m, L_n\}_{1,\mu}^{\mathcal{P}_3^a},$$
$$\{L_m, H_n\}_{5,\mu}^{\mathcal{P}_3^a} = \begin{cases} (n-m)H_{m+n}, & m > 0, n \geqslant -1, \\ (m-n)H_{m+n}, & m < 0, n \leqslant -2, \\ (-na+\mu)H_n, & m = 0, n \geqslant -1, \\ (-n(a+1)+\mu)H_n, & m = 0, n \leqslant -2, \\ 0, & \text{otherwise}; \end{cases}$$

$$(\mathcal{LW}_1^{\mathcal{P}_5}):\ \{L_m,L_n\}_1^{\mathcal{P}_5}=\begin{cases}(n-m)L_{m+n}, & m,n\geqslant 2,\\(m-n)L_{m+n}, & m,n\leqslant 1,\\0, & otherwise;\end{cases}$$

$$\{L_m,H_n\}_1^{\mathcal{P}_5}=\begin{cases}0, & m\geqslant 2,\\(m-n)H_{m+n}, & m\leqslant 1;\end{cases}$$

$$(\mathcal{LW}_2^{\mathcal{P}_5}):\ \{L_m,L_n\}_2^{\mathcal{P}_5}=\{L_m,L_n\}_1^{\mathcal{P}_5},$$

$$\{L_m,H_n\}_2^{\mathcal{P}_5}=\begin{cases}(n-m)H_{m+n}, & m\geqslant 2,\\0, & m\leqslant 1;\end{cases}$$

$$(\mathcal{LW}_3^{\mathcal{P}_5}):\ \{L_m,L_n\}_3^{\mathcal{P}_5}=\{L_m,L_n\}_1^{\mathcal{P}_5},$$

$$\{L_m,H_n\}_3^{\mathcal{P}_5}=\begin{cases}(n-m)H_{m+n}, & m,n\geqslant 2,\\(m-n)H_{m+n}, & m,n\leqslant 1,\\0, & otherwise;\end{cases}$$

$$(\mathcal{LW}_4^{\mathcal{P}_5}):\ \{L_m,L_n\}_4^{\mathcal{P}_5}=\{L_m,L_n\}_1^{\mathcal{P}_5},$$

$$\{L_m,H_n\}_4^{\mathcal{P}_5}=\begin{cases}(n-m)H_{m+n}, & m\geqslant 2, n>0,\\(m-n)H_{m+n}, & m\leqslant 1, n\leqslant 0,\\0, & otherwise\end{cases}$$

where $a,b,\mu\in\mathbb{C}$.

Proof. Theorem 2.6 tells us that, up to isomorphism, there are 17 types of graded post-Lie algebra structure on $W(2,2)$ satisfying (5)-(8), which induced 17 types of Lie algebras by Proposition 1.2, and here are denoted by $\mathcal{LW}_k^{\mathcal{P}_1}$, $\mathcal{LW}_k^{\mathcal{P}_2}$, $\mathcal{LW}_{i,\mu}^{\mathcal{P}_3^a}$, $\mathcal{LW}_j^{\mathcal{P}_5}$ and $\mathcal{LW}_j^{\mathcal{P}_6}$ where $k\in\{1,2\}$, $i\in\{1,2,3,4,5\}$ and $j\in\{1,2,3,4\}$. On the other hand, the Lie algebras $\mathcal{LW}_k^{\mathcal{P}_1}$, $\mathcal{LW}_j^{\mathcal{P}_5}$ are isomorphic to the Lie algebras $\mathcal{LW}_k^{\mathcal{P}_2}$, $\mathcal{LW}_j^{\mathcal{P}_6}$ respectively through the linear transformation $L_m\to -L_{-m}, H_m\to -H_{-m}$. The conclusions are easily deducible. □

3. Application to Rota-Baxter operators

Lemma 3.1 (see [1])**.** *Let L be a complex Lie algebra and $R:L\to L$ a Rota-Baxter operator of weight* 1. *Define a new operation $x\circ y=[R(x),y]$ on L. Then $(L,[,],\circ)$ is a post-Lie algebra.*

In this section, by using Lemma 3.1 and Theorem 2.6, we mainly consider the homogeneous Rota-Baxter operator R of weight 1 on the W-algebra $W(2,2)$ given by

$$R(L_m)=f(m)L_m,\quad R(H_m)=g(m)H_m \tag{36}$$

for all $m\in\mathbb{Z}$, where f,g are complex-valued functions on $\mathbb{Z}$. We will prove the following.

Theorem 3.2. *A homogeneous Rota-Baxter operator R of weight* 1 *satisfying* (36) *on the W-algebra $W(2,2)$ must be one of the following types* (*where $a,b\in\mathbb{C}$*) *for all $m,n\in\mathbb{Z}$,*

$(\mathcal{R}_1^{\mathcal{P}_1}):\ R(L_m)=0, R(H_m)=0;$

$(\mathcal{R}_2^{\mathcal{P}_1}):\ R(L_m)=0, R(H_m)=-H_m;$
$(\mathcal{R}_1^{\mathcal{P}_2}):\ R(L_m)=-L_m, R(H_m)=0;$
$(\mathcal{R}_2^{\mathcal{P}_2}):\ R(L_m)=-L_m, R(H_m)=-H_m;$

$$(\mathcal{R}_1^{\mathcal{P}_3^a}):\ R(L_m)=\begin{cases}-L_m, & m>0,\\ aL_0, & m=0,\\ 0, & m<0;\end{cases} \qquad R(H_n)=0;$$

$$(\mathcal{R}_2^{\mathcal{P}_3^a}):\ R(L_m)=\begin{cases}-L_m, & m>0,\\ aL_0, & m=0,\\ 0, & m<0;\end{cases} \qquad R(H_n)=-H_n;$$

$$(\mathcal{R}_3^{\mathcal{P}_3^{a,b}}):\ R(L_m)=\begin{cases}-L_m, & m>0,\\ aL_0, & m=0,\\ 0, & m<0;\end{cases} \qquad R(H_n)=\begin{cases}-H_n, & n>0,\\ bH_0, & n=0,\\ 0, & n<0;\end{cases}$$

$$(\mathcal{R}_4^{\mathcal{P}_3^a}):\ R(L_m)=\begin{cases}-L_m, & m>0,\\ aL_0, & m=0,\\ 0, & m<0;\end{cases} \qquad R(H_n)=\begin{cases}-H_n, & n\geqslant 2,\\ 0, & n\leqslant 1;\end{cases}$$

$$(\mathcal{R}_5^{\mathcal{P}_3^a}):\ R(L_m)=\begin{cases}-L_m, & m>0,\\ aL_0, & m=0,\\ 0, & m<0;\end{cases} \qquad R(H_n)=\begin{cases}-H_n, & n\geqslant -1,\\ 0, & n\leqslant -2;\end{cases}$$

$$(\mathcal{R}_1^{\mathcal{P}_4^a}):\ R(L_m)=\begin{cases}-L_m, & m<0,\\ aL_0, & m=0,\\ 0, & m>0;\end{cases} \qquad R(H_n)=0;$$

$$(\mathcal{R}_2^{\mathcal{P}_4^a}):\ R(L_m)=\begin{cases}-L_m, & m<0,\\ aL_0, & m=0,\\ 0, & m>0;\end{cases} \qquad R(H_n)=-H_n;$$

$$(\mathcal{R}_3^{\mathcal{P}_4^a}):\ R(L_m)=\begin{cases}-L_m, & m<0,\\ aL_0, & m=0,\\ 0, & m>0;\end{cases} \qquad R(H_n)=\begin{cases}-H_n, & n<0,\\ bH_0, & n=0,\\ 0, & m>0;\end{cases}$$

$$(\mathcal{R}_4^{\mathcal{P}_4^a}):\ R(L_m)=\begin{cases}-L_m, & m<0,\\ aL_0, & m=0,\\ 0, & m>0;\end{cases} \qquad R(H_n)=\begin{cases}-H_n, & n\leqslant -2,\\ 0, & n\geqslant -1;\end{cases}$$

$$(\mathcal{R}_5^{\mathcal{P}_4^a}):\ R(L_m)=\begin{cases}-L_m, & m<0,\\ aL_0, & m=0,\\ 0, & m>0;\end{cases} \qquad R(H_n)=\begin{cases}-H_n, & n\leqslant 1,\\ 0, & n\geqslant 2;\end{cases}$$

$$(\mathcal{R}_1^{\mathcal{P}_5}):\ R(L_m)=\begin{cases}-L_m, & m\geqslant 2,\\ 0, & m\leqslant 1;\end{cases} \qquad R(H_n)=0;$$

$$(\mathcal{R}_2^{\mathcal{P}_5}):\ R(L_m)=\begin{cases}-L_m, & m\geqslant 2,\\ 0, & m\leqslant 1;\end{cases}\qquad R(H_n)=-H_n;$$

$$(\mathcal{R}_3^{\mathcal{P}_5}):\ R(L_m)=\begin{cases}-L_m, & m\geqslant 2,\\ 0, & m\leqslant 1;\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n\geqslant 2,\\ 0, & n\leqslant 1;\end{cases}$$

$$(\mathcal{R}_4^{\mathcal{P}_5}):\ R(L_m)=\begin{cases}-L_m, & m\geqslant 2,\\ 0, & m\leqslant 1;\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n>0,\\ 0, & n\leqslant 0;\end{cases}$$

$$(\mathcal{R}_1^{\mathcal{P}_6}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant 1,\\ 0, & m\geqslant 2;\end{cases}\qquad R(H_n)=0;$$

$$(\mathcal{R}_2^{\mathcal{P}_6}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant 1,\\ 0, & m\geqslant 2;\end{cases}\qquad R(H_n)=-H_n;$$

$$(\mathcal{R}_3^{\mathcal{P}_6}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant 1,\\ 0, & m\geqslant 2;\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n\leqslant 1,\\ 0, & n\geqslant 2;\end{cases}$$

$$(\mathcal{R}_4^{\mathcal{P}_6}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant 1,\\ 0, & m\geqslant 2;\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n\leqslant 0,\\ 0, & n>0;\end{cases}$$

$$(\mathcal{R}_1^{\mathcal{P}_7}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant -2,\\ 0, & m\geqslant -1;\end{cases}\qquad R(H_n)=0;$$

$$(\mathcal{R}_2^{\mathcal{P}_7}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant -2,\\ 0, & m\geqslant -1;\end{cases}\qquad R(H_n)=-H_n;$$

$$(\mathcal{R}_3^{\mathcal{P}_7}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant -2,\\ 0, & m\geqslant -1;\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n\geqslant -1,\\ 0, & n\leqslant -2;\end{cases}$$

$$(\mathcal{R}_4^{\mathcal{P}_7}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant -2,\\ 0, & m\geqslant -1;\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n<0,\\ 0, & n\geqslant 0;\end{cases}$$

$$(\mathcal{R}_1^{\mathcal{P}_8}):\ R(L_m)=\begin{cases}-L_m, & m\geqslant -1,\\ 0, & m\leqslant -2;\end{cases}\qquad R(H_n)=0;$$

$$(\mathcal{R}_2^{\mathcal{P}_8}):\ R(L_m)=\begin{cases}-L_m, & m\geqslant -1,\\ 0, & m\leqslant -2,\end{cases}\qquad R(H_n)=-H_n;$$

$$(\mathcal{R}_3^{\mathcal{P}_8}):\ R(L_m)=\begin{cases}-L_m, & m\geqslant -1,\\ 0, & m\leqslant -2,\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n\geqslant -1,\\ 0, & n\leqslant -2,\end{cases}$$

$$(\mathcal{R}_4^{\mathcal{P}_8}):\ R(L_m)=\begin{cases}-L_m, & m\geqslant -1,\\ 0, & m\leqslant -2,\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n\geqslant 0,\\ 0, & n<0.\end{cases}$$

Proof. In view of Lemma 3.1, if we define a new operation $x\circ y=[R(x),y]$ on $W(2,2)$, then $(W(2,2),[,],\circ)$ is a post-Lie algebra. By (36), we have

$$L_m\circ L_n=[R(L_m),L_n]=(m-n)f(m)L_{m+n},$$

$$L_m\circ H_n=[R(L_m),H_n]=(m-n)f(m)H_{m+n},$$

$$H_m \circ L_n = [R(H_m), L_n] = (m-n)g(m)H_{m+n},$$

and $H_m \circ H_n = [R(H_m), H_n] = 0$ for all $m, n \in \mathbb{Z}$. This means that $(W(2,2), [,], \circ)$ is a graded post-Lie algebra structure satisfying (5)-(8) with $\phi(m,n) = (m-n)f(m)$, $\varphi(m,n) = (m-n)f(m)$ and $\theta(m,n) = (m-n)g(m)$. By Theorem 2.6, we see that f, g must be of the 30 cases listed in Table 2, which can yield the 30 forms of R one by one. It is easy to verify that every form of R listed in the above is a Rota-Baxter operator of weight 1 satisfying (36). The proof is completed. □

4. Application to Yang-Baxter equation

First we give some notations. Let $\mathrm{ad} : \mathfrak{g} \to gl(\mathfrak{g})$ be the adjoint representation of a Lie algebra $\mathfrak{g}$ defined by $\mathrm{ad}(x)(y) = [x, y]$ for any $x, y \in \mathfrak{g}$. Let $\mathrm{ad}^* : \mathfrak{g} \to gl(\mathfrak{g}^*)$ be the dual representation of the adjoint representation of $\mathfrak{g}$. On the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$, there is a natural Lie algebra structure (denoted by $\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*$) given by

$$[x_1 + f_1, x_2 + f_2] = [x_1, x_2] + \mathrm{ad}^*(x_1)f_2 - \mathrm{ad}^*(x_2)f_1, \ \forall x_1, x_2 \in \mathfrak{g}, \ f_1, f_2 \in \mathfrak{g}^*.$$

A linear map is said to be of finite rank if its image has finite dimension. A linear operator R on $\mathfrak{g}$ of finite rank can be identified as an element in $\mathfrak{g} \otimes \mathfrak{g}^* \subset (\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*) \otimes (\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*)$ as follows. Let $\{e_i\}_{i\in I}$ be a basis of $\mathrm{Im}R$, then for $x \in g$, $R(x)$ can be written as a linear combination of the basis. Namely, for each $i \in I$ there exists a unique linear functional $R_i \in g^*$ such that

$$R(x) = \sum_{i\in I} R_i(x)e_i, \quad \forall x \in \mathfrak{g}.$$

From R is of finite rank we know that I is finite. Then we have

$$R = \sum_{i\in I} e_i \otimes R_i \in \mathfrak{g} \otimes \mathfrak{g}^* \subset (\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*) \otimes (\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*).$$

Lemma 4.1 ([13])**.** *Let $\mathfrak{g}$ be a Lie algebra and $R : \mathfrak{g} \to \mathfrak{g}$ a balanced linear map. Then R is a Rota-Baxter operator of weight 1 on $\mathfrak{g}$ if and only if both $(R - R^{21}) + \mathrm{Id}$ and $(R - R^{21}) - \mathrm{Id}^{21}$ are solutions of the formal CYBE on $\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*$.*

Lemma 4.2 ([13])**.** *R is a Rota-Baxter operator of weight 1 on a Lie algebra $\mathfrak{g}$ if and only if so is $-R - \mathrm{Id}$ on $\mathfrak{g}$ and*

$$((-R - \mathrm{Id}) - (-R - \mathrm{Id})^{21}) + \mathrm{Id} = -((R - R^{21}) - \mathrm{Id}^{21}).$$

In this paper, we only list the solutions of the CYBE obtained from $(R - R^{21}) + \mathrm{Id}$. Note that $\mathrm{Id} = \sum_{m\in\mathbb{Z}} L_m \otimes L_m^* + \sum_{n\in\mathbb{Z}} H_n \otimes H_n^*$ for $W(2,2)$.

By [13], a formal tensor $r = \sum_{i,j\in I} a_{ij} e_i \otimes e_j \in \mathfrak{g}\widehat{\otimes}\mathfrak{g}$, is called a solution of the formal CYBE if it is row-finite and column-finite and satisfies

$$[[r]](e_i, e_j, e_k) := \sum_{s,t\in I} (C^i_{st} a_{sj} a_{tk} + a_{is} C^j_{st} a_{tk} + a_{is} a_{jt} C^k_{st}) = 0$$

for all $i, j, k \in I$, where C_{rs}^i are the structural coefficients of $\mathfrak{g}$. A linear operator R on $\mathfrak{g}$ can be identified as an element in $\mathfrak{g}\widehat{\otimes}\mathfrak{g}^* \subset (\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*)\widehat{\otimes}(\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*)$ as follows. Let $\{e_i\}_{i\in I}$ be a basis of $\mathfrak{g}$ and $\{e_i^*\}_{i\in I}$ be its dual defined by

$$e_i^*(e_j) = \delta_{ij}, \quad \forall i, j \in I.$$

By Zorn's lemma, $\{e_i^*\}_{i\in I}$ can be extended to a basis of $\mathfrak{g}^*$, say $\{e_i^*\}_{i\in I} \cup \{f_i\}_{i\in J}$. Then we have

$$R = \sum_{i\in I} R(e_i) \otimes e_i^* + \sum_{j\in J} 0 \otimes f_j \in \mathfrak{g}\widehat{\otimes}\mathfrak{g}^* \subset (\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*)\widehat{\otimes}(\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*).$$

By a similar argument as in [13], we have the following theorem.

Theorem 4.3. *Lemma 4.2 gives the following solutions of the formal CYBE on $W(2,2)\ltimes_{\mathrm{ad}^*} W(2,2)^*$ from the Rota-Baxter operators of weight 1 on $W(2,2)$ given in Theorem 3.2, for some where $a, b \in \mathbb{C}$:*

$(\mathcal{Y}_1^{\mathcal{P}_1}):\ r_1^{\mathcal{P}_1} = \sum_{m\in\mathbb{Z}} L_m \otimes L_m^* + \sum_{n\in\mathbb{Z}} H_n \otimes H_n^*;$

$(\mathcal{Y}_2^{\mathcal{P}_1}):\ r_2^{\mathcal{P}_1} = \sum_{m\in\mathbb{Z}} L_m \otimes L_m^* + \sum_{n\in\mathbb{Z}} H_n^* \otimes H_n;$

$(\mathcal{Y}_1^{\mathcal{P}_2}):\ r_1^{\mathcal{P}_2} = \sum_{m\in\mathbb{Z}} L_m^* \otimes L_m + \sum_{n\in\mathbb{Z}} H_n \otimes H_n^*;$

$(\mathcal{Y}_2^{\mathcal{P}_2}):\ r_2^{\mathcal{P}_2} = \sum_{m\in\mathbb{Z}} L_m^* \otimes L_m + \sum_{n\in\mathbb{Z}} H_n^* \otimes H_n;$

$$(\mathcal{Y}_1^{\mathcal{P}_3^a}):\ r_1^{\mathcal{P}_3^a} = \sum_{m<0} L_m \otimes L_m^* + (a+1)L_0 \otimes L_0^* + \sum_{m>0} L_m^* \otimes L_m - aL_0^* \otimes L_0 + \sum_{n\in\mathbb{Z}} H_n \otimes H_n^*;$$

$$(\mathcal{Y}_2^{\mathcal{P}_3^a}):\ r_2^{\mathcal{P}_3^a} = \sum_{m<0} L_m \otimes L_m^* + (a+1)L_0 \otimes L_0^* + \sum_{m>0} L_m^* \otimes L_m - aL_0^* \otimes L_0 + \sum_{n\in\mathbb{Z}} H_n^* \otimes H_n;$$

$$\begin{aligned}(\mathcal{Y}_3^{\mathcal{P}_3^{a,b}}):\ r_3^{\mathcal{P}_3^{a,b}} = &\sum_{m<0} L_m \otimes L_m^* + (a+1)L_0 \otimes L_0^* \\ &+ \sum_{m>0} L_m^* \otimes L_m - aL_0^* \otimes L_0 \\ &+ \sum_{n<0} H_n \otimes H_n^* + (b+1)H_0 \otimes L_0^* \\ &+ \sum_{n>0} H_n^* \otimes H_n - bH_0^* \otimes H_0;\end{aligned}$$

$$(\mathcal{Y}_4^{\mathcal{P}_3^a}):\ r_4^{\mathcal{P}_3^a} = \sum_{m<0} L_m \otimes L_m^* + (a+1)L_0 \otimes L_0^* + \sum_{m>0} L_m^* \otimes L_m - aL_0^* \otimes L_0 + \sum_{n\leq 1} H_n \otimes H_n^* + \sum_{n\geqslant 2} H_n^* \otimes H_n;$$

$$(\mathcal{Y}_5^{\mathcal{P}_3^a}):\ r_5^{\mathcal{P}_3^a} = \sum_{m<0} L_m \otimes L_m^* + (a+1)L_0 \otimes L_0^* + \sum_{m>0} L_m^* \otimes L_m - aL_0^* \otimes L_0 + \sum_{n\leqslant -2} H_n \otimes H_n^* + \sum_{n\geq -1} H_n^* \otimes H_n;$$

$$(\mathcal{Y}_1^{\mathcal{P}_4^a}):\ r_1^{\mathcal{P}_4^a} = \sum_{m>0} L_m \otimes L_m^* + (a+1)L_0 \otimes L_0^* + \sum_{m<0} L_m^* \otimes L_m - aL_0^* \otimes L_0 + \sum_{n\in\mathbb{Z}} H_n \otimes H_n^*;$$

$$(\mathcal{Y}_2^{\mathcal{P}_4^a}):\ r_2^{\mathcal{P}_4^a} = \sum_{m>0} L_m \otimes L_m^* + (a+1)L_0 \otimes L_0^* + \sum_{m<0} L_m^* \otimes L_m - aL_0^* \otimes L_0 + \sum_{n\in\mathbb{Z}} H_n^* \otimes H_n;$$

$$(\mathcal{Y}_3^{\mathcal{P}_4^{a,b}}):\ r_3^{\mathcal{P}_4^{a,b}} = \sum_{m>0} L_m \otimes L_m^* + (a+1)L_0 \otimes L_0^* + \sum_{m<0} L_m^* \otimes L_m - aL_0^* \otimes L_0 + \sum_{n>0} H_n \otimes H_n^* + (b+1)H_0 \otimes H_0^* + \sum_{n<0} H_n^* \otimes H_n - bH_0^* \otimes H_0;$$

$$(\mathcal{Y}_4^{\mathcal{P}_4^a}):\ r_4^{\mathcal{P}_4^a} = \sum_{m>0} L_m \otimes L_m^* + (a+1)L_0 \otimes L_0^* + \sum_{m<0} L_m^* \otimes L_m - aL_0^* \otimes L_0 + \sum_{n\geqslant -1} H_n \otimes H_n^* + \sum_{n\leq -2} H_n^* \otimes H_n;$$

$$(\mathcal{Y}_5^{\mathcal{P}_4^a}):\ r_5^{\mathcal{P}_4^a} = \sum_{m>0} L_m \otimes L_m^* + (a+1)L_0 \otimes L_0^* + \sum_{m<0} L_m^* \otimes L_m - aL_0^* \otimes L_0 + \sum_{n\geq 2} H_n \otimes H_n^* + \sum_{n\leqslant 1} H_n^* \otimes H_n;$$

$$(\mathcal{Y}_1^{\mathcal{P}_5}):\ r_1^{\mathcal{P}_5} = \sum_{m\leqslant 1} L_m \otimes L_m^* + \sum_{m\geq 2} L_m^* \otimes L_m + \sum_{n\in\mathbb{Z}} H_n \otimes H_n^*;$$

$$(\mathcal{Y}_2^{\mathcal{P}_5}):\ r_2^{\mathcal{P}_5} = \sum_{m\leqslant 1} L_m \otimes L_m^* + \sum_{m\geq 2} L_m^* \otimes L_m + \sum_{n\in\mathbb{Z}} H_n^* \otimes H_n;$$

$$(\mathcal{Y}_3^{\mathcal{P}_5}):\ r_3^{\mathcal{P}_5} = \sum_{m\leqslant 1} L_m \otimes L_m^* + \sum_{m\geq 2} L_m^* \otimes L_m + \sum_{n\leqslant 1} H_n \otimes H_n^* + \sum_{n\geq 2} H_n^* \otimes H_n;$$

$$(\mathcal{Y}_4^{\mathcal{P}_5}):\ r_4^{\mathcal{P}_5} = \sum_{m\leqslant 1} L_m \otimes L_m^* + \sum_{m\geq 2} L_m^* \otimes L_m + \sum_{n\leq 0} H_n \otimes H_n^* + \sum_{n>0} H_n^* \otimes H_n;$$

$$
\begin{aligned}
(\mathcal{Y}_1^{\mathcal{P}_6}):\ r_1^{\mathcal{P}_6} &= \sum_{m\geqslant 2} L_m\otimes L_m^* + \sum_{m\leq 1} L_m^*\otimes L_m + \sum_{n\in\mathbb{Z}} H_n\otimes H_n^*;\\
(\mathcal{Y}_2^{\mathcal{P}_6}):\ r_2^{\mathcal{P}_6} &= \sum_{m\geqslant 2} L_m\otimes L_m^* + \sum_{m\leq 1} L_m^*\otimes L_m + \sum_{n\in\mathbb{Z}} H_n^*\otimes H_n;\\
(\mathcal{Y}_3^{\mathcal{P}_6}):\ r_3^{\mathcal{P}_6} &= \sum_{m\geqslant 2} L_m\otimes L_m^* + \sum_{m\leq 1} L_m^*\otimes L_m + \sum_{n\geqslant 2} H_n\otimes H_n^*\\
&\quad + \sum_{n\leq 1} H_n^*\otimes H_n;\\
(\mathcal{Y}_4^{\mathcal{P}_6}):\ r_4^{\mathcal{P}_6} &= \sum_{m\geqslant 2} L_m\otimes L_m^* + \sum_{m\leq 1} L_m^*\otimes L_m + \sum_{n>0} H_n\otimes H_n^*\\
&\quad + \sum_{n\leq 0} H_n^*\otimes H_n;\\
(\mathcal{Y}_1^{\mathcal{P}_7}):\ r_1^{\mathcal{P}_7} &= \sum_{m\geqslant -1} L_m\otimes L_m^* + \sum_{m\leq -2} L_m^*\otimes L_m + \sum_{n\in\mathbb{Z}} H_n\otimes H_n^*;\\
(\mathcal{Y}_2^{\mathcal{P}_7}):\ r_2^{\mathcal{P}_7} &= \sum_{m\geqslant -1} L_m\otimes L_m^* + \sum_{m\leq -2} L_m^*\otimes L_m + \sum_{n\in\mathbb{Z}} H_n^*\otimes H_n;\\
(\mathcal{Y}_3^{\mathcal{P}_7}):\ r_3^{\mathcal{P}_7} &= \sum_{m\geqslant -1} L_m\otimes L_m^* + \sum_{m\leq -2} L_m^*\otimes L_m + \sum_{n\leq -2} H_n\otimes H_n^*\\
&\quad + \sum_{n\geqslant -1} H_n^*\otimes H_n;\\
(\mathcal{Y}_4^{\mathcal{P}_7}):\ r_4^{\mathcal{P}_7} &= \sum_{m\geqslant -1} L_m\otimes L_m^* + \sum_{m\leq -2} L_m^*\otimes L_m + \sum_{n\geq 0} H_n\otimes H_n^*\\
&\quad + \sum_{n<0} H_n^*\otimes H_n;\\
(\mathcal{Y}_1^{\mathcal{P}_8}):\ r_1^{\mathcal{P}_8} &= \sum_{m\leqslant -2} L_m\otimes L_m^* + \sum_{m\geq -1} L_m^*\otimes L_m + \sum_{n\in\mathbb{Z}} H_n\otimes H_n^*;\\
(\mathcal{Y}_2^{\mathcal{P}_8}):\ r_2^{\mathcal{P}_8} &= \sum_{m\leqslant -2} L_m\otimes L_m^* + \sum_{m\geq -1} L_m^*\otimes L_m + \sum_{n\in\mathbb{Z}} H_n^*\otimes H_n;\\
(\mathcal{Y}_3^{\mathcal{P}_8}):\ r_3^{\mathcal{P}_8} &= \sum_{m\leqslant -2} L_m\otimes L_m^* + \sum_{m\geq -1} L_m^*\otimes L_m + \sum_{n\leqslant -2} H_n\otimes H_n^*\\
&\quad + \sum_{n\geq -1} H_n^*\otimes H_n;\\
(\mathcal{Y}_4^{\mathcal{P}_8}):\ r_4^{\mathcal{P}_8} &= \sum_{m\leqslant -2} L_m\otimes L_m^* + \sum_{m\geq -1} L_m^*\otimes L_m + \sum_{n<0} H_n\otimes H_n^*\\
&\quad + \sum_{n\geq 0} H_n^*\otimes H_n.
\end{aligned}
$$

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